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A Systematic Approach to the Instantaneous Duality of Mechanisms and its Application

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Abstract: The instantaneous duality (known also as symmetry) between serial chain manipulators and fully parallel systems is well known in the literature. This paper takes this idea one step further, by introducing a systematic method that transforms one mechanical system into another. This duality concept rests on the concept of dual graphs to define the kinematics of the dual system. The mechanism structure can be represented in two essentially different ways: its kinematic topology or its constraint system. The first is embodied by the topological graph and the second by the constraint graph. The dual to a topological graph is a constraint graph and vice versa. Hence there are various ways to introduce a dual topology. The dual kinematics is defined by instantaneously identifying the twist screws in the original system with the wrench screws of the dual. This identification allows carrying over statements from the original to the dual system. In particular it is shown that the singularities can be easily established in the dual if they are known in the original system. This concept of transference is a powerful tool since a variety of dual systems can be assigned to a given system. This idea is demonstrated for a Bricard mechanism that is instantaneously dual to a 6/6 Stewart platform at a singular position, and in a another configuration (resembling the cyclohexane molecule) is dual to the 6/3 Stewart platform at the singular position. This provides another perspective of the known mobility of this molecule.

Keywords: dual mechanisms, dual screw systems, dual graphs, singularities

1. Introduction

Duality has been proposed as a way to analyze a mechanism by considering another mechanism that is referred to as its *dual*. There is no unique mechanism that can be called ‘dual’ to a given mechanism, however. Moreover there are various ways to associate a mechanism dual to a given one. The aim of this paper is to propose a systematic duality concept that combines the concept of dual graphs and self-dual screw systems.

The aim of introducing a mechanism dual to a given mechanism is to transfer statements that can be made for one to the other. These statements are specific to particular the kinematics, i.e. the topology and geometry.

The construction of a dual kinematics involves two steps:

1. Designation of the *dual screw system*, and the
2. Designation of a *dual topology*.

Since the ‘duality’ in question is a design concept, rather than a mechanical principle, none of the two steps has a unique solution.

This freedom in defining dual objects allows for introducing a dual mechanism may be helpful for synthesis and understanding the original mechanism.

The notion of duality of serial and parallel manipulators was presented by Waldron & Hunt. In their original paper [1] they identified the joint screws of a 6R serial manipulator with the linear actuator forces of a fully parallel platform. This particular assignment implies that the screw system of the original manipulator and its dual are identical. It also implies that the dual topology of an open serial chain is a parallel platform. The latter assumption was concluded by observing the role the Jacobian, i.e. the screw system, plays for the velocity transformation in the serial and the force transformation in the parallel manipulator. In order to establish a systematic foundation for introducing the dual topologies, Shai [2] applied the concept of dual graphs, where mechanisms were represented as bar-joint graphs. Along this line the special class of parallel manipulators with self-dual topologies was investigated by Lambert & Herder in [3], and it was shown how results obtained for the dual manipulator can be employed for the analysis of the original one.

In all publications the (instantaneous) screw system of the dual mechanism is identified with the screw system of the original mechanism. The reason for this, and the actual motivation for introducing the duality concept, is that any statement about the instantaneous kinematics of a mechanism can be transferred to its dual, and vice versa. From the outset the duality is only instantaneous. Consequently it is in general not possible to transfer about global properties like the finite mobility or singularity loci. In [4] Bruyninckx classified two serial manipulators as being instantaneously dual if their screw systems are instantaneously equal. In [5] Gosselin & Lallemand analyzed the duality of redundant serial and parallel manipulators and suggested to adopt control schemes established for redundant serial manipulators to their parallel duals.

In this paper the notion of dual graph is used to define the dual mechanism. The concept rests on the representation of mechanism topology by a topological graph and a constraint graph introduced in section 2. The definition of the dual graph to a planar graph is recalled in section 4.1. This is used to introduce the dual to the topological graph in section 4.2 and to the constraint graph in section 4.3. The dual to a topological graph is interpreted as constraint graph, and the dual to a constraint is interpreted as topological graph. Together with the dual screw system they define the dual mechanism. In section 5 three types of duality are introduced: instantaneous, local, and global. This paper deals with instantaneous duality. A few examples are presented in section 6 that show how duality can be used to identify instantaneous kinematic properties of a mechanism upon its dual.

