Finding Dead-Point Positions of Planar Pin-Connected Linkages Through Graph Theoretical Duality Principle

O. Shai
e-mail: shai@eng.tau.ac.il

I. Polansky

Department of Mechanics Materials and Systems,
Faculty of Engineering,
Tel-Aviv University,
Tel-Aviv, 69978 Israel

The paper brings another view on detecting the dead-point positions of an arbitrary planar pin-connected linkage by employing the duality principle of graph theory. It is first shown how the dead-point positions are derived through the interplay between the linkage and its dual determinate truss—the relation developed in the previous works by means of graph theory. At the next stage, the process is shown to be performed solely upon the linkage by employing a new variable, the dual of potential, termed face force. Since the mathematical foundation of the presented method is discrete mathematics, the paper points to possible computerization of the method. [DOI: 10.1115/1.2179461]

Keywords: dead-point position, duality, graph theory, double-butterfly linkage, face force

1 Introduction

Dead-point position of a linkage is such a geometrical configuration at which the linkage loses its mobility [1]. In the field of machine design it is important to be aware of the possible dead-point configurations of the given planar linkage, as in some cases such configurations constitute an undesirable obstacle for the valid operation of the linkage, while in other cases they constitute a part of the linkage functionality [2,3]. Additional applications include the synthesis of mechanisms, where the criterion of the mechanism motion between the dead-point positions is used to reduce the considered space of solutions [4].

The problem of finding dead-point positions of engineering systems is widely reported in the literature [5–7], due to its importance in many fields of engineering practice [8] and engineering design [9]. Some works consider the problem of finding the dead-point positions for the mechanism of a given geometry, while others relate to enumerating all the possible dead-point configurations of a mechanism where only the topology is given. The first problem is also known as a problem of limit positions. Analytical solutions for this problem have been obtained for various types of linkages [10]. Some methods for solution of the first type of problem often involve analysis of the corresponding Jacobian matrix [11] which is known to provide an efficient solution.

The challenge of the second problem, much more ill defined, is that it gives the engineer the possible dead-point positions that can occur in potential in the specific topology of the linkage. Knowing these positions can assist the engineering designers in taking advantage of the dead points in such applications as toggle mechanisms [12], or in preventing the mechanism from entering the locked positions. One of the known works done in this field is by Yan and Wu [5], who employed the properties of linkage instant centers to find the dead-point configurations of the mechanism. This method employs the principles of instant centers of the linkage, thus it is subject to the limitation that there are linkages where the instant centers are difficult to be found by employing Kennedy's theorem [13], such as in the known double-butterfly mechanism [14].

The idea underlying this paper is to bring another view to this problem, by employing the knowledge existing in statics through the duality relation between the planar pin-connected linkages, for brevity referred here as linkages, and planar determinate trusses, which are briefly described in Sec. 2. The latter duality relation was established by employing graph-theoretic representations of the corresponding systems, as is thoroughly described in the previous publications of [15,16] and other works done in the field. The use of graph-theoretic duality in engineering is well known in the literature, in particular in electrical networks [17], and in one-dimensional engineering systems, such as one-dimensional dynamic systems, hydraulics and more [18]. The pioneer in establishing the graph-theoretic duality in mechanical applications was Andrews who established the vector-network model [19], with which he represented multidimensional dynamic systems, and for them constructed the dual systems [20]. Recent developments in this research direction are due to McPhee [21] and Lang [22].

The approach presented in the paper sees the problem of finding the dead-point positions for a given topology as a problem in statics, where one needs to find a geometric configuration capable of sustaining forces applied upon it.

As is depicted schematically in Fig. 1, such configurations constitute isolated points in the overall space of all the possible geometric configurations corresponding to the given topology. The problem of enumerating these configurations is therefore a sophisticated problem that does not possess an apparent solution. In Sec. 3 this problem is transformed through the mathematical basis of graph theory into an isomorphic problem in the dual engineering domain—kinematics. This new kinematical problem is a problem of finding possible sets of feasible deformations of the rods in the dual structure. It will be shown that the sets of feasible deformations form a subspace within the space of deformation and it can be straightforwardly spanned by means of the joint displacement variables corresponding to the structure (Fig. 1). Accordingly, it is argued in the paper that upon transforming to the second engineering domain, the solution to the original problem of detecting the dead-point positions becomes transparent. The methodology employing this transformation for finding the dead-point positions of a given linkage is presented in Sec. 4.

For the sake of convenience of implementation of the methodology, in Sec. 5, the method is developed solely in the terminology of linkages. In order to make this possible, a new variable,

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received August 31, 2004; final manuscript received June 4, 2005. Review conducted by Q. Jeffrey Ge.

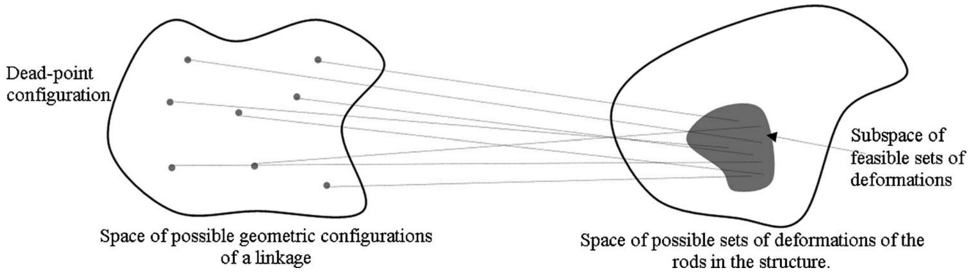


Fig. 1 Schematic illustration of the essence of the suggested methodology.

which is the dual of the potential in the original graph, is employed. In physical implementation, the variable is the dual of the absolute displacement of the joint in the dual truss, and is termed "face force" [23].

Section 6 is dedicated to the study of the size of the finite space of the possible dead-point positions on the basis of the techniques developed in the paper.

2 Duality Between Kinematics of Planar Linkages and Statics of Planar Trusses

The term duality (sometimes referred as reciprocity) employed in the paper corresponds to a powerful concept impacting science in general [24], and mechanics in particular [25,26]. The term is widely used in the literature, sometimes for defining completely different issues, thus to avoid confusion a special notion for it is provided below.

One of the widely known duality principles in engineering finds its origin in the virtual work theorems which result in the orthogonality between the kinematic and the static variables underlying the behavior of a structure. The mathematical basis in this case is linear algebra, and the relation can be traced through similarities in the corresponding matrices. The approach adopted in this paper, although it also relates between the domains of statics and kinematics, is utterly different. The mathematical principle we adopt is the duality between graphs [17]. Furthermore, the resulting relation is between two distinct engineering systems, possessing dual topology, dual geometry—but the same behavior of the variables. In this respect, the presented approach resonates with other duality developed in mechanical engineering research based on screw theory, as appears in [27–29]. In these works it was shown that the equations underlying the statics and the kinematics of these two engineering domains are the same. This correspondence makes them dual to one another, as in the case of the duality relation between parallel and serial manipulators.

In the previous publications of the author [15] it was proved, on the basis of graph theory, that there exists a tight mathematical relation between linkages and trusses. This relation is referred as a "duality relation," since it is based on the duality principle of graph theory as is defined in Proposition 1.

PROPOSITION 1 (Shai, 2001). *For every planar determinate truss, T, there exists a dual planar pin-connected linkage, L, satisfying*:

- Each link in *L* corresponds to a truss element (rod, external force, reaction) of *T* and vice versa.
- The vector of the relative velocity of each link in *L* is equal to the vector of the force acting in the corresponding element in *T*.

