A Study of the Duality Between Planar Kinematics and Statics

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This paper provides geometric insight into the correlation between basic concepts underlying the kinematics of planar mechanisms and the statics of simple trusses. The implication of this correlation, referred to here as duality, is that the science of kinematics can be utilized in a systematic manner to yield insight into statics, and vice versa. The paper begins by introducing a unique line, referred to as the equimomental line, which exists for two arbitrary coplanar forces. This line, where the moments caused by the two forces at each point on the line are equal, is used to define the direction of a face force which is a force variable acting in a face of a truss. The dual concept of an equimomental line in kinematics is the instantaneous center of zero velocity (or instant center) and the paper presents two theorems based on the duality between equimomental lines and instant centers. The first theorem, referred to as the equimomental line theorem, states that the three equimomental lines defined by three coplanar forces must intersect at a unique point. The second theorem states that the equimomental line for two coplanar forces acting on a truss, with two degrees of indeterminacy, must pass through a unique point. The paper then presents the dual Kennedy theorem for statics which is analogous to the well-known Aronhold-Kennedy theorem in kinematics. This theorem is believed to be an original contribution and provides a general perspective of the importance of the duality between the kinematics of mechanisms and the statics of trusses. Finally, the paper presents examples to demonstrate how this duality provides geometric insight into a simple truss and a planar linkage. The concepts are used to identify special configurations where the truss is not stable and where the linkage loses mobility (i.e., dead-center positions). [DOI: 10.1115/1.2181600]

Keywords: duality, equimomental lines, equimomental line theorem, face force, dual Kennedy theorem, dual Kennedy circle, indeterminate mechanisms, synthesis, unstable trusses, dead-center positions

1 Introduction

Engineering science provides a number of duality principles [1], for example, the principle of duality between the points and lines of the projective plane, are elements of a two-dimensional projective domain, so that there is a projective geometry of points as well as of lines [2]. A principle of duality is used in the theory of screws to map homogeneous coordinates to screw coordinates [3,4]. This technique was employed by Pennock and Yang [5] to obtain closed-form kinematic equations for serial robot manipulators and by Tarnai [6] to prove the duality between plane trusses and grillages. Davies [7] used the principle of duality in a study of mechanical networks relating to wrenches on circuit screws. Another example is the duality between serial robot manipulators and parallel, or platform-type, manipulators [8,9]. The principles underlying the kinematics and the statics of these two types of manipulators are the same which makes them dual to each other. An in-depth study of first-order instantaneous kinematics and statics by Duffy [10] contributed to a better understanding of both serial and parallel robot manipulators. Davidson and Hunt [11] extended this work to the kinetostatics of spatial robots and presented relationships between kinematically equivalent serial and parallel manipulators. A more recent investigation [12] illustrates the duality between the statics of a variety of systems and the kinematics of planar, spherical, and spatial mechanisms.

It is well known that instantaneous kinematics and statics proceed alongside one another, the important principle of reciprocity linking the two together [13]. The duality between first-order kinematics and statics has proved to be an important concept in engineering practice [14]. The correlation is based on the principle of virtual work which implies an orthogonality between the kinematic and static variables underlying the behavior of a mechanism and a truss. In general, the mathematical basis is linear algebra and the relations can be traced through similarities in the corresponding matrices. Shai [15] presented relationships between planar linkages and determinate trusses based on graph theory. The duality between linkages and trusses was described in some detail and insight into the statics of a truss was obtained from the kinematics of a planar linkage, and vice versa. The focus of this current work is the correlation between basic concepts underlying kinematics and statics and not between specific linkages and trusses. The goal is to provide the reader with a more general perspective on the duality between the sciences of kinematics and statics.

The paper begins with a definition of a new entity in statics; namely, the equimomental line. For two arbitrary coplanar forces there is a unique straight line in the plane where the two forces apply the same moment; i.e., the moments caused by the two forces at each point on this line are equal. This line is referred to throughout this paper as the equimomental line. A correlation between planar kinematics and statics is observed based on the dual relation between the instantaneous centers of zero velocity (henceforth referred to as instant centers) of a planar mechanism and the equimomental lines of a simple truss. This correlation provides a new approach to transforming knowledge between the domains of kinematics and statics. The theorems related to instant centers in single-degree-of-freedom and two-degree-of-freedom linkages can be extended to theorems in statics. Moreover, by employing the duality relation between a linkage and a simple truss, special properties of a linkage can be related to the equimomental lines of the dual truss. This will be demonstrated by transforming a prop-

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received December 14, 2004; final manuscript received October 12, 2005. Review conducted by Thomas R. Chase.

erty of the secondary instant centers for mechanisms with kinematic indeterminacy [16,17] to a counterpart property in statics. Then the result will be used to identify dead-center configurations of a linkage and unstable configurations of the dual truss.

The paper is arranged as follows. Section 2 presents an original theorem in statics regarding the existence of an equimomental line; i.e., a unique line where the moments about each point on the line, due to two coplanar forces, are equal. The authors believe that the concept of an equimomental line is a basic concept and the importance of equimomental lines in statics can be compared to the importance of instant centers in kinematics. Section 3 develops the correspondence between kinematic and static principles based on the duality between instant centers and equimomental lines. The paper then shows that well-known theorems in the kinematics of mechanisms can be transformed to original theorems in the statics of trusses. Section 4 explains the duality relation between planar linkages and determinate trusses using the concept of a face force [18]. The face forces in a truss are analogous to the linear velocities of the corresponding joints in the dual mechanism. A kinematic joint corresponds to a face (i.e., a non-bisected area) closed by the truss elements; i.e., the rods, external forces, or internal reaction forces in the rods. Face forces allow equimomental lines to be used in a direct manner in the static analysis of a determinate truss.