2. Graph Representation of Mechanisms

2.1 Topological Graph Γ

The topology, i.e. the arrangement of bodies and joints can be represented by a topological graph $\Gamma(V, E_\Gamma)$ where the vertices $v \in V$ represent bodies, and an edge $e = (u, v) \in E_\Gamma$ represents the existence of a joint between the bodies represented by u and v. In most mechanisms there is no more than one joint between two bodies, hence there is at maximum one edge between two vertices of Γ. The number of edges is denoted with $n_E^\Gamma = |E_\Gamma|$.

If moreover only 1-DOF joint are assumed, the topological graph represents possible 1-DOF relative motions of adjacent bodies. The edges of the topological graph thus have the physical meaning of relative twists of adjacent bodies. Hence to any edge e can be associated an instantaneous relative twist $\mathbf{V}_e = \dot{q}^e \mathbf{Y}_e$, where $\mathbf{Y}_e$ is the instantaneous screw coordinate vector of the relative twist (commonly expressed in the spatial frame), and $\dot{q}^e$ is the generalized speed of the joint represented by the edge e (here q^e is the joint variable). In other words the topological graph is a basic representation of the mechanism kinematics.

2.2 Constraint Graph G

The topological graph provides information about the relative DOF of adjacent bodies. The freedom of a 1-DOF joint is represented by one edge. Alternatively to representing the 'motion elements' of a mechanism, as in the topological graph, the constraint system can be represented the *constraint graph* $G(V, E_G)$. Vertices of G still represent bodies, but now a vertex $e = (u, v) \in E_G$ indicates the existence of one constraint between the bodies u and v. A (lower pair) 1-DOF joint imposes five constraints, i.e. edges, between adjacent bodies. It should be remarked that the constraint graph with vertices representing bodies is commonly called body-bar graph [6]. The number of edges is denoted with $n_E^G = |E_G|$. For mechanisms with 1-DOF joints $n_E^G = 5n_E^\Gamma$.

As an example the constraint graph for the overconstrained Bricard mechanism is shown in figure ??.

The edges of G have the physical meaning of constraint wrenches. Hence to an edge $e \in E_G$ can be assigned a wrench $\mathbf{W}_e = f_e \mathbf{Y}_e^*$, with instantaneous screw coordinate vector $\mathbf{Y}_e^*$ and intensity f_e.

Edges of G represent kinetostatic variables (wrenches) that are dual to these of Γ (twists). Both graphs represent the same mechanism, however. Moreover the relative twist screw of the joint connecting two bodies, represented by edge e in the topological graph Γ, is reciprocal to the five constraint wrenches represented by the edges between these bodies in the constraint graph G. Since there is a 5-dimensional variety of constraint wrenches reciprocal to a joint screw, this assignment is not unique. On the other hand the twist screw of a joint is reciprocal to itself and thus provides one unique reciprocal screw giving rise to one unique constraint wrench.

3. The Dual Screw Systems

3.1 Self-Dual Twist and Wrench Systems

A twist is represented by a 6-vector of the form $\mathbf{V} = (\boldsymbol{\omega}, \boldsymbol{v})^T$, where $\boldsymbol{\omega} \in R^3$ is the angular and $\mathbf{v} \in R^3$ the translational velocity. A twist can be expressed in terms of the speed magnitude ω and screw coordinates $\mathbf{Y} = (\mathbf{e}, \mathbf{p} \times \mathbf{e} + h\mathbf{e})^T$, as $\mathbf{V} = \omega \mathbf{Y}$, where h is the pitch.

Wrenches are represented by 6-vectors of the form $\mathbf{W} = (\mathbf{f}, \mathbf{m})^T$, where $\mathbf{m} \in R^3$ is the torque and $\mathbf{f} \in R^3$ the force of the couple. The screw coordinates of the couple generated by a force $\mathbf{f} = f\mathbf{e}$ acting at the point $\mathbf{p} \in R^3$ is $\mathbf{W} = f(\mathbf{e}, \mathbf{p} \times \mathbf{e} + h\mathbf{e})^T$. The wrench is thus determined by the magnitude f and the screw coordinate vector.