Table 1 The correspondence between the terminologies of trusses and linkages.

Terminology of trusses	Terminology of linkages
Truss element (rod, external force, reaction)	Link
Area closed by rods—face	Kinematical pair—link joint
Internal force of the element	Relative linear velocity of the link

Table 1 lists the transformation rules [15] defining a complete correspondence between the determinate truss and the corresponding dual linkage.

From Table 1 one can see that the process of constructing the truss dual to a given linkage or vice versa is actually quite similar to that of constructing a dual graph [17]. This relation gave rise to the correspondence between the equations stating the equilibrium of forces in the truss and the equations of compatibility of the dual linkage as will be shown below.

Figure 2(*a*) shows an example of a simple truss and Fig. 2(*b*) shows its corresponding dual linkage that is obtained by applying the transformation rules listed in Table 1.

The mathematical correspondence between these two simple systems will now be examined, to facilitate the understanding of the truss-linkage duality.

The truss of Fig. 2(*a*) is comprised of three structural elements: rods 1 and 2 and the external force 3. The linkage, on the other hand, is built of three kinematical elements—links $1'$, $2'$ and $3'$, while $3'$ is the driving link of the linkage. Link $3'$ is chosen to be the driving link due to its correspondence to the external force 3 in the truss, which in contrast to other truss elements is associated with a predetermined force vector. Elements $1'$, $2'$ and $3'$ in the linkage correspond respectively to elements 1, 2 and 3 in the truss. Elements 1, 2 and 3 in the truss meet at the same joint, thus yielding the following vector equation for the internal static forces (West, 1993):

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_3 \tag{1}$$

In terms of network theory, Eq. (1) is called the "cutset equation" (Swamy and Thulasiraman, 1981).

Links $1'$, $2'$ and $3'$, on the other hand, form a contour in the linkage, thus their corresponding relative linear velocities comply with the following kinematical vector equation (Norton, 1992):

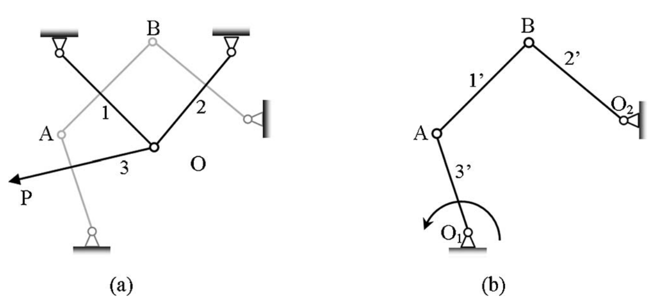


Fig. 2 (*a*) A simple truss with the dual linkage superimposed and (*b*) the dual linkage

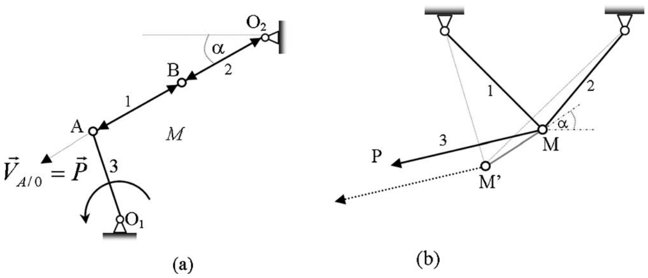


Fig. 3 **(*a*) Linkage in a dead-point position and (*b*) the corresponding deformation configuration of the truss with the dual topology**

$$\mathbf{V}_{1'} + \mathbf{V}_{2'} = \mathbf{V}_{3'} \tag{2}$$

In terms of network theory, Eq. (2) is called the "circuit equation" [17]. As statical equations underlying the truss (Eq. (1)) possess the same form as the kinematical equations underlying the linkage (Eq. (2)), it was proved [15] that there is a *topological isomorphism* between these two engineering systems.

One can see from Fig. 2 that the angles of the linkages are chosen to be perpendicular to the corresponding rods in the dual truss. Doing that guarantees that the direction of the force in every truss rod is the same as the direction of the relative linear velocity in the corresponding link of the linkage. This correspondence constitutes the *geometrical isomorphism* between the two systems. Accordingly, the numerical solutions of Eqs. (1) and (2) are the same, thus

$$\mathbf{F}_i = \mathbf{V}_{i'} \tag{3}$$

This demonstrates that the static behavior of the truss is isomorphic to the kinematical behavior of its dual linkage.

In the next section, based on this result, we will derive a new hypothesis that constitutes the mathematical foundation of the proposed method.

3 Duality Between Statics of Dead-Point Positions of Planar Linkages and Kinematics of Trusses

The linkage in a dead-point position is immobile, which means that if trying to rotate the driving link, the linkage will produce a counter force that will cause the system to maintain its original position. Therefore, the linkage in a dead-point position can be considered as a static system, not different from a regular truss that satisfies the law of force equilibrium at each of its joints (Fig. 3(*a*)).

The same duality principle used in the previous section, can be applied now to this truss-like configuration of the linkage. According to Table 1 the system dual to the dead-point configuration of a linkage should possess a topology of a truss. In addition, the forces in the links of the linkage should be equal to some variable corresponding to the truss rods possessing kinematical properties, i.e., satisfying the laws of compatibility. As is shown in the Appendix, a variable suitable for such a role is the vector of rod deformation, which is defined to be the vector difference between the original and the deformed states of the rod [30]. On the basis of this reasoning, Proposition 2 has been developed (the mathematical proof of Proposition 2 is provided in the Appendix).

PROPOSITION 2 (Appendix). *For each planar pin-joint linkage in a dead-point position* L_{dead}, *there corresponds a deformed state of its planar dual truss*, T_{deform}, *where the deformation vector of each rod i defines the angle of the corresponding link in the linkage*.

We shall now examine Proposition 2 upon a simple example appearing in Fig. 3.

The linkage of Fig. 3(*a*) is obviously found in a dead-point position, due to the collinearity of links 1 and 2, thus trying to move the driving link will induce forces within the links. In contrast to the other links of the linkage, the driving link, as the link which applies the force upon the rest of the linkage, is drawn in perpendicular to the corresponding displacement vector in the truss. There are two joints in the linkage—A and B; at each the equilibrium of these forces is satisfied, as is formulated in the following vector equations:

$$A\colon \mathbf{F}_3 = \mathbf{F}_1$$

$$B\colon \mathbf{F}_1 = \mathbf{F}_2 \tag{4}$$

In the dual truss (Fig. 3(*b*)) there are three corresponding rods: 1, 2 and 3. For the truss some deformed state is considered, which presents some geometrical deviation from the original configuration. In order for the truss to maintain its structural integrity, the deformation vectors corresponding to its elements are subject to the compatibility law stating that the vector sum of the deformation of each contour of the truss equals zero. In the truss of Fig. 3(*b*) there are two contours: one formed by force 3 and rod 1, other formed by rods 1 and 2. Thus, the behavior of the truss deformations can be described through the following vector equations:

$$A\colon \mathbf{\Delta}_3 = \mathbf{\Delta}_1$$

$$B\colon \mathbf{\Delta}_1 = \mathbf{\Delta}_2 \tag{5}$$

where $\mathbf{\Delta}_i$ is the deformation vector of element i of the truss.

Equation (5) can be explained in an intuitive way, as follows: in Fig. 3(*b*) only joint M is displaced, while the joints corresponding to the fixed supports remained at place. All the elements of the truss are connected at one end to M and on the other end to some fixed point. Therefore the deformations of all the elements are equal to the displacement of joint M and accordingly equal to one another as was obtained in Eq. (5).