Section 5 presents the dual Kennedy theorem which is an original graphical technique for the static analysis of a truss. The section also includes a kinematic synthesis technique using instant centers which is then transformed to a synthesis technique in statics. The approach in statics remains the same as in kinematics with the important difference that equimomental lines are used in place of instant centers. Section 6 investigates special configurations of a truss and dead-center positions of a linkage. To illustrate the correspondence between the kinematic and static theorems presented in this paper, two practical examples are included; namely, a determinate truss and the double flier eight-bar linkage. For a special configuration, the truss is not stable which indicates that the dual linkage is in a dead-center position. The double flier eight-bar linkage is used to illustrate the correlation between instant centers and equimomental lines when a linkage is in a dead-center position. Finally, Section 7 presents an overview of the work in the paper, important conclusions and suggestions for future research.

2 The Equimomental Line

A rigid body moving in a single plane can be defined as an entity that determines the vector field of linear velocities; i.e., each point in the body is associated with a linear velocity vector. However, there is a point fixed in the body which has zero velocity, referred to as the instant center. The concept of an instant center for two rigid bodies in planar motion was discovered by Johann Bernoulli [19] and is a powerful graphical tool for the analysis and design of planar mechanisms [20,21]. Instant centers are useful for determining both the velocity distribution in a given link and the motion transmission between links [22] and are helpful in the kinematic analysis of mechanisms containing higher pairs, for example, gear trains and cam mechanisms [23]. The method of instant centers has proved to be very efficient in finding the input-output velocity relationships of complex linkages [24]. When combined with the conservation of energy, instant centers also provide an efficient method to obtain the input-output force or torque relationships.

The statics of a rigid body can be investigated using an approach similar to the method of instant centers in kinematics. A force acting on a rigid body can be defined as an entity that determines the vector field of moments; i.e., a moment vector is associated with each point fixed in the body. A new concept, defining the locus of points where two coplanar forces apply the same moment, is presented here in the form of a theorem.

THEOREM. *For two arbitrary forces acting in a single plane, there exists a unique line in the plane where the moments about each point on the line, due to the two forces, are equal.*

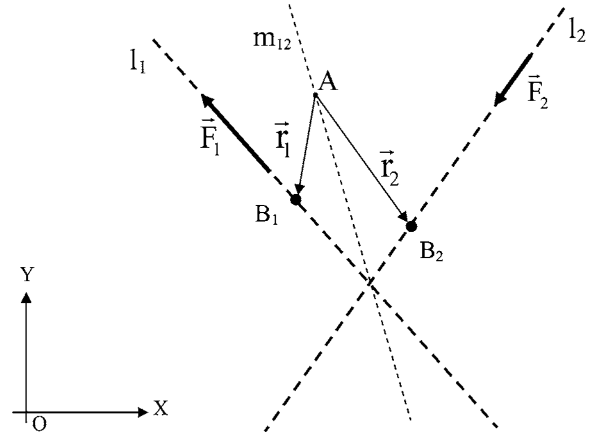


Fig. 1 Forces $\vec{F}_1$ and $\vec{F}_2$ acting along lines of action l_1 and l_2

Proof. Consider the two coplanar forces $\vec{F}_1$ and $\vec{F}_2$ which act along the skewed lines of action l_1 and l_2, respectively, as shown in Fig. 1. The plane is represented by the fixed Cartesian reference frame XOY where O denotes the origin of the reference frame.

Consider the point A in the plane where the moments due to the two forces are equal; i.e., the moments satisfy the constraint

$$\vec{M}_{1\rightarrow A} = \vec{M}_{2\rightarrow A} \tag{1a}$$

Equation (1a) can be written as

$$\vec{r}_1 \times \vec{F}_1 = \vec{r}_2 \times \vec{F}_2 \tag{1b}$$

where $\vec{r}_i$ (i=1 and 2) is the vector from point A to an arbitrary point B_i on line l_i. Equation (1b) can be written as

$$F_{1x}(y_A - y_1) - F_{1y}(x_A - x_1) = F_{2x}(y_A - y_2) - F_{2y}(x_A - x_2) \tag{2}$$

where F_{ix} and F_{iy} are the x and y components of force $\vec{F}_i$, (x_A, y_A) are the Cartesian coordinates of point A, and (x_i, y_i) are the Cartesian coordinates of point B_i. Rearranging Eq. (2) gives the equation of a unique straight line; i.e.,

$$y_A = a x_A + b \tag{3a}$$

where the slope and the intercept, respectively, are

$$a = \frac{F_{1y} - F_{2y}}{F_{1x} - F_{2x}} \quad \text{and} \quad b = \frac{F_{1x}y_1 - F_{1y}x_1 - F_{2x}y_2 + F_{2y}x_2}{F_{1x} - F_{2x}} \tag{3b}$$

Note that this line, denoted as m_{12} in Fig. 1, must pass through the point of intersection of the lines of action l_1 and l_2. The conclusion is that the locus of all points, where two coplanar forces exert the same moment, is a unique straight line henceforth referred to as the equimomental line. In the special case where the two coplanar forces are parallel then the equimomental line must also be parallel, and in the plane of, the two forces.

Table 1 presents a summary of important correlations between three basic concepts in kinematics and statics. In particular, note that the new concept of an equimomental line in statics is analogous to the well-known concept of an instant center in kinematics.

Additional correlations between the domains of kinematics and statics will be presented in the following section. Two new theorems and formal proofs of the theorems will also be detailed in this section.

3 Relationship Between Instant Centers and Equimomental Lines

The Aronhold-Kennedy theorem, also referred to as the theorem of three instant centers, is commonly used to locate the sec-

Table 1 Correlations between planar kinematics and statics

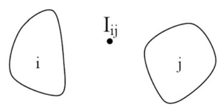
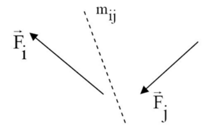
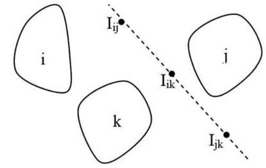
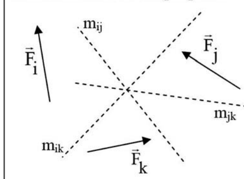
A link moving in a plane.	A force acting in a plane.
Velocity field defined by motion of the link.	Moment field defined by the force.
The instant center I_{ij} for links i and j. 	The equimomental line m_{ij} for forces $\vec{F}_i$ and $\vec{F}_j$. 