A screw coordinate vector $\mathbf{Y}$ can either represents a twist or a wrench. Using the above notation convention the rotational and translational part of twists and wrenches are interchanged, since the direction unit vector $\mathbf{e}$ of the screw represents the angular velocity and force, respectively.

Twists are elements of the vector space $se(3) \cong R^6$ –the Lie algebra of screws. As to any vector space, there is a dual space, denoted $se^*(3)$, consisting of linear operators on $se(3)$. These can be identified with wrenches that are represented by screw coordinate vectors $\mathbf{W} \in R^6 \cong se^*(3)$. Therefore when considered as linear operators the wrench screws are occasionally called co-screws. The pairing of $\mathbf{W} \in R^6$ and $\mathbf{V} \in R^6$ is given by the reciprocal product $(\mathbf{W}, \mathbf{V}) = \boldsymbol{\omega} \cdot \boldsymbol{\tau} + \mathbf{v} \cdot \mathbf{f}$ which has the physical meaning of power.

With the dual meaning of screws the notion of dual screw system was introduced in [1]. The conceptual idea is to identify the screws $\mathbf{Y}_e$ associated to the joint twists with the screws $\mathbf{Y}_e^*$ defining the wrench system of another mechanism that is called the 'dual mechanism'. In the following a wrench and twist system that have an identical screw system, i.e. $\mathbf{Y}_e = \mathbf{Y}_e^*$, are called *self-dual*.

Self-dual screw systems have the property that by definition the wrench $\mathbf{W}_e = f_e \mathbf{Y}_e$ is reciprocal to the twist $\mathbf{V}_e = \dot{q}^e \mathbf{Y}_e^*$. Moreover $\mathbf{W}_e$ is a unique constraint wrench reciprocal to the joint twist $\mathbf{V}_e$.

3.2 Introducing a Dual Screw System

As described in section 2 a mechanism can be represented by the topological graph Γ or the constraint graph G. Edges of Γ stand for joint twists, and edges of G for joint constraint wrenches. In either case screws are assigned to the edges that constitute a screw system. The dual screw system is defined as the self-dual system.

Given a screw system S (S^*) representing joint twists (constraint wrenches) of a mechanism, the *dual screw system* S^* (S) representing constraint wrenches (joint twists) is defined is defined via the identification $S = S^*$.

Yet nothing is said about the kinematics and topology. If the dual screw system is supposed to be the screw system of a (yet to be defined) dual mechanism, the dual topology must necessarily contain the same number of edges as the original (n_E^{Γ} or n_E^G). This can be achieved making use of the concept of dual graphs described below.

4. Dual Topologies

4.1 Duality of Planar Graphs

A planar graph is a graph G that can be embedded in the plane without any edge intersecting. To a planar graph G can be associated a *geometric dual graph* G^* with the same number of edges. This rests upon the notion of faces (a face is a region surrounded by a cycle without any edges reaching from the cycle into the region). As the plane embedding of the topological graph of a mechanism is not unique there can be different non-isomorphic geometric dual graphs. However, it is desirable that this relation is in a sense one-one. The question is whether the dual of the dual G^* is isomorphic to the original G. It is known ([7], theorem 5.10) that this is only so if G^* is the *combinatorial dual* of G. This is so if there is a one-one correspondence of the edges in G with the edges in G^* such that a subset of the edges in G forms a cycle if and only if the corresponding set of edges in G^* forms a cut-set. This is a more general notion of a dual graph since it does not rely on the faces of G but is based on the cycles in G. Therefore the combinatorial dual is used in the following.

This applies to any graph, and in the following the dual of the topological graph Γ and of the constraint graph G will be used.

Accepting that the vertices in the original graph and in its dual represent bodies, the idea is to *define* a dual mechanism with topology defined by the dual graph. Notice that this duality is a conceptual construct rather than a mechanical principle.

4.2 The Dual of a Topological Graph $\boldsymbol{\Gamma}$

Yet the dual screw system is not assigned to a dual kinematics. The dual kinematics is determined by the dual to the graph representing the mechanism.