As the statical equations underlying the linkage (Eq. (4)) possess the same form as the kinematical equations underlying the truss (Eq. (5)), there is a complete correspondence between the behaviors of these two systems.

One can see from Fig. 3 that the angles of the forces acting in the links are chosen to be parallel to the deformations of the corresponding rods in the dual truss. Doing that guarantees that the direction of the force in every link of the immobile linkage is the same as the direction of the deformation of the corresponding rod in the truss. This correspondence constitutes the *geometrical isomorphism* between the two systems. Accordingly, the numerical solutions of Eqs. (4) and (5) are the same, thus:

$$\mathbf{F}_i = \mathbf{\Delta}_i \tag{6}$$

In the following section this relation will be employed to systematically yield dead-point configurations of the linkage with a given topology.

4 Method for Establishing Possible Dead-Point Positions of Planar Linkages

In the previous section it was shown that each feasible set of deformations of truss rods corresponds to a dead-point position of its dual linkage. Therefore, in order to find a dead-point position of a linkage it is enough to find a set of feasible deformations of the rods in the dual truss.

An arbitrary set of vectors will not necessarily constitute a valid set of truss rod deformation vectors, as those vectors might not satisfy the geometric compatibility requirements [30]. The rod deformation equals the vector difference between the vectors of displacement of its end joints. In contrast to the deformation vectors, the joint displacements are not subject to compatibility requirements—one can "move" truss joints in an arbitrary fashion and then obtain the deformed states of the rods by drawing a line between the corresponding end joints. In other words, unlike deformation vectors, any set of joint displacements corresponds to a

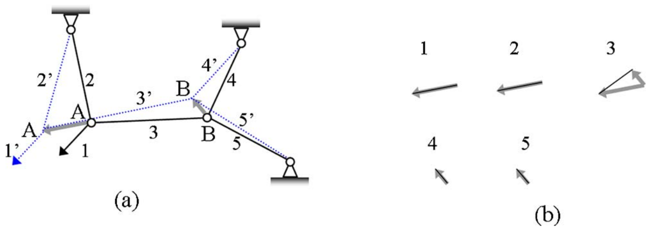


Fig. 4 (*a*) **Arbitrary displacement vectors associated with truss joints,** (*b*) **yield feasible sets of rod deformations**

valid deformed state of a truss. Therefore, as is demonstrated in Fig. 4, in order to obtain a feasible set of rod deformations for a given truss, it is enough to associate each of the truss joints with an arbitrary vector. The deformation of a specific rod is evaluated by subtracting the vectors associated with its end joints. Different sets of feasible deformations are obtained by coming up with different sets of arbitrary vectors associated with the truss joints.

On the basis of the relation developed in the previous section, the following method is formulated for finding the dead-point configurations for a given linkage:

1. Use Table 1 to establish a truss dual to the linkage.
2. Associate each joint of the truss with an arbitrary vector.
3. For each truss rod obtain a deformation vector equal to the vector difference between the vectors associated with its end joints.
4. Redraw the linkage so that the angle of each of its links is the same as that of the deformation vector associated with the corresponding rod in the dual truss. According to Proposition 2, the resultant configuration of the linkage is a dead-point configuration.
5. Repeat the process for different arbitrary vector sets in 2 to obtain other dead-point configurations of the original linkage. A discussion about the ways of choosing these random vector sets, consequently about the finiteness of the proposed method, is provided in Sec. 6.

The method is demonstrated on the known double-butterfly (or double-flier) linkage [14] shown in Fig. 5(*a*).

The truss dual to the double-butterfly linkage is shown in Fig. 5(*c*). In Fig. 6(*a*), arbitrary vectors are associated with the joints of the dual truss and the corresponding rod deformations are evaluated. In Fig. 6(*b*) the dead-point position configuration of the double-butterfly linkage is drawn by taking the angles of the links from the corresponding rod deformation vectors.

As shown in Fig. 7, additional dead-point configurations of the double-butterfly linkage are obtained when considering other arbitrary sets of displacement vectors.

Double-butterfly linkages are known to be kinematically indeterminate [14], indicating the impossibility of finding their secondary instant centers by conventional methods of machine theory. Therefore, the study of double-butterfly linkages requires engagement of advanced mathematical tools [31]. Verification of the dead-point positions of the double-butterfly linkages, presented in Figs. 6 and 7, was made in [32]. It should be noted that the three dead-point positions of this linkage are not the only dead-point positions of the linkage, as additional positions can be obtained by applying the proposed method to additional displacement vector sets [33].

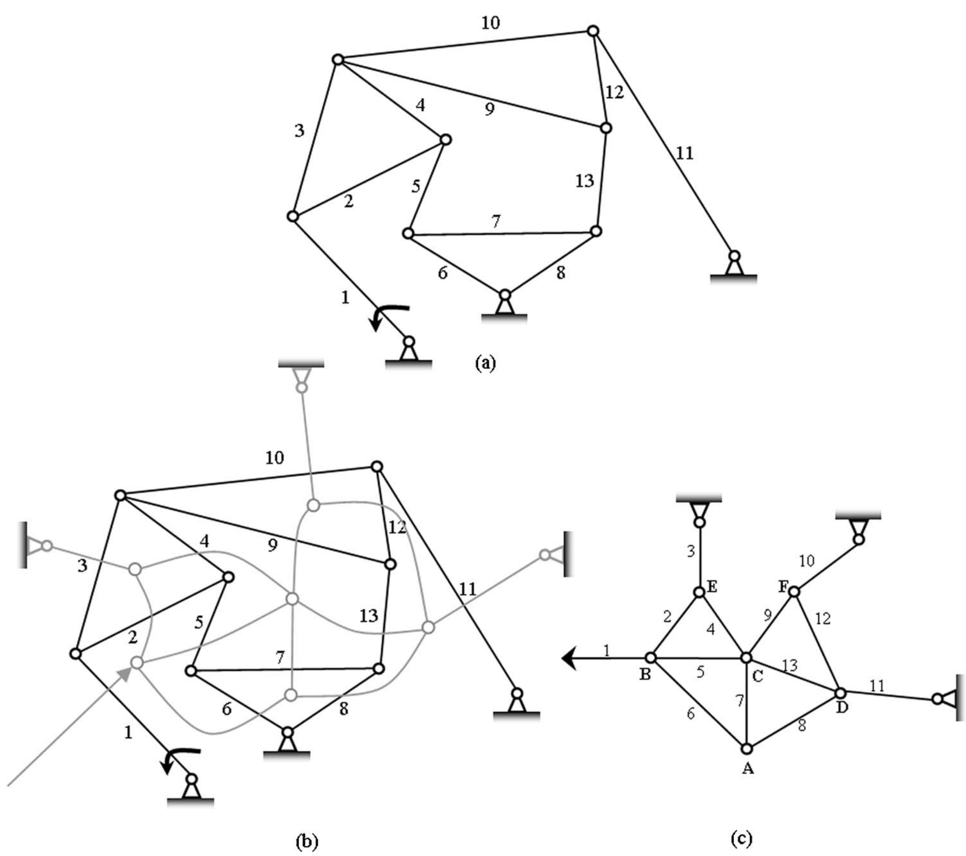


Fig. 5 (*a*) **Double-butterfly linkage;** (*b*) **the structures of the dual truss superimposed upon the linkage;** (*c*) **the truss possessing the dual topology**