Table 2 The Aronhold-Kennedy theorem and the equimomental line theorem

The instant centers I_{ij}, I_{ik}, and I_{jk} associated with links i, j, and k must lie on a unique straight line. 	The equimomental lines m_{ij}, m_{ik}, and m_{jk} defined by co-planar forces $\vec{F}_i$, $\vec{F}_j$, and $\vec{F}_k$ must intersect at a unique point. 

ondary (or the unknown) instant centers of a single-degree-of-freedom planar mechanism [22]. The theorem states that the instant centers associated with links i, j, and k of the mechanism (denoted here as I_{ij}, I_{ik}, and I_{jk}) must lie on a unique straight line. This theorem will now be transformed to the domain of statics resulting in a new theorem, henceforth referred to as the equimomental line theorem.

THE EQUIMOMENTAL LINE THEOREM. *The three equimomental lines defined by three arbitrary coplanar forces must intersect at a unique point*.

Proof. Consider the three equimomental lines m_{12}, m_{13}, and m_{23}, defined by the three coplanar forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, see Fig. 2. The point of intersection of lines m_{12} and m_{13} will be denoted as point D. Since this point lies on the equimomental line m_{12} then the moment exerted about point D due to the forces $\vec{F}_1$ and $\vec{F}_2$ must satisfy the relation

$$\vec{M}_{1D} = \vec{M}_{2D} \tag{4}$$

Similarly, since point D lies on the equimomental line m_{13} then the moment exerted about point D due to the forces $\vec{F}_1$ and $\vec{F}_3$ must satisfy the relation

$$\vec{M}_{1D} = \vec{M}_{3D} \tag{5}$$

Therefore, the moment exerted about point D due to the forces $\vec{F}_2$ and $\vec{F}_3$ must satisfy the relation

$$\vec{M}_{2D} = \vec{M}_{3D} \tag{6}$$

The conclusion is that the equimomental line m_{23} must pass through point D as shown in Fig. 2.

The authors believe that the equimomental line theorem (documented in the right-hand column of Table 2) can play an important role in a static analysis, or synthesis, of trusses. For the purpose of comparison, the Aronhold-Kennedy theorem is presented in the left-hand column of Table 2.

3.1 Statically Indeterminate Trusses and Indeterminate Mechanisms. A structure is referred to as statically indeterminate when the unknown force variables (for example, an external force or an internal reaction force) cannot be determined uniquely from the force equilibrium equations. The forces cannot be determined without consideration of the deformations within the elements constituting the structure. The difference between the unknown variables and the number of independent equilibrium equations is commonly referred to as the degrees of indeterminacy of the structure. An indeterminate structure has the important property of self-equilibrating forces [25], i.e., when an internal force is acting in the redundant element, the forces in the structure can still be in self-equilibrium. This property is also referred to as a state of self-stress, widely used by the community investigating tensegrity systems [26], and adopted in this paper. From a more general perspective, a statically determinate system can be regarded as containing a self-stress when the external force is viewed as a regular rod that contains an internal force. For the purposes of this paper, the degrees of static indeterminacy are considered to be the number of independent self-stresses, including those originating from the external forces. The term, "the degree of indeterminacy," will be used consistently to designate the number of independent force variables that are required to uniquely define the static behavior of a truss.

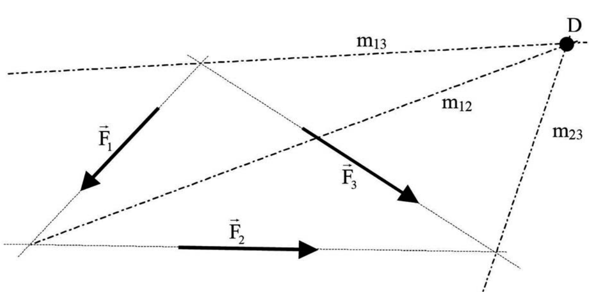


Fig. 2 The equimomental lines for the three coplanar forces

There is a class of single-degree-of-freedom planar mechanisms where some, or all, of the secondary instant centers cannot be located from the direct application of the Aronhold-Kennedy theorem [27,28]. These mechanisms are commonly referred to as mechanisms with kinematic indeterminacy or as indeterminate mechanisms [29]. A graphical technique to locate the secondary instant centers for an indeterminate mechanism was presented by Foster and Pennock [16,17]. The technique is based on the concept that a secondary instant center of a two-degrees-of-freedom planar mechanism (i.e., two independent inputs are required for a unique output) must lie on a unique straight line. The kinematics of an indeterminate mechanism is dual to the statics of a truss with two degrees of indeterminacy whose behavior is uniquely determined by two independent forces.

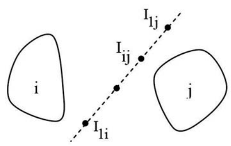
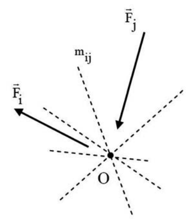
The dual static theorem states that all of the equimomental lines for a truss, with two degrees of indeterminacy, must pass through the same point. For the convenience of the reader, the kinematic theorem (for links i and j) and the dual static theorem (for forces $\vec{F}_i$ and $\vec{F}_j$) are presented in the left-hand and the right-hand columns of Table 3, respectively.

THEOREM. *The equimomental line for two arbitrary coplanar forces, in a truss with two degrees of indeterminacy, must pass through a unique point*.