The topological graph indicates the existence of joints between adjacent bodies. An edge in $\boldsymbol{\Gamma}$ gives rise to an edge in the dual $\boldsymbol{\Gamma}^*$. The dual of a topological graph is a constraint graph. Hence there are as many constraint wrenches in the dual as there are joints in the topological graph, namely n_E^{Γ}. Vertices of the dual constraint graph correspond to bodies in the dual mechanism. Clearly the number of vertices (bodies) in the topological graph differs from the number of vertices in the dual constraint graph.

This is shown for the Bricard mechanism in **Figure 1**. The topological graph comprises 6 vertices (bodies) mutually connected by 1-DOF joints. Hence the dual topology consists of two vertices (bodies) and 6 wrenches between them. This mechanism is identified as a Stewart platform. The constraint graph of the latter is the dual to the topological graph of the former.

Remark: Waldron & Hunt [1] introduced the concept of serial-parallel duality, and a platform connected to the ground by n kinematic chains was identified as the dual mechanism to a serial chain with n joints. This is not conforming to the dual topology deduced from the dual of the topological graph. The dual topology of an open chain with n joints is a single body where n wrenches acting within the body. Within the dual graph approach the end-effector of the serial chain must be connected to ground so forming a loop, i.e. the end-effector configuration is specified.

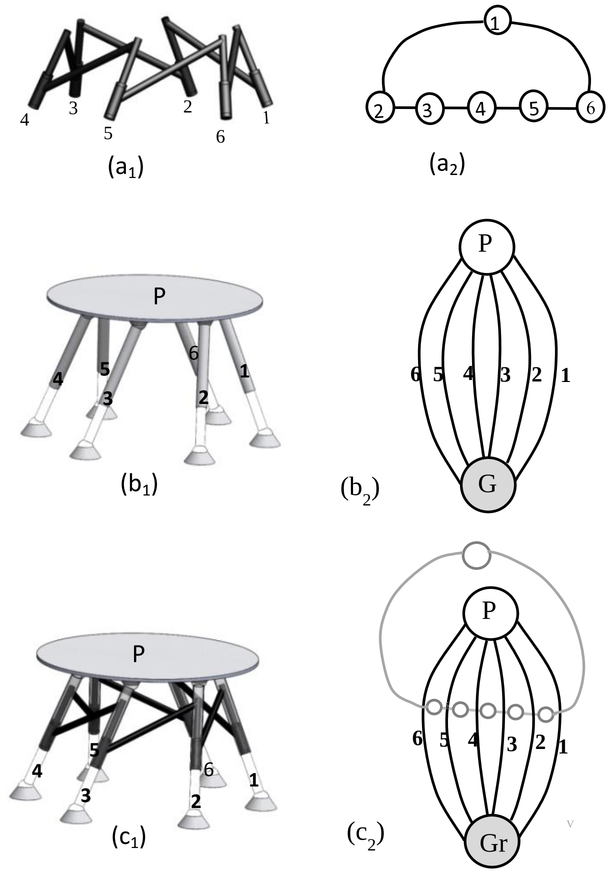

Figure 1 Bricard mechanism a1) and its topological graph a2). Steward platform b1) and its constraint graph b2) when only the platform P and the ground Gr are considered. Superposition of the topological and constraint graph of the mechanisms c).

4.3 The Dual of a Constraint Graph $\boldsymbol{G}$

The constraint graph contains one edge for each constraint wrench. The dual of a constraint graph G is a topological graph G^*. An edge in G corresponds to an edge in G^*, and both have n_E^G edges. Indeed the number of vertices, i.e. bodies is different.

Figure 2 shows this for the Bricard mechanism. The constraint graph consists of 6 vertices (bodies) mutually connected by 5 edges (constraints). The dual topology consists of two vertices (bodies) that are connected by 6 kinematic chains, each containing 5 joints and 4 bodies.