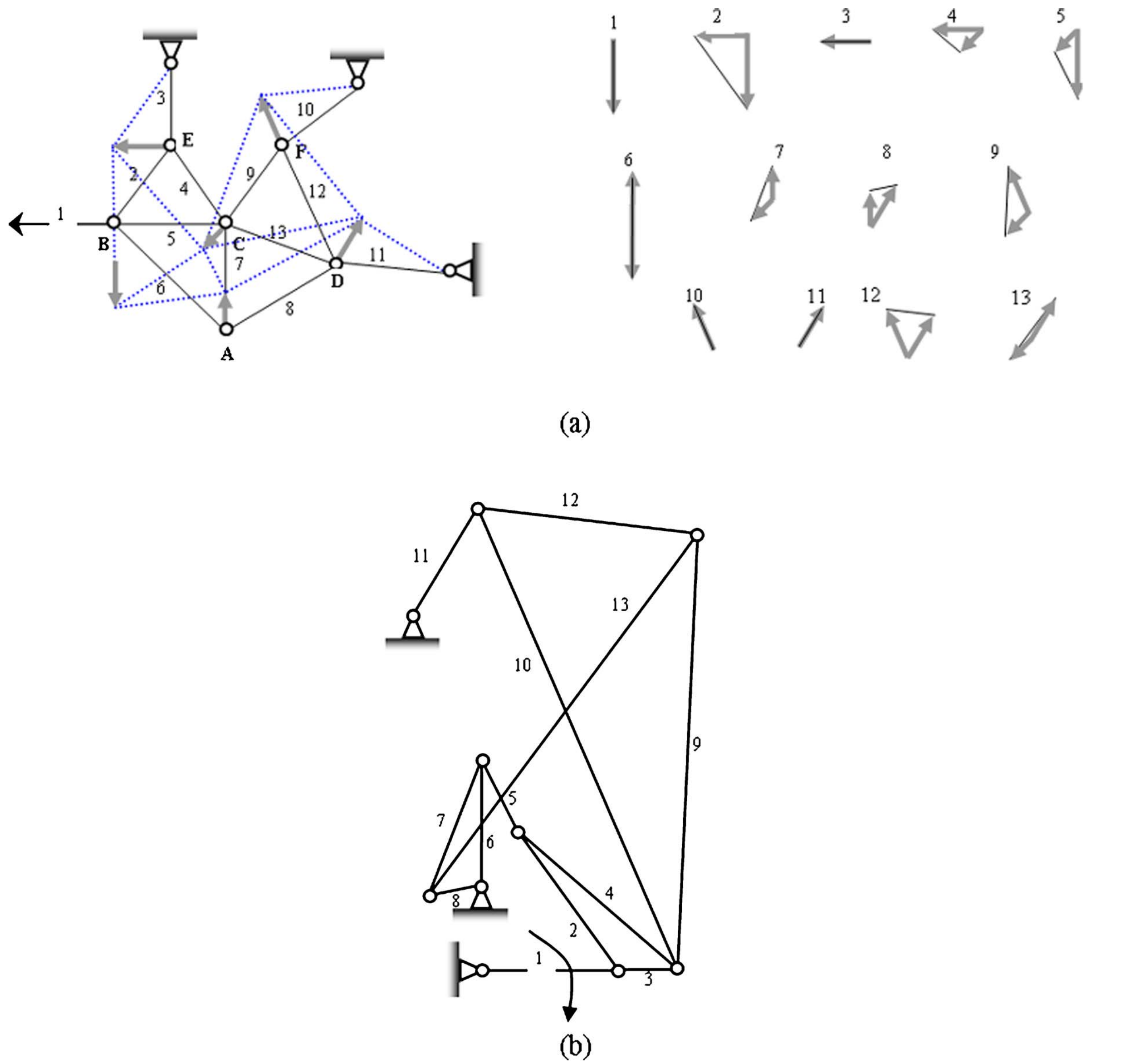


Fig. 6 (*a*) **Arbitrary displacements assigned to all the joints of the truss and the corresponding deformation vectors.** (*b*) **The double-butterfly linkage redrawn in accordance to the angles of the deformations.**

5 Method for Detection of Dead-Point Positions Directly Upon Linkages

The methodology formulated in the previous section, although efficient, requires one to maneuver through the terminologies of both structural mechanics and kinematics. In this section we will employ the correspondence existing between linkages and trusses to derive the dead-point positions directly upon the linkage without the need to perform the transformation to the dual truss.

The only term from structural analysis appearing in the method formulated in the previous section and not possessing an apparent counterpart in linkages is the vector of the joint displacements of the truss. It was found [16] that this variable corresponds to a new type of variable describing the static behavior of a structure.

The properties of this variable appear in Table 2. According to Table 2, the new variable has the same units as a force, but, in contrast to a regular through variable, is associated with the faces of the statical system, thus is termed face-force [16].

Following is the description of the properties listed in Table 2 applied to the linkage of Fig. 8:

Property 1 of Table 2 implies that faces A, B and O of the linkage shown in Fig. 8 are associated with a vector variables designated by $\mathbf{F}_A$, $\mathbf{F}_B$, $\mathbf{F}_O$.

According to property 2, the following relation exists between the face forces and the forces in the links of the linkage:

$$\mathbf{F}_4=\mathbf{F}_5=\mathbf{F}_O-\mathbf{F}_B;\mathbf{F}_3=\mathbf{F}_A-\mathbf{F}_B;\mathbf{F}_1=\mathbf{F}_2=\mathbf{F}_A-\mathbf{F}_O \qquad (7)$$

According to property 3, the face-force $\mathbf{F}_O$ is zero.

It can be concluded from the properties listed in Table 2 and the example that the new variable established in static structures that face force can be thought of as a multidimensional generalization of the "mesh current" in electrical circuits [33,34]. Both variables are known to "circulate" within the faces of the engineering system and to be the components of the real engineering system variable. The main difference is that while mesh currents are scalars and produce real currents in circuit elements through scalar subtraction, the face forces are multidimensional vectors. This