Proof. Consider the two forces $\vec{F}_i$ and $\vec{F}_j$ in a plane defined by the fixed Cartesian reference frame XOY where O denotes the origin, see Fig. 3.

The vectors $\vec{r}_{Oi}$ and $\vec{r}_{Oj}$ point from the origin O to the arbitrary points A and B on the lines of action of the forces $\vec{F}_i$ and $\vec{F}_j$, respectively. The vectors $\vec{r}_{iI}$ and $\vec{r}_{jI}$ point from A and B, respectively, to a point which is assumed to lie on the equimomental line m_{ij} for the two forces (the point is denoted here as E_{ij}). Finally,

Table 3 Kinematic theorem and the dual static theorem

In a two-degree-of-freedom planar mechanism, the relative instant center I_{ij} (for the two links i and j) must lie on a unique straight line (defined by the absolute instant centers I_{1i} and I_{1j}).	In a truss with two degrees of indeterminacy, the equimomental line m_{ij}, for the two forces $\vec{F}_i$ and $\vec{F}_j$, must pass through a unique point (denoted here as O).
	
A special case is where one link is in pure translation relative to the other link. For this case, the relative instant center is located on a line at infinity (perpendicular to the direction of translation).	

the vector $\vec{r}_I$ $(=x_I\hat{i}+y_I\hat{j})$ points from the origin O to the point E_{ij}.

The moment due to the two forces $\vec{F}_i$ and $\vec{F}_j$, with respect to the point E_{ij}, can be written as

$$\vec{M}_{E_{ij}} = \vec{M}_O - (x_I\hat{i} + y_I\hat{j}) \times (\vec{F}_i - \vec{F}_j) \tag{7}$$

where $\vec{M}_O$ is the moment about point O, and x_I and y_I are the Cartesian coordinates of point E_{ij}. For convenience, Eq. (7) can be written as

$$\vec{M}_{E_{ij}} = \vec{M}_O - (x_I\hat{i} + y_I\hat{j}) \times \vec{F}_{ij} \tag{8a}$$

where

$$\vec{F}_{ij} = \vec{F}_i - \vec{F}_j \tag{8b}$$

is referred to as the force difference vector. Since point E_{ij} is assumed to lie on the equimomental line m_{ij} then the moment about this point due to the two forces is zero; i.e.,

$$\vec{M}_{E_{ij}} = \vec{M}_{E_i} - \vec{M}_{E_j} = 0 \tag{9}$$

where $\vec{M}_{E_i}$ is the moment about E_{ij} due to $\vec{F}_i$ and $\vec{M}_{E_j}$ is the moment about E_{ij} due to $\vec{F}_j$. Substituting Eq. (9) into Eq. (8a), and rearranging, the moment about point O can be written as

$$\vec{M}_O = (x_I\hat{i} + y_I\hat{j}) \times \vec{F}_{ij} \tag{10}$$

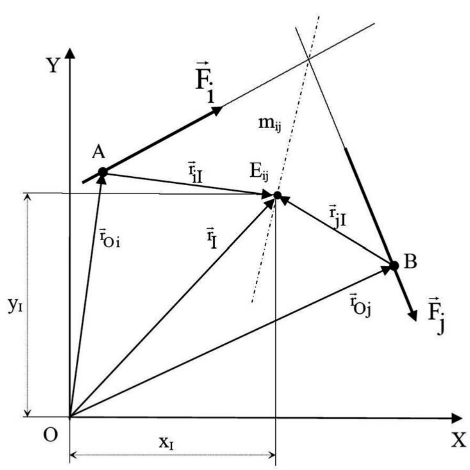


Fig. 3 Point E_{ij} which lies on the equimomental line of two forces $\vec{F}_i$ and $\vec{F}_j$

As stated previously, the static behavior of a truss with two states of self-stress (i.e., two degrees of indeterminacy) can be defined by two independent force variables. For the purposes of introducing the dual of the kinematic theorem [17] into statics, these two forces will be referred to here as generalized forces and denoted as p_1 and p_2. The generalized forces are analogous to the generalized velocities that are used in the kinematics theorem. Therefore, the moment about point O can be expressed as a linear combination of the two generalized forces; i.e.,

$$M_O = h_{M_1}p_1 + h_{M_2}p_2 \tag{11a}$$

where the coefficients

$$h_{M_1} = \frac{\partial M_O}{\partial p_1} \text{ and } h_{M_2} = \frac{\partial M_O}{\partial p_2} \tag{11b}$$

Sign convention: Since Eq. (11a) is a scalar equation then the equation gives a positive value if the moment is counterclockwise and the equation gives a negative value if the moment is clockwise.

The Cartesian components of the force difference vector, see Eq. (8b), can be expressed as linear combinations of the two generalized forces; i.e.,

$$F_{x_{ij}} = f_{x_1}p_1 + f_{x_2}p_2 \tag{12a}$$

and

$$F_{y_{ij}} = f_{y_1}p_1 + f_{y_2}p_2 \tag{12b}$$

where the coefficients

$$f_{x_1} = \frac{\partial F_{x_{ij}}}{\partial p_1}, \quad f_{x_2} = \frac{\partial F_{x_{ij}}}{\partial p_2}, \quad f_{y_1} = \frac{\partial F_{y_{ij}}}{\partial p_1} \text{ and } f_{y_2} = \frac{\partial F_{y_{ij}}}{\partial p_2} \tag{13}$$

The equation of the equimomental line, see Eq. (3a), can be written as

$$y_I = ax_I + b \tag{14a}$$

where the slope and the intercept can be written, respectively, as

$$a = \frac{F_{y_{ij}}}{F_{x_{ij}}} \text{ and } b = -\frac{M_O}{F_{x_{ij}}} \tag{14b}$$