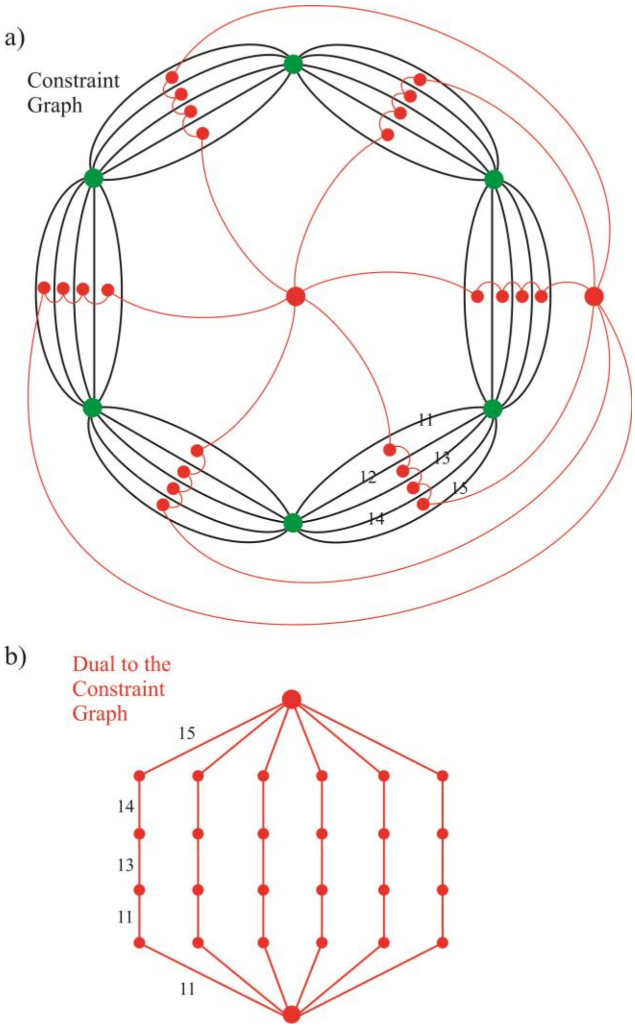

Figure 2 Constraint graph of the Bricard mechanism a), and the dual graph b) (also superposed to the constraint graph).

4.4 Fundamental Cycle and Cut Sets

The graph G possess $\gamma = n_E^G - |V| + 1$ fundamental cycles (this is the cyclomatic number). The number of fundamental cycles of G is equal to the number of fundamental cut sets of its dual G^*. Figure 3 shows this for the topological graph of the Bricard mechanism. That is, to any fundamental cycle corresponds a fundamental cut set.

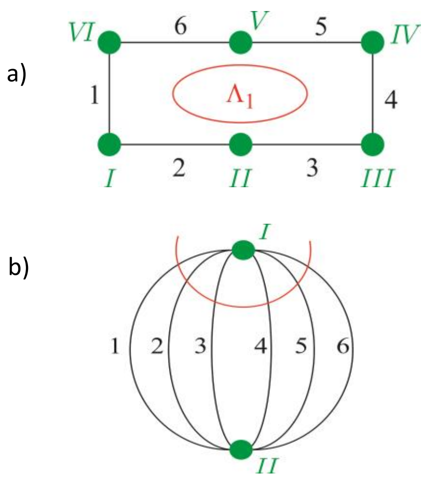

Figure 3 a) The topological graph G of the Bricard mechanism has one fundamental cycle Λ_1. b) Its dual graph G^* has one fundamental cut.

5. The Dual Mechanism

Combining the dual topology and the dual screw system gives rise to the dual mechanism. That is, the dual topology determines how the dual screw system is arranged in the dual mechanism.

The mechanism dual to the topological graph of the Bricard in **Figure 1**a) turns out to be a structure equivalent to a Stewart platform in **Figure 1**b) where the wrenches are actuation forces. Notice, however, that only the forces acting on the platform are specified, the actual joints connecting the linear actuators are arbitrary.

When starting from the constraint graph of the Bricard mechanism the dual mechanism is a parallel manipulator in Figure 2 consisting of a moving platform connected to a fixed platform by 6 kinematic chains. Each of the 6 chains contains consists of 5 1-DOF joints.