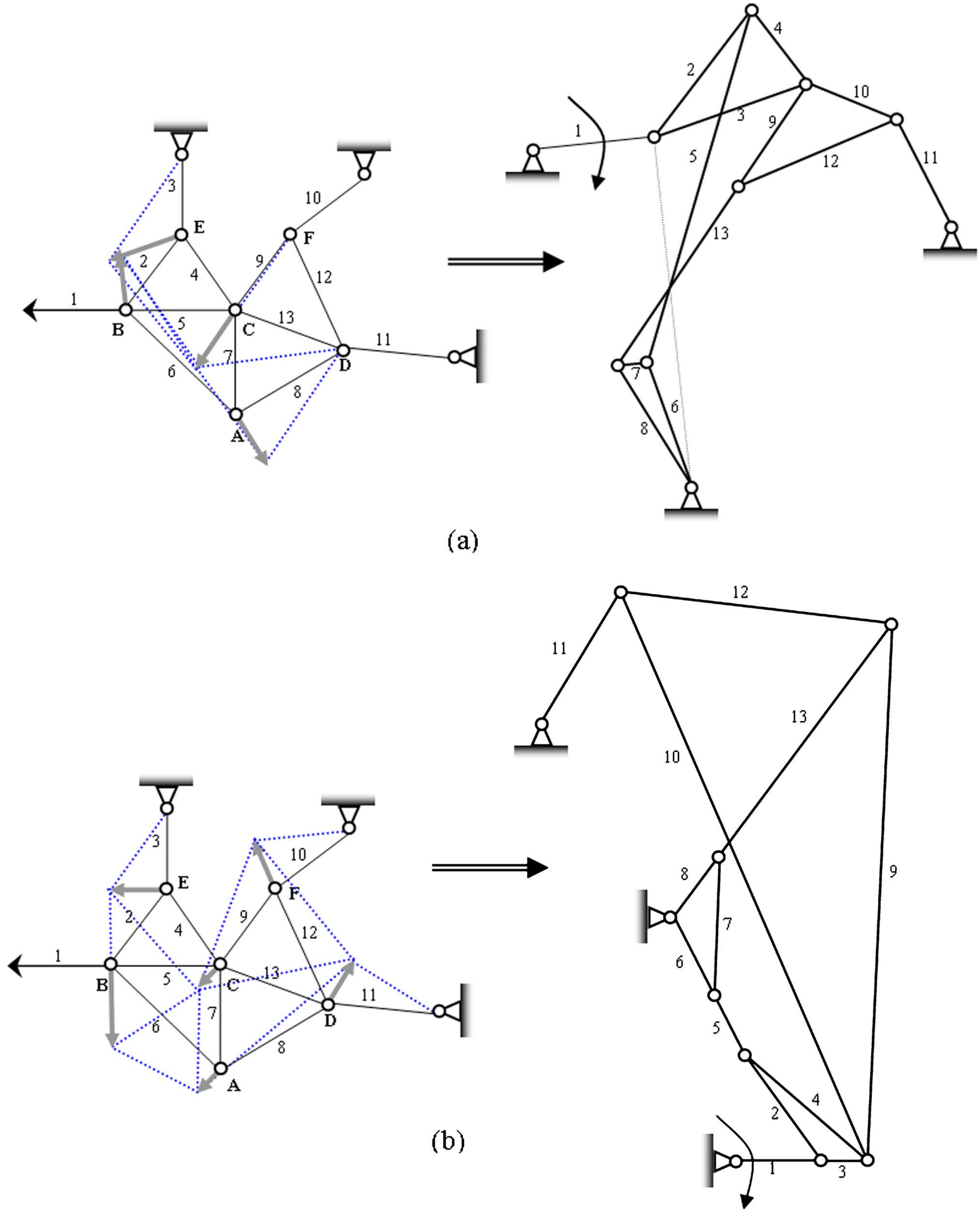


Fig. 7 Additional dead-point configurations of the double-butterfly linkage obtained by means of the suggested method (Step 5). (*a*) The dead-point configuration obtained, when two of the dual truss joints, D and F, are assigned zero displacement, (*b*) the dead-point configuration obtained, when two of the dual truss joints, A and C, are assigned the same displacement vector.

implies that the face forces possess geometrical properties and produce the forces in the links. Thus, if the face forces are applied to a linkage then they would define a geometry in which real forces are produced in the immobile linkage configuration through vector subtraction of the corresponding face forces. Since face-forces variable is the dual of the absolute linear velocity variable, in network theory terminology it can be considered also as the dual of the nodal variable [17].

Table 2 Establishing the properties of new statical variable.

Properties of the joint displacement in truss	Translation to the terminology of the linkage found in a dead-point position
1. The variable is a two-dimensional vector corresponding to each joint in the truss.	1. The new variable is a two-dimensional vector corresponding to each face (area bounded by the truss elements) of the linkage.
2. Deformation of a truss rod is equal to the difference between the displacements corresponding to the end joints of this rod.	2. The force in the linkage link is equal to the difference between the new variables corresponding to the faces separated by this element.
3. The displacement of the fixed joint is equal to zero.	3. The new variable corresponding to the external face of the linkage is equal to zero.

Applying the face forces to the method described in the previous section yields the following method for finding the dead-point positions:

1. Associate each face of the linkage with an arbitrary vector of its face force.
2. For each link obtain a force vector equal to the vector difference between the face forces associated with the faces separated by this link.
3. Redraw the linkage so that the angle of each of its links is the same as that of the force vector obtained for it in 2. The resultant configuration of the linkage is a dead-point configuration.
4. Repeat the process for different arbitrary vector sets in 1 to obtain other dead-point configurations of the original linkage. The ways of choosing these vector sets are discussed in Sec. 6.

Example for employing the above algorithm upon Stephenson II linkage and yielding its dead-point configuration is shown in Fig. 9.

Applying instant center method [13] can be used to verify whether the configuration of the Stephenson II linkage presented in Fig. 9(*c*) is the dead-point position. Straightforward application of Kennedy theorem to this configuration yields that in this specific geometry, the relative instant center between links 4-5-7 and the driving link 1 coincides with the absolute instant center of link 4-5-7. By definition of instant center, such a coincidence, indicates that the configuration is indeed a dead-point position. The configuration of Fig. 9(*c*) was obtained by applying the technique to a single set of random face force vectors. Additional dead-point positions of this linkage [5] can be obtained by applying the technique to other vector sets, as is explained in Sec. 6.

6 Finiteness of the Suggested Methodology

The final steps of the methods presented in the previous two sections were concerned with repeating the process of finding a dead-point linkage configuration, each time originating from another random vector set. The current section provides a possible algorithm for choosing a finite number of such vector sets, leading to the establishing of all the possible dead-point positions of the given linkage topology.

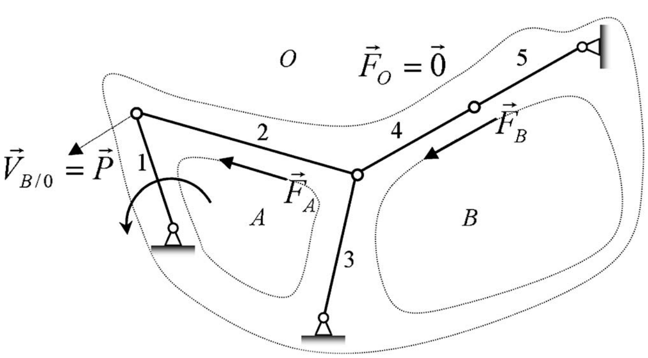


Fig. 8 The face forces in the linkage in a dead-point position

As was shown in the previous section, the space of the dead-point positions of a mechanism can be spanned by the face-force vectors. It is customary [5] to consider a specific dead-point position as a unique combination of kinematical/statical constraints leading the mechanism to becoming immobile. Accordingly a specific dead-point position is not determined by a single set of face force vectors, but by some finite subspace of the above overall vector space.

Current section demonstrates one of the possible techniques for establishing these subspaces and thus reducing the output of the methods presented in Secs. 4 and 5 to a finite, but at the same time complete, set of dead-point configurations.

Consider a mechanism, M, possessing n faces: $\mathbf{F}=\{1,2,\ldots,n\}$. A subset $\mathbf{L}=\{l_1,\ldots,l_k | l_i \in \mathbf{F}, k \leq n\}$ defines a subspace $S_{\mathbf{L}}$ of face-force vectors through Eq. (8)

$$\{\mathbf{F}_1,\ldots,\mathbf{F}_n\} \in S_{\mathbf{L}} \quad \text{if} \begin{cases} \mathbf{F}_i \neq 0 & \text{for } i \in \mathbf{L} \\ \mathbf{F}_j = 0 & \text{for } i \notin \mathbf{L} \end{cases} \tag{8}$$

By Eq. (8) each possible subspace $S_{\mathbf{L}}$ can contain face-force vector sets corresponding to at most one unique configuration of static forces in the mechanism, and therefore corresponds to at most one dead-point configuration of the mechanism.