Substituting Eqs. (11a) and (12) into Eqs. (14b), the slope and the intercept of the equimomental line can be written as

$$a = \frac{f_{y_1}p_1 + f_{y_2}p_2}{f_{x_1}p_1 + f_{x_2}p_2} \text{ and } b = -\frac{h_{M_1}p_1 + h_{M_2}p_2}{f_{x_1}p_1 + f_{x_2}p_2} \tag{15}$$

The ratio of the two generalized forces, henceforth referred as the force ratio, will be defined as

$$r_f = \frac{p_2}{p_1} \tag{16}$$

Substituting Eq. (16) into Eq. (15), and simplifying, the slope and the intercept of the equimomental line can be written as

$$a = \frac{f_{y_1} + f_{y_2}r_f}{f_{x_1} + f_{x_2}r_f} \quad \text{and} \quad b = -\frac{h_{M_1} + h_{M_2}r_f}{f_{x_1} + f_{x_2}r_f} \tag{17}$$

Rearranging these two equations, the force ratio can be written as

$$r_f = \frac{af_{x_1} - f_{y_1}}{f_{y_2} - af_{x_2}} \quad \text{or as} \quad r_f = -\frac{bf_{x_1} + h_{M_1}}{h_{M_2} + bf_{x_2}} \tag{18}$$

Then equating these two relations, and rearranging, gives

$$\frac{f_{y_1}h_{M_2} - f_{y_2}h_{M_1}}{f_{x_1}f_{y_2} - f_{y_1}f_{x_2}} = a\left(\frac{f_{x_1}h_{M_2} - f_{x_2}h_{M_1}}{f_{x_1}f_{y_2} - f_{y_1}f_{x_2}}\right) + b \tag{19}$$

Note that Eq. (19) is the equation of a straight line (in the fixed Cartesian reference frame in the plane of the forces). From inspection of Eq. (19), a point with the Cartesian coordinates

Table 4 The kinematic theorem and the dual static theorem

Stage	Kinematic theorem	Dual static theorem
1	For links i and j of a two-degrees-of-freedom planar mechanism, the instant center I_{ij} must lie on a unique straight line. The velocity of instant center I_{ij}, in terms of the velocity of a point A in link i and the angular velocity of link i relative to link j, is $\vec{v}_{I_{ij}}=\dot{x}_{A_i}\hat{i}+\dot{y}_{A_i}\hat{j}+\omega_{ij}\hat{k}\times(x_I\hat{i}+y_I\hat{j})$	In a truss with two degrees of indeterminacy, the equimomental line must pass through a unique point. The moment due to the forces $\vec{F}_i$ and $\vec{F}_j$, in terms of the moment about the origin (denoted as point O), is $\vec{M}_{E_{ij}}=\vec{M}_O-(x_I\hat{i}+y_I\hat{j})\times\vec{F}_{ij}$ $(8a')$
2	Setting the above equation to zero, the coordinates of the instant center I_{ij} are $x_I=-\dfrac{\dot{y}_{A_i}}{\omega_{ij}}$ and $y_I=\dfrac{\dot{x}_{A_I}}{\omega_{ij}}$	Setting Eq. $(8a)$ to zero, the equation describing the equimomental line is $y_I=ax_I+b$ $(14a')$ where $a=F_{y_{ij}}/F_{x_{ij}}$ and $b=-M_O/F_{x_{ij}}$ $(14b')$
3	Writing the above equations as a linear combination of two generalized velocities ($\dot{q}_1$ and $\dot{q}_2$) defines the velocity of the mechanism; i.e., $\omega_{ij}=h_{ij1}\dot{q}_1+h_{jj2}\dot{q}_2$ $\dot{x}_{Ai}=f_{xA1}\dot{q}_1+f_{xA2}\dot{q}_2$ $\dot{y}_{Ai}=f_{yA1}\dot{q}_1+f_{yA2}\dot{q}_2$	Writing Eq. $(8a)$ as a linear combination of two generalized forces (p_1 and p_2) defines the statics of the truss; i.e., $M_O=h_{M1}p_1+h_{M2}p_2$ $(11a')$ $F_{x_{ij}}=f_{x1}p_1+f_{x2}p_2$ $(12a')$ $F_{y_{ij}}=f_{y1}p_1+f_{y2}p_2$ $(12b')$
4	Introducing the generalized velocity ratio $r_V(=\dot{q}_2/\dot{q}_1)$, the Cartesian coordinates of the instant center I_{ij} can be written as $x_I=-(f_{y_{A1}}+f_{y_{A2}}r_v)/(h_{ij_1}+h_{ij_2}r_v)$ and $y_I=-(f_{x_{A1}}+f_{x_{A2}}r_v)/(h_{ij_1}+h_{ij_2}r_v)$	Introducing the generalized force ratio $r_f(=p_2/p_1)$, the slope and the intercept of the equimomental line can be written as $a=(f_{y_1}+f_{y_2}r_f)/(f_{x_1}+f_{x_2}r_f)$ and $b=-(h_{M_1}+h_{M_2}r_f)/(f_{x_1}+f_{x_2}r_f)$ $(17')$
5	The relations in Stage 4 constrain the instant center I_{ij} to lie on the unique straight line $(h_{ij1}f_{x_{A_2}}-h_{ij2}f_{x_{A_1}})x_1+(h_{ij1}f_{y_{A_2}}-h_{ij2}f_{y_{A_1}})y_1$ $=f_{x_{A_1}}f_{y_{A_2}}-f_{x_{A_2}}f_{y_{A_1}}$	Equation (17) constrains the equimomental lines to pass through the unique point with the coordinates $x_o=(f_{x_1}h_{M_2}-f_{x_2}h_{M_1})/(f_{x_1}f_{y_2}-f_{y_1}f_{x_2})$, $y_o=(f_{y_1}h_{M_2}-f_{y_2}h_{M_1})/(f_{x_1}f_{y_2}-f_{y_1}f_{x_2})$ $(20')$

$$x_0=\frac{f_{x_1}h_{M_2}-f_{x_2}h_{M_1}}{f_{x_1}f_{y_2}-f_{y_1}f_{x_2}} \quad \text{and} \quad y_0=\frac{f_{y_1}h_{M_2}-f_{y_2}h_{M_1}}{f_{x_1}f_{y_2}-f_{y_1}f_{x_2}} \tag{20}$$

must lie on this unique straight line. Therefore, independent of the generalized forces p_1 and p_2 the equimomental line for the two forces $\vec{F}_i$ and $\vec{F}_j$ must pass through the point whose coordinates are defined by Eq. (20). The slope and the intercept of the line, however, will change in accordance with the force ratio, see Eq. (17). The right-hand column of Table 4 presents a summary of this proof. For the sake of completeness and for the purpose of comparison, the left-hand column presents a summary of the kinematic theorem.