Obviously, for a given mechanism, the actual dual mechanism depends on the graph representation that is used to define the dual graph. Moreover, the identification of the screw dual system is generally only possible in a specific configuration. For instance, since the Bricard and its dual Stewart platform have a different kinematic it is clear that the identification of the joint twist screws of the Bricard with the constraint/actuator wrench screws is only valid in particular configurations.

To generalize this a classification of the duality is in order. From a kinematic viewpoint instantaneously a mechanism is uniquely defined by its joint screw system S and its topological graph Γ. A mechanism is thus symbolized by $M(S,\Gamma)$. Γ describes the topology, i.e. the arrangement of joints and bodies. From a static viewpoint instantaneously a mechanism is uniquely defined by the screw system S^* defining (constraint) wrenches between interconnected bodies and its constraint graph G. This is symbolized by $M^*(S^*,G)$. G describes the presence of constraints between the bodies.

These are two equivalent descriptions of mechanisms since a given mechanism can be described by M or M^*. Denote with $\mathbf{q}$ the vector of generalized coordinates (joint angles) describing the mechanism's configuration, and with V the mechanism's configuration space (joint space). Assume that the topological graph is planar.

Definition 1:

$M(S,\Gamma)$ is *instantaneously dual* to $M^*(S^*,G)$ at the configuration $\mathbf{q} \in V$, iff $S(\mathbf{q}) = S^*(\mathbf{q})$ and G is the dual graph of Γ.

$M(S,\Gamma)$ is *locally dual* to $M^*(S^*,G)$ at $\mathbf{q} \in V$, iff its instantaneously dual to $M^*(S^*,G)$ on a subvariety of V containing $\mathbf{q}$, and G is the dual graph of Γ.

$M(S,\Gamma)$ is (globally) *dual* to $M^*(S^*,G)$, iff it is locally dual to $M^*(S^*,G)$ at any configuration $\mathbf{q} \in V$.

The dual mechanism to $M(S,\Gamma)$ is thus $M^*(S^*,\Gamma^*)$.

Analogously M^* can be defined to be dual to M by simply interchanging them, and G and Γ, in the above definition. This duality is not reciprocal, i.e. M being dual to M^* does not necessarily imply that M^* is dual to M. This is so because the dual of a graph is not unique.

In this paper only instantaneous duality is considered.

6. The Dual Instantaneous Kinematics

6.1 Kinematic-Static Equivalence

By construction the (instantaneous) screw system of the original mechanism and its dual are identical, i.e. the twists of one and the wrenches of the other have the same geometry. This was called self-duality. The significance of this instantaneous duality of two mechanism is that (instantaneous) statements made for one mechanism can be carried over to the other.

Consider the instantaneous duality of Bricard mechanism and Stewart platform for instance. The Bricard mechanism has one kinematic loop and satisfies the loop closure condition

$$\sum_{e\in\Gamma} \dot{q}^e \mathbf{Y}_e = \mathbf{0} \qquad (1)$$

The Stewart platform satisfies the wrench closure condition

$$\sum_{e\in\Gamma^*} f^e \mathbf{Y}_e = \mathbf{0} \qquad (2)$$

In general, as explained in section 4.4, (with appropriate labeling) the fundamental circuits Λ_i of Γ are the fundamental cut sets of Γ^*. Then the kinematic loop closure condition for $M(S,\Gamma)$ is

$$\sum_{e\in\Lambda_\mathrm{i}} \dot{q}^e \mathbf{Y}_e = \mathbf{0}, \qquad i = 1 \dots \gamma \qquad (3)$$

and the static balance for $M^*(S^*,G)$ is

$$\sum_{e\in\Lambda_\mathrm{i}} f_e \mathbf{Y}_e = \mathbf{0}, \qquad i = 1 \dots \gamma. \qquad (4)$$

Hence the rank of the screw system, i.e. the dependence of $\mathbf{Y}_e$ is crucial. This reveals in particular the instantaneous mobility of the mechanism $M(S,\Gamma)$ and the wrench closure of $M^*(S^*,\Gamma^*)$.