Some of the subsets $\mathbf{L}$ would not contain a new dead-point configuration and thus can be eliminated in the process. Such subsets include $\mathbf{L}=\{\phi\}$ and those $\mathbf{L}$ whose size equals $n-1$, as those configurations can be proved to be equivalent to $\mathbf{L}=\mathbf{F}$, due to the potential-like behavior of the face-force variable (Sec. 3). Also, if the driving link of the mechanism is defined, the force in it cannot be zero, thus at least one of its adjacent faces (say 1 and 2) should possess a non-zero face-force, which makes it possible to eliminate all the 2^{n-2} subsets $\mathbf{L}$, where $(1 \notin \mathbf{L}) \cap (2 \notin \mathbf{L})$.

Thus, the upper boundary for the number of subsets $\mathbf{L}$ that can correspond to the dead-point positions is

$$2^n - 2^{n-2} - n = 3 \cdot 2^{n-2} - n \tag{9}$$

Consequently, for each subset $\mathbf{L}$, the algorithms presented in Secs. 4 and 5 should account for only one random vector set $\{\mathbf{F}_1,\ldots,\mathbf{F}_n\} \in S_{\mathbf{L}}$.

Some heuristic rules can be developed to reduce the complexity of the methods, such as eliminating from the process those subsets $\mathbf{L}$ which yield dead positions that are equivalent [35]. Guided by this principle, following is an example for finding all the dead-point configurations of the known Stephenson III mechanism, shown in Fig. 10.

The mechanism of Fig. 10 possesses four faces that can be associated with a set of four face forces: $\{\mathbf{F}_A,\mathbf{F}_B,\mathbf{F}_C,\mathbf{F}_D\}$. Faces

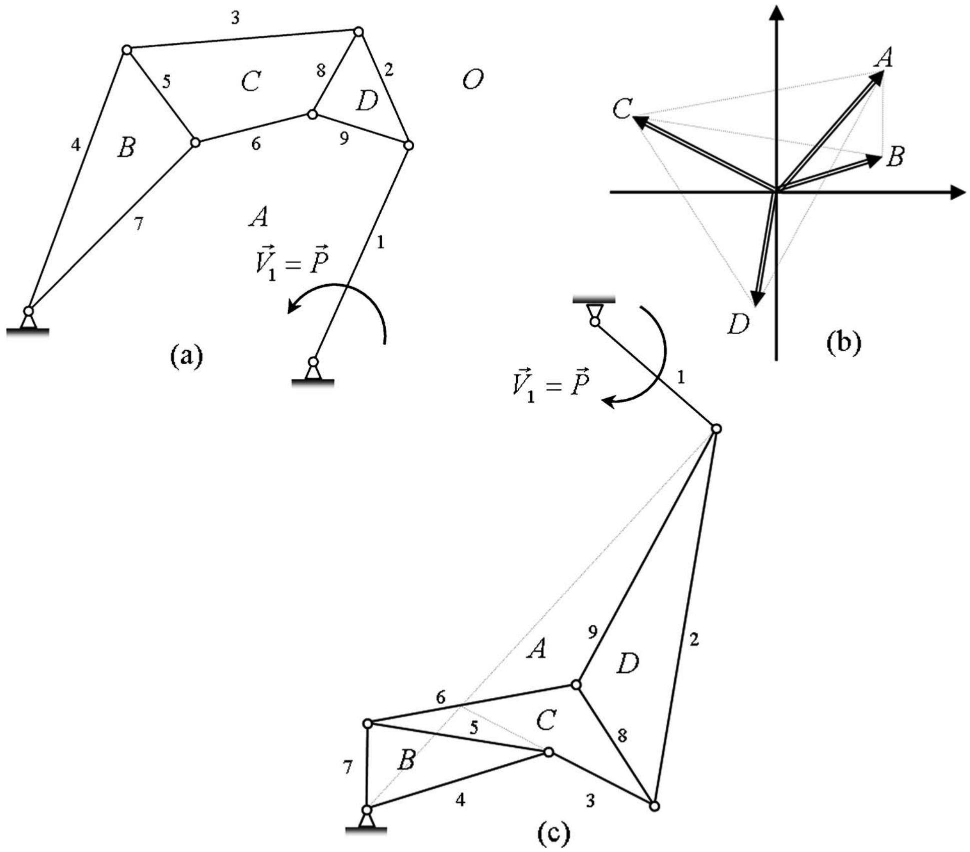


Fig. 9 Example for employing face forces in detecting dead-point positions of the Stephenson II linkage. (*a*) Stephenson II linkage, (*b*) randomly chosen face-force vectors, (*c*) the resultant dead-point position.

adjacent to the driving link, 7, are A and D. The possible nontrivial subsets **L** for this mechanism are listed in the truth table (Table 3).

Accordingly, the method of Sec. 5 can be applied through eight iterations, yielding eight positions, as is listed in Table 4.

The right-most column of Table 4 provides the general geometrical property identifying the specific type of the corresponding dead-point position. The positions presented in the paper constitute all the dead positions for this linkage with driving link 7 that appear in [5] including additional special-case positions. Furthermore, an explanation is provided on how such a description has been obtained on the basis of the properties of the face-force variables. In the course of the research, additional kinematical/statical tools have been developed for effective identification of found configurations [32] which enable to find all the configurations corresponding to the subsets **L** in a fully systematic manner [35].

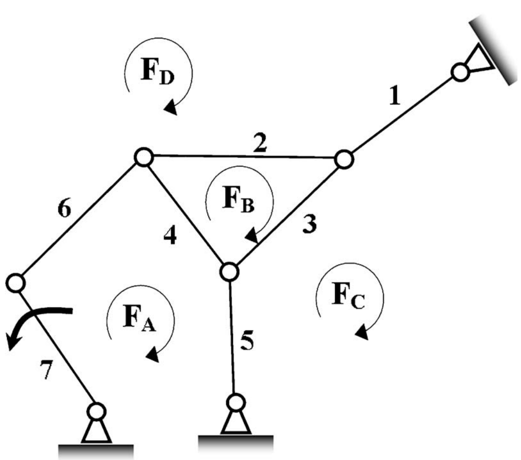


Fig. 10 Stephenson III mechanism, for which all the possible dead-point positions are to be found

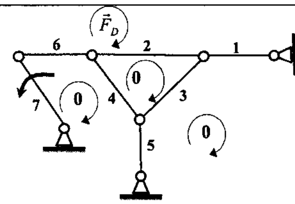
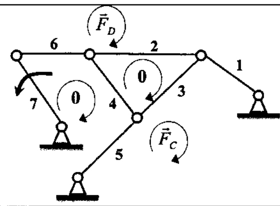
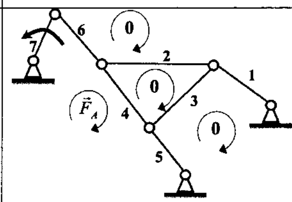
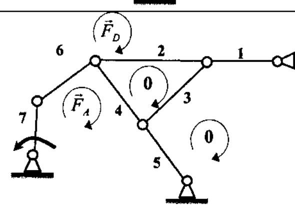
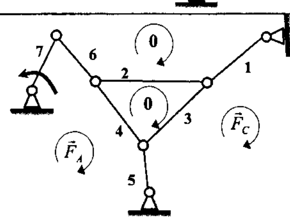
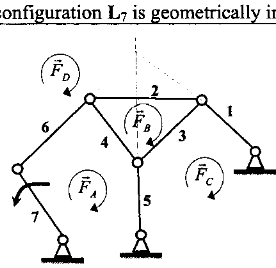
7 Conclusions and Further Research

The paper has demonstrated another view on the problem of finding the dead-point positions of linkages from the perspective of graph theory duality. This perspective has led to the develop-

Table 3 The truth table listing the dead-point positions of the Stephenson III mechanism. Unit entry of the table stands for the corresponding face-force belonging to the subset defining the dead-point position.