The following section shows that an expansion of the duality principle yields a correlation between linkages and trusses. The topology of the two systems are related by the graph theory duality principle [15].

4 The Duality Between Kinematics and Statics

The correlation between basic laws in kinematics and statics leads to the duality between the kinematics of mechanisms and the statics of structures. For the first-order kinematic properties of a mechanism, there is a corresponding property for the dual truss. The basic proposition of the duality relationship between a linkage and a determinate truss is as follows. For a linkage L there exists a truss T satisfying: (i) each element of T (for example, a member or a rod, an externally applied force of known magnitude, or an internal reaction force) corresponds to a link of L; and (ii) a force acting on an element of T corresponds to the velocity of a point fixed in a link of L. Note that the links in a planar linkage can be transformed into a number of interconnected binary links [21], therefore, this paper will regard these linkages to be composed solely of binary links. Table 5 lists the transformation rules defining the correspondence between a linkage and the dual truss (for more details, and a formal proof of these rules, the reader is referred to Shai [18]).

The process of constructing a truss that is dual to a given linkage, or vice versa, is similar to that of constructing a dual graph [30]. This relationship gives rise to the correspondence between the equations of static equilibrium in the truss [31] and the velocity equations of the dual linkage [15]. To facilitate an understanding of this duality, consider the simple truss shown in Fig. 4(a) which is comprised of two members (or rods) pinned together at O and subjected to the external force $\vec{P}_2$ acting at this pin. Since the internal reaction forces in the truss must satisfy the conditions of static equilibrium then the external force can be written as

$$\vec{P}_2=\vec{F}_3+\vec{F}_4 \tag{21}$$

where $\vec{F}_3$ and $\vec{F}_4$ are the internal reaction forces in rods 3 and 4, respectively.

The three elements of the truss are rod 3, rod 4, and the external force $\vec{P}_2$ and the three faces of the truss are denoted as A', B', and O', as shown in Fig. 4(a). Note that the face O' is s special face, referred to as the reference face, and is analogous to the ground link $O_2'O_4'$ (also denoted as link 1) of the dual linkage. This link-

Table 5 Transformation rules for a linkage and the dual truss

Linkage terminology	Truss terminology
A link; i.e., a rigid body which can be regarded as infinite in extent.	A truss element (e.g., a rod, an external force, or an internal reaction force).
A kinematic pair or a joint (e.g., a revolute joint or a prismatic joint).	A face; i.e., the area of a plane enclosed by the truss elements.
The relative velocity between the two kinematic pairs of a binary link; i.e., the vector difference of the absolute velocities of the two kinematic pairs.	The force in a truss element.
The absolute linear velocity of a joint.	The face force (defined as the force associated with each face of the truss).
The number of degrees of freedom of the linkage.	The number of degrees of indeterminacy of the truss.

age can be obtained from the transformation rules that are presented in Table 5. The result is a planar four-bar linkage comprised of the ground link 1, the input (or driving) link 2′, the coupler link 3′, and the output link 4′, as shown in Fig. 4(*b*). Links 2′ and 4′ are pinned to the ground link 1 at $O_{2'}$ and $O_{4'}$, respectively, and pinned to link 3′ at *A* and *B*. Note that joints *A* and *B* of the dual linkage are analogous to faces *A*′ and *B*′ of the simple truss. Also, note that links 2′, 3′, and 4′ correspond to the elements 2, 3, and 4 of the truss, respectively, such that each link is perpendicular to the corresponding element in the truss, as illustrated in Fig. 4(*a*). This construction guarantees that the direction of a force in the truss is parallel to either the absolute velocity of a joint or the relative velocity between two joints in the dual linkage; i.e., there is a geometrical isomorphism between the truss and the dual linkage.

In addition, there is a topological isomorphism between the elements of the truss and the links of the dual linkage. To illustrate this, consider pin *A* which connects the input link 2′ and the coupler link 3′, see Fig. 4(*b*). The linear velocity of pin *A* (i.e., the input velocity) can be written as

$$\vec{V}_A = \vec{V}_{\mathrm{B}} + \vec{V}_{A/\mathrm{B}} \tag{22}$$

A comparison of Equations (21) and (22) shows the topological isomorphism between the elements in the truss and the links in the dual linkage. The conclusion is that there is a correspondence between the forces in the truss elements (due to the external force

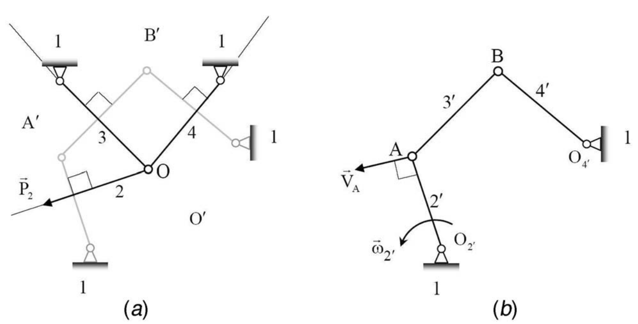


Fig. 4 Example of the truss-linkage duality. (*a*) A simple truss (the dual linkage is superimposed). (*b*) The dual linkage.