6.2 Instantaneous Mobility and Singular Configurations

The (instantaneous) constraint system of dual mechanisms is deduced from the (instantaneous) kinematics of the original mechanism. Duality can hence be used to conclude the (instantaneous) kinematic properties of a given mechanism by looking at its dual and vice versa.

Conditions for screw systems having a certain rank have been the subject in kinematics and geometry, and according classifications of n-systems exist, e.g. [8, 9, 10, 11]. On the other hand the conditions for the dual mechanism being in a special configuration where the screws become linear dependent can possibly be identified by inspection or may be known already. Then this allows identification of special configurations of the original mechanism. A few examples are presented in the following.

1.) Consider again the Bricard mechanism in **Figure 1**a). The condition for the overconstrained Bricard mechanism to be mobile is that the 6 joint axes intersect a common line; they form a 5 system. This property is (instantaneously) shared by its dual, the Stewart platform in **Figure 1**b). Hence only 5 actuation wrenches of the latter are linearly independent, and the manipulator is in a singularity (force singularity, parallel singularity). However, the kinematics of both mechanisms is different. The geometry of the Bricard mechanism is such that the dependence is preserved in all configurations, which is the condition for mobility. For the Stewart platform this is a special configuration where the platform cannot withstand a torque about the line that all actuator forces intersect. In this example the Stewart platform is constructed as the dual $M^*(S^*,\Gamma^*)$ of the Bricard mechanism $M(S,\Gamma)$.

2.) Starting from the constraint graph description $M(S,G)$ of the Stewart platform (modeled as two platforms), a Bricard mechanism can be defined as its dual $M^*(S^*,G^*)$. Consider the 6/3 Stewart platform (6 anchor points on the ground and 3 at the platform) in Figure 4a). The six screws correspond to the actuator forces. The dual mechanisms is the single-loop 6R Bricard mechanism in Figure 4b). The condition for a force singularity are indicated in Figure 4a). This condition is retained by the closed loop 6R mechanism which ensures its mobility. This particular mechanism is kinematically equivalent to the cyclohexane molecule (Figure 4d), which is known to be mobile with 1 DOF.