Position No.	F_A	F_B	F_C	F_D
L_1	0	0	0	1
L_2	0	0	1	1
L_3	0	1	0	1
L_4	1	0	0	0
L_5	1	0	0	1
L_6	1	0	1	0
L_7	1	1	0	0
L_8	1	1	1	1

Table 4 Applying the method for finding the dead-point configurations of Stephenson III mechanism.

Subset **L**	Resultant position	Properties identifying the configuration
$\mathbf{L}_1 = \{D\}$		By the definition of the face force (section 5, Shai 2002), in this dead point position, the forces in links 1,2 and 6 are all equal to the face force $\vec{F}_D$. Therefore, the property identifying the position $\mathbf{L}_1$ is that links 1, 2 and 6 are collinear.
$\mathbf{L}_2 = \{C, D\}$		Links 2 and 6 are collinear (forces equal to $\vec{F}_D$); links 3 and 5 are collinear (forces equal to $\vec{F}_C$).
$\mathbf{L}_3 = \{B, D\}$	Forces in links 4 and 3 have to be the same (equal to $\vec{F}_B$) and thus have to be parallel. The force in 2 is equal to $\vec{F}_B - \vec{F}_D$. Thus, due to the geometrical constraint for the links 2,3 and 4 to form a triangle, position $\mathbf{L}_3$ is geometrically infeasible.	
$\mathbf{L}_4 = \{A\}$		Links 4,5 and 6 are collinear (the forces in these links are the same and equal to $\vec{F}_A$).
$\mathbf{L}_5 = \{A, D\}$		Links 4 and 5 are collinear (force equal to $\vec{F}_A$); links 1 and 2 are collinear (force equal to $\vec{F}_D$).
$\mathbf{L}_6 = \{A, C\}$		Links 4 and 6 are collinear (force equal to $\vec{F}_A$); links 1 and 3 are collinear (force equal to $\vec{F}_C$).
$\mathbf{L}_7 = \{A, B\}$	Forces in links 2 and 3 are equal to $\vec{F}_B$, while force in 4 is equal to $\vec{F}_B - \vec{F}_A$. As it was in $\mathbf{L}_3$, links 2 and 3 cannot be located in parallel since links 2,3 and 4 form a triangle in the linkage, thus the configuration $\mathbf{L}_7$ is geometrically infeasible.	
$\mathbf{L}_8 = \{A, B, C, D\}$		The lines of continuation of the links 1, 5 and 6 should intersect at the same point. *Explanation:* $\left.\begin{array}{l}\vec{F}_6 = \vec{F}_A - \vec{F}_D \\ \vec{F}_1 = \vec{F}_D - \vec{F}_C \\ \vec{F}_5 = \vec{F}_C - \vec{F}_A\end{array}\right\} \Rightarrow \vec{F}_1 + \vec{F}_5 + \vec{F}_6 = 0$ As is known from statics (Timoshenko and Young, 1965), if three forces are in static equilibrium their vectors should intersect at the same point.

ment of two methods introduced in the paper, which employ properties of the dual static system to detect the dead-point configurations.

Although the paper is aimed at introducing the derivation of the methods for the specific problem stated above, the results indicate the potential to yield more general results. Specifically, one can derive knowledge about the original engineering system by means of knowledge and properties of another system from different domain. In the paper, this idea was applied to find the dead-point positions of the given linkage through the knowledge and properties of the dual system—trusses, from structural mechanics. Additional duality relations existing between different engineering systems, such as duality between Stewart platforms and serial robots [23,27], planetary and beam systems [23] and others [15] that are based on the duality principle of graph theory, indicate the possibility of applying the view introduced in the paper to other engineering fields.

The method of finding the dead-point positions can be also stated as a way to come up with geometries appropriate for sustaining static force equilibrium, while originally given only the topology. This problem is tightly related to the form-finding problem for tensegrity structures [37]. The latter systems present interest not only to the engineering, but to the biological community as well [38].

Appendix. Proof of Proposition 2

Relevant terminology of network graph theory [15,17,19]: **Flow (vectorial through variable)**—a vector variable associated with the edges of the graph and satisfying the "flow law," stating that the vector sum of the flows at each graph cutset is zero. The special case of the flow law is the known "vertex postulate," stating that the vector sum of the flows at each graph vertex is zero. **Potential (vectorial nodal variable)**—a vector variable associated with the vertices of the graph.

Potential difference (vectorial across variable)—a vector variable associated with the edges of the graph and satisfying the "potential law," that can be formulated though two equivalent statements: (a) the vector sum of potential differences in each circuit of the graph is zero or (b) the potential difference of an edge is equal to the vector difference between the potentials of the end vertices of the edges.

Flow graph (designated $\mathbf{G_F}$)—a graph where the flows through the edges are of interest. **Potential graph (designated $\mathbf{G_\Delta}$)**—a graph where the potential differences of the edges are of interest.

LEMMA 1. *For each linkage in a dead-point position there exists a flow graph G_F that reflects its static behavior.*

Proof. The immobility of the linkage in a dead-point position is reflected in the fact that the driving link cannot cause motion, thus it can statically sustain forces applied upon it. Therefore a linkage in a dead-point position can be considered as a static structure [30]. As was proved in [15], such a structure can be represented by a flow graph.

LEMMA 2. *For each truss T there exists a potential graph, G_Δ, that reflects the kinematical behavior of the deformations of its rods.*

Proof. Given the truss T, construct a graph G_Δ, as follows [15]: associate each pinned joint in T with a vertex in G_Δ and each rod/external force/reaction in T with an edge in G_Δ associate each vertex k with a vector $\boldsymbol{\pi}_k$ equal to some linear displacement of the pinned joint, associate each edge i with a vector $\boldsymbol{\Delta}_i$ equal to the resultant deformation vector of rod i. By definition of deformations [30] vectors $\boldsymbol{\Delta}_i$ and $\boldsymbol{\pi}_k$ satisfy the potential law [15,19], thus G_Δ can be considered a valid potential graph representation.

PROPOSITION 2. *For each linkage in a dead-point position L_{dead}, there exists a deformed state of its dual truss, T_{deform} where the deformation vector of each rod i is parallel to the corresponding link in the linkage.*

Proof. Given the linkage L_{dead} construct its corresponding flow graph G_F (Lemma 1). Construct a graph G_Δ, which is dual to G_F [17]. Associate each edge in G_Δ with a vector $\boldsymbol{\Delta}_i$, so that

$$\boldsymbol{\Delta}_i = \mathbf{F}_i \tag{10}$$

where $\mathbf{F}_i$ is the flow in the corresponding edge in G_F.

Since G_Δ and G_F are dual, the edges forming a circuit C in G_Δ form a cutset in G_F. Substituting Eq. (7) into the flow law for G_F yields an equation of potential law for the circuits of G_Δ, thus, G_Δ can be considered a valid potential graph, with potential differences defined by $\boldsymbol{\Delta}_i$.