$\vec{P}_2$) and the linear velocities of the joints in the dual linkage (due to the input velocity $\vec{V}_A$). Since the truss can be scaled up or down without affecting the magnitudes of the internal forces in members 3 and 4 then the lengths of links 3′ and 4′ can also be scaled up or down without affecting the velocity of pin *B* or the velocity of pin *A* relative to pin *B*.

A face force is the most suitable entity to introduce equimomental lines into the static analysis of a truss. This force variable, which is defined in Table 5, is associated with each face of the truss and can be considered the multidimensional expansion of the mesh currents in an electronic circuit [18]. An internal reaction force acting in a member of a truss, denoted as member *j*, can be written as

$$\vec{F}_j = \vec{F}_R - \vec{F}_L \tag{23}$$

where $\vec{F}_R$ and $\vec{F}_L$ are the face forces of the two adjacent faces on the right and left sides (or planes), respectively, of member *j*. The right and left planes are defined according to the direction of the edge; i.e., the unit vector pointing from the head vertex to the tail vertex [15]. The equimomental line of the two face forces coincides with the vector difference of the two forces (see Sec. 2). Therefore, the equimomental line of two adjacent face forces is parallel to the line of the rod, the line of action of the external force, or the line of action of the internal reaction force separating the two faces. To indicate the importance of a face force, Table 6 lists a number of dual properties between a planar mechanism and a truss.

The following section will present an original graphical technique, referred to here as the dual Kennedy theorem, to locate the unknown equimomental lines of a simple truss. After the locations of the equimomental lines are known then the face forces and the unknown reaction forces acting in the truss can be determined. An advantage of this graphical technique, compared to an analytical technique, is that the internal reaction force in a specified rod can be obtained directly without the need to evaluate the internal reaction forces in other rods of the truss.

5 The Dual Kennedy Theorem

The dual Kennedy theorem and the face force relationship of a determinate truss are summarized in the right-hand column of

Table 6 Dual properties of a mechanism and a truss

The relative instant center of two links connected by a revolute joint is coincident with the revolute joint.	The equimomental line of two adjacent face forces is the line which separates the two face forces.
The ground link.	The reference face. The face of the truss where, without loss of generality, the face force is taken to be zero. It is common practice to chose the outside area of the truss to be the reference face.
The input (or driving) link.	The input (or specified) face force. In general, this is the force in the face located on one side of the external force.
If the relative instant center I_{ij} is coincident with the absolute instant center I_{1i} then the linkage is immobile (or locked).	If the corresponding equimomental line m_{xy} is coincident with the equimomental line m_{zy} then the truss is not rigid.
The linkage is mobile if, and only if, there exists a set of all possible instant centers that satisfy the Aronhold-Kennedy theorem.	A truss is stable if, and only if, there exists a set of all possible equimomental lines that satisfy the equimomental line theorem.

Table 7 The Aronhold-Kennedy theorem and the dual Kennedy theorem

Summary of the Aronhold-Kennedy theorem to find the instant centers of a planar mechanism: 1. Map all the kinematic pairs as the instant centers of the connected links. 2. Find a set of four links (say i, j, k, and l) for which the relative instant centers I_{ij}, I_{ik}, I_{kl}, and I_{il} are known. 3. The point of intersection of the line connecting instant centers I_{ij} and I_{jk}, and the line connecting instant centers I_{i1} and I_{k1} is the instant center I_{ik}. 4. Repeat this procedure to locate all the unknown instant centers. To facilitate the application of this method, use the Kennedy circle as follows. Associate each link with a vertex on the circle. If the location of the instant center between two links is known then connect the two corresponding vertices with an edge. Step 2 is the search for a quadrangle formed by the edges in the circle, while step 3 results in the addition of a diagonal to that quadrangle.	Summary of the dual Kennedy theorem to find the equimomental lines of a determinate truss: 1. Map all the rods as the equimomental lines of the faces they separate. 2. Find a set of four face forces (say x, y, z, and w) for which the equimomental lines m_{xy}, m_{yz}, m_{wx}, and m_{wz}, are known. 3. The line connecting the point of intersection of equimomental lines m_{xy} and m_{yz}, and the point of intersection of equimomental lines m_{wx} and m_{wz} is the equimomental line is m_{xz}. 4. Repeat this procedure to locate all the unknown equimomental lines. To facilitate the application of this method, use the dual Kennedy circle as follows. Associate each face with a vertex on the circle. If the location of the equimomental line between two face forces is known then connect the two corresponding vertices with an edge. Step 2 is the search for a quadrangle formed by the edges in the circle, while step 3 results in the addition of a diagonal to the quadrangle.
The angular velocity of link i can be written in terms of the input angular velocity ω_2 as $$\omega_i = \pm \frac{I_{2i}I_{21}}{I_{2i}I_{i1}}\omega_2 \quad (24)$$ where 1 denotes the ground link and $I_{2i}I_{i1}$ and $I_{2i}I_{21}$ are the distances between instant centers I_{2i} and I_{i1} and between instant centers I_{2i} and I_{2l}, respectively. Equation (24) takes advantage of the fact that the points on two links, which are coincident with the relative instant center, have the same linear velocity.	The face force of face K can be written in terms of the face force F_P (i.e., the known external force) as $$F_K = \pm \frac{m_{\mathrm{PK}}m_{\mathrm{PO}}}{m_{\mathrm{PK}}m_{\mathrm{KO}}}F_P \quad (25)$$ where O denotes the reference face and $m_{\mathrm{PK}}m_{\mathrm{KO}}$ and $m_{\mathrm{PK}}m_{\mathrm{PO}}$ are the perpendicular distances from an arbitrary point on the relative equimomental line m_{PK} to the absolute equimomental lines m_{KO}, and m_{PO}, respectively. Equation (25) takes advantage of the fact that two coplanar forces apply the same moment at each point on the equimomental line of these two forces.