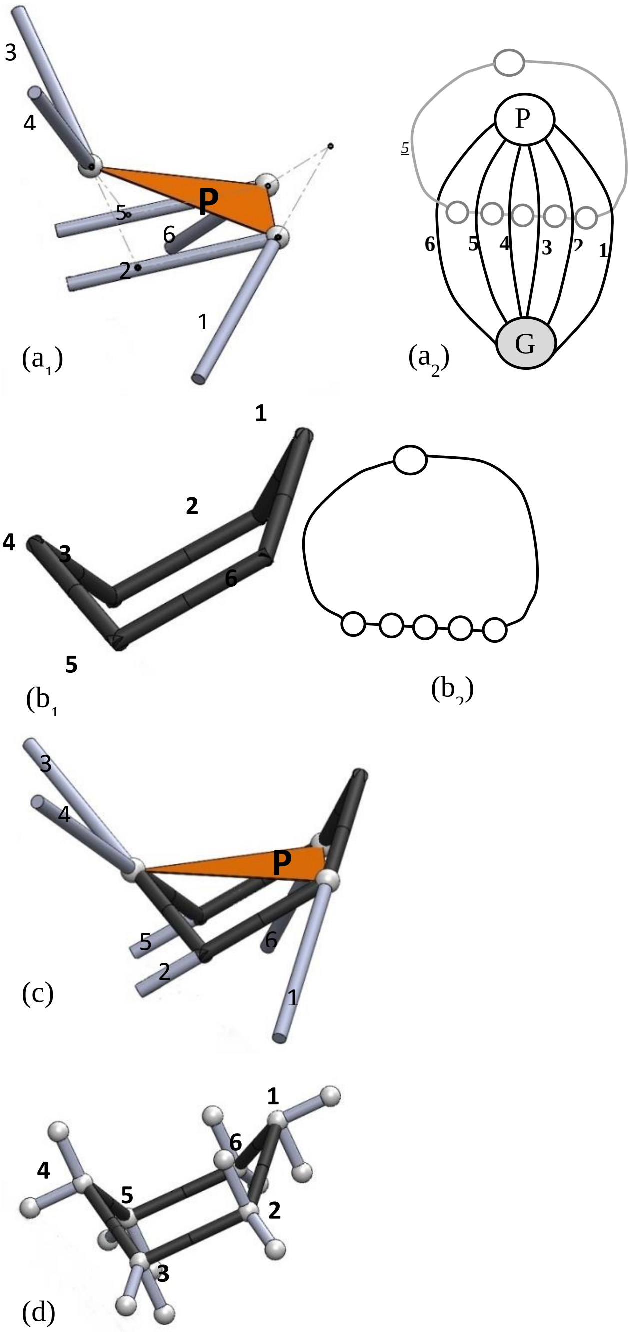

Figure 4 6/3 Stewart platform (a_1) and its constraint graph (a_2). Its dual 6R Bricard mechanism (b_1) and its topological graph (b_2). The cyclohexane molecule (d) is a physical realization of the 6R mechanism.

3.) Instead of representing the 6R Bricard mechanism as $M(S, \Gamma)$ using the topological graph, it can also be represented as $M(S, G)$ using the constraint graph in Figure 2a). The dual topology is shown in Figure 2b). Combined with the screw system this leads to the dual mechanism in Figure 5. This is a moving platform connect with 6 chains to the fixed platform/ground. Since the constraint wrenches for the revolute joints in the Bricard mechanism are not unique, the joint screws, and thus the joints in the chains are not uniquely specified. Any combination of joints can be used as long as their screws are reciprocal to those of the revolute joints in the Bricard mechanism. Consequently all statements about the instantaneous kinematics of the Bricard mechanism apply to all platforms of this class.

By construction all 5 joint screws of a chain of the platform are reciprocal to the corresponding screw of the revolute joint in Bricard mechanism. The fact that all revolute joint axes intersect a common line implies that the platform cannot withstand a torque about this common line. This holds for any such platform dual to the Bricard mechanism $M(S, G)$ regardless of the particular assignment of the 5 joints within the respective chain.

This example shows the potential of the duality concept.

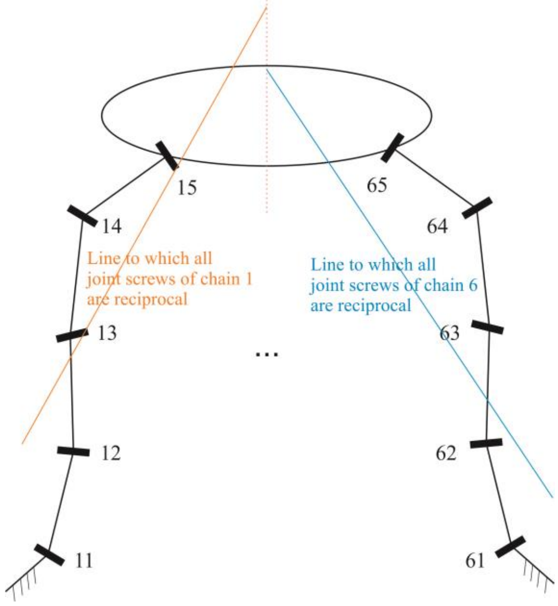

Figure 5 Dual mechanism deduced from the constraint graph of the Bricard mechanism. The joint kinematics of the six chains is arbitrary, but since the joint axes of the Bricard mechanism intersect a common line the platform cannot withstand a torque about that line.

7. Conclusions and Further Research

The paper introduces a systematic approach for constructing dual mechanisms resting on dual graphs and self-dual screw systems. The approach employs the concept of dual graphs to define the topology of the dual mechanism. Since there is no natural dual topology this is a design principle rather than an inherent mechanical principle. This work is part of a general combinatorial approach, in which combinatorial methods are used to solve mechanical engineering problems in a systematic way, such as: determining the generic/topological mobility, decomposing any mechanism into building blocks and more. The combinatorial and graph theoretical approach enables dealing with complicated systems in a systematic way. Future work will address multiloop systems, whose duals are cooperating parallel manipulators, and thus allows studying the singularities of complex mechanical systems in a systematic manner. The duality principle is also applicable to the analysis of gears, which has been discussed in [12, 13, 14].

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