By Lemma 2, the potential graph representation can be interpreted as a representation of some deformed state of truss T, where each rod corresponds to an edge in G_Δ, which by the duality principle corresponds to an edge in G_F. By Lemma 2 the latter corresponds to a truss element. Similarly, by Lemmas 1 and 2, and Eq. (10), the deformation vector of each truss rod is equal to the force vector acting in the corresponding link of the linkage in the dead position. As the linkage in the dead-point position presents a static structure, the forces in its links are in parallel to the orientation of the links. Thus the vectors of the truss rod deformations are parallel to the links in the corresponding dual dead-pointed linkage.

References

[1] Ionesku, T., 2003, Mech. Mach. Theory, **38**, pp. 777–785.
[2] Goodman, T. P., 1965, "Toggle Linkage Applications in Different Mechanisms," in *Mechanism, Linkages, and Mechanical Controls*, McGraw–Hill, New York.
[3] Young, D. W., 1986, "Aircraft Landing Gears—the Past, Present and Future," Proc. Inst. Mech. Eng., Part D (J. Automob. Eng.), **200**(D2), pp. 75–92.
[4] Balli, S. S., and Chand, S., 2003, "Synthesis of a Planar Sever-Link Mechanism With Variable Topology for Motion Between Two Dead-Center Positions," Mech. Mach. Theory, **38**, pp. 1271–1287.
[5] Yan, H.-S., and Wu, L.-L., 1989, "On the Dead-Center Positions of Planar Linkage Mechanisms," ASME J. Mech., Transm., Autom. Des., **111**, pp. 40–46.
[6] Hain, K., 1967, *Applied Kinematics*, 2nd ed., McGraw–Hill, New York.
[7] Soni, A. H., 1974, *Mechanism Synthesis and Analysis*, McGraw–Hill, New York.
[8] Qin, Y. Y., Duan., B. Y., Wei, Q., and Du, J. L., 2001, "Elimination of Force Singularity of the Cable and Cabin Structure for the Next Generation Large Radio Telescope," Mechatronics, **12**, pp. 905–918.
[9] Kohli, D., Cheng, Jc., Tsai, Ky, 1994, "Assemblability, Circuits, Branches, Locking Positions, and Rotatability of Input Links of Mechanisms With 4 Closures," ASME J. Mech. Des., **116**(1), pp. 92–98.
[10] Watanabe, K., and Funabashi, H., 1984 "Kinematic Analysis of Stephenson Six-Link Mechanisms (1st Report, Discrimination of Composition Loops)," Bull. JSME, **27**(234), pp. 2863–2870.
[11] Yang, G., Chen., I.-M., Lin, W., and Angeles, J., 2001, "Singularity Analysis of Three-Legged Parallel Robots Based on Passive-Joint Velocities," IEEE Trans. Rob. Autom., **17**(4), pp. 413–422.
[12] Sclater, N., and Chironis, N. P., 2001, *Mechanisms and Mechanical Devices—Source Book*, 3rd ed., McGraw–Hill, New York.
[13] Norton, R. L., 1992, *Design of Machinery*, McGraw–Hill, New York.
[14] Foster, D. E., and Pennock, G. R., 2003, "A Graphical Method to Find the Secondary Instantaneous Centers of Zero Velocity for the Double Butterfly Linkage," Trans. ASME, **125**, pp. 268–274.
[15] Shai, O., 2001, "The Duality Relation Between Mechanisms and Trusses," Mech. Mach. Theory, **36**(3), pp. 343–369.
[16] Shai, O., 2002, "Utilization of the Dualism Between Determinate Trusses and Mechanisms," Mech. Mach. Theory, **37**(11), pp. 1307–1323.
[17] Swamy, M. N., and Thulasiraman, K., 1981, *Graphs: Networks and Algorithms*, Wiley, New York.
[18] Shearer, L. J., Murphy, A. T., and Richardson, H. H., 1971, *Introduction to System Dynamics*, Addison-Wesley, Reading, MA.
[19] Andrews, G. C., 1971, "The Vector-Network Model—A Topological Approach to Mechanics," Ph.D. thesis, University of Waterloo.
[20] Andrews, G. C., 1977, "General Re-Statement of the Laws of Dynamics Based on Graph Theory," in *Problem Analysis in Science and Engineering*, Academic, New York.
[21] McPhee, J. J., 1996, "On the Use of Linear Graph Theory in Multibody System Dynamics," Nonlinear Dyn., **9**, pp. 73–90.
[22] Lang, S. Y. T., 2005, "Graph-Theoretic Modeling of Epicyclic Gear Systems," to appear in Mech. Mach. Theory.
[23] Shai, O., 2002a, "Duality Between Statical and Kinematical Engineering Systems," *Sixth International Conference on Computational Structures Technology*, Prague, Czech Republic, September 4–6.
[24] Girvin, S. M., 1996, "Duality in Perspective," Science, **274**, pp. 524–525.
[25] Maxwell, J. C., 1864, "On Reciprocal Figures and Diagrams of Forces,"

Philos. Mag., **27**, pp. 250–261.
[26] Davies, T. H., 1983, “Mechanical Networks—III, Wrenches on Circuit Screws,” Mech. Mach. Theory, **18**(2), pp. 107–112.
[27] Waldron, K. J., and Hunt, K. H., 1991, “Series-Parallel Dualities in Actively Coordinated Mechanisms,” Int. J. Robot. Res., **10**(5), pp. 473–480.
[28] Gosselin, F., and Lallemand, J.-P., 2001, “A New Insight Into the Duality Between Serial and Parallel Non-Redundant and Redundant Manipulators,” Robotica, **19**, pp. 365–370.
[29] Collins, C., and Long, G., 1995, “On the Duality Twist/Wrench Distributions in Serial and Parallel Chain Robot Manipulators,” *IEEE International Conference on Robotics and Automation*, Nagoya, Japan, May 21–27, pp. 526–531.
[30] West, H. H., 1993, *Fundamentals of Structural Analysis*, Wiley, New York.
[31] Foster, D. E., and Pennock, G. R., 2005, “Graphical Methods to Locate the Secondary Instant Centers of Single-Degree-of-Freedom Indeterminate Linkages,” ASME J. Mech. Des., **127**(2), pp. 249–256.
[32] Shai, O., and Pennock, G. R., 2005, “The Duality Between Kinematics and Statics,” submitted to ASME J. Mech. Des.
[33] Balabanian, N., and Bickart, T. A., 1969, *Electrical Network Theory*, Wiley, New York.
[34] Kron, G., 1963, *Diakoptics—The Piecewise Solution of Large Scale Systems*, Macdonald, London.
[35] Polansky, I., 2005, “Deriving Mechanism Singularity Positions Through the Graph Theory Duality Principles,” MS Thesis, Tel-Aviv University.
[36] Timoshenko, S. P., and Young, D. H., 1965, *Theory of Structures*, 2nd ed., McGraw–Hill, Auckland.
[37] Pellegrino, S., and Tibert, G., 2004, “Review of Form-Finding Methods for Tensegrity Structures,” accepted for publication in Int. J. Space Structures.
[38] Ingber, D. E., 1998, “The Architecture of Life,” Sci. Am., **278**(1), pp. 48–57.
[39] Erdman, A. G., and Sandor, G. N., 1997, *Mechanism Design—Analysis and Synthesis*, 3rd ed., Prentice–Hall International, London.
[40] Shai, O., 2003, “Transforming Engineering Problems Through Graph Representations,” Adv. Eng. Inf., **17**(2), pp. 77–93.
[41] Tibert, A. G., and Pellegrino, S., 2003, “Review of Form-Finding Methods for Tensegrity Structures,” Space Struct., **18**(4), pp. 209–223.