Table 7. For the convenience of the reader and for the purpose of comparison, the Aronhold-Kennedy theorem and the angular velocity relationship of a single-degree-of-freedom planar mechanism are summarized in the left-hand column.

The sign convention for Eq. (24) is well-known; i.e., use the negative sign if the relative instant center I_{2i} lies between the two absolute instant centers I_{21} and I_{i1} and use the positive sign if the relative instant center lies outside the two absolute instant centers.

The sign convention for Eq. (25) can be obtained in a systematic manner from the sign convention for Eq. (24) using the duality relation. The location of the relative instant center with respect to the absolute instant centers is replaced by the direction of the face forces with respect to the two half-planes formed by the relative equimomental line. Therefore, use the positive sign in Eq. (25) if the face forces $\vec{F}_{\mathbf{K}}$ and $\vec{F}_{\mathbf{P}}$ acting along the absolute equimomental lines $m_{\mathbf{KO}}$ and $m_{\mathbf{PO}}$, respectively, are both directed from one side of the half-plane created by the relative equimomental line $m_{\mathbf{PK}}$ to the other side of the half-plane. Similarly, use the negative sign in Eq. (25) if the two forces are not both directed from one side of the half-plane created by the relative equimomental line to the other side of the half-plane.

To illustrate the dual Kennedy theorem and the sign convention consider the simple truss (commonly referred to as the Howe truss) shown in Fig. 5(*a*), in which the common assumption of pinned joints is implied. The truss is subjected to a known external force $\vec{P}$ which acts at the pin connecting the five rods 5, 6, 8, 9, and 11.

A typical statics problem is to determine the internal reaction force in a particular rod of the idealized truss. For the purposes of illustration, assume that the problem is to determine the internal reaction force in the lower rod 7.

Recall that a conventional analytical approach to solve a statically determinate truss problem is to write the force balance equations for the members of the truss. This produces a set of simultaneous linear equations which can be solved in a straight-forward manner using a computer software package, such as MATLAB or MATHEMATICA. However, this procedure affords no geometric insight and the internal forces in several of the rods must be evaluated before the internal force in rod 7 can be obtained. Note that the graphical technique proposed here does not require a knowledge of the internal forces in the other rods in order to determine the internal force in a specified rod. This graphical technique is believed to be an original contribution to the literature and is similar to drawing the Aronhold-Kennedy circle to locate the secondary instant centers of a single-degree-of-freedom planar

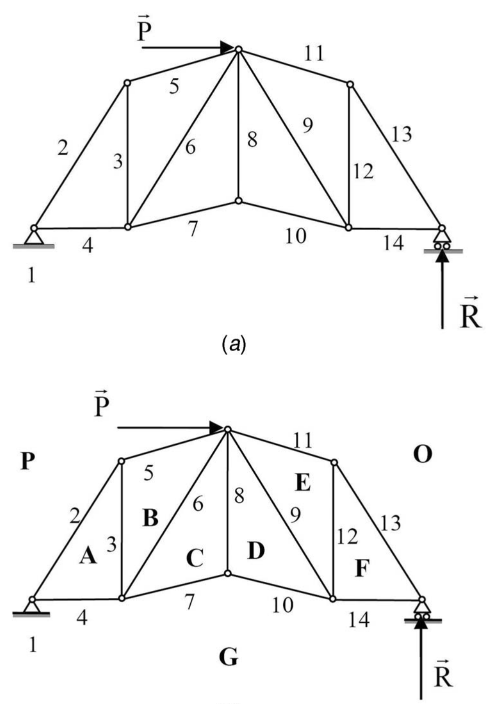


Fig. 5 (*a*) **A simple truss.** (*b*) **The faces of the simple truss.**

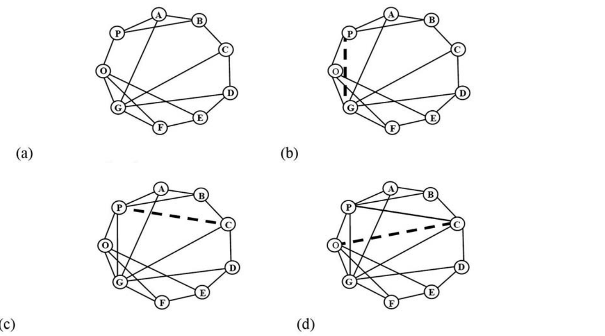


Fig. 6 The dual Kennedy circle (*a*) The primary equimomental lines. (*b*) The secondary equimomental line m_{PG}. (*c*) The secondary equimomental line m_{PC}. (*d*) The secondary equimomental line m_{CO}.

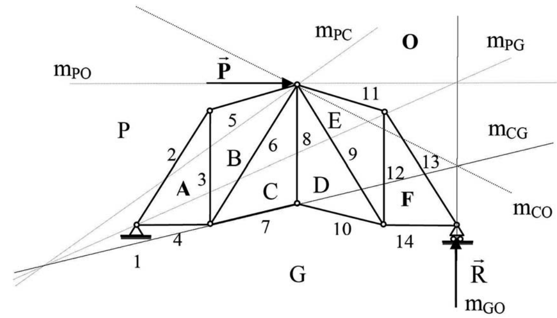


Fig. 7 Six of the equimomental lines

mechanism [20,22]. For this reason the technique is referred to here as drawing the dual Kennedy circle.

The number of faces of this truss is $n=9$ and are denoted as A, B, C, D, E, F, G, P, and O as shown in Fig. 5(*b*). The reference face is denoted as O and the input face is denoted as P. Note that the external force is between the input face P on the left and the reference face O on the right. Since the face force in the reference face is defined as zero then the external force $\vec{P}$ is the face force of face P; i.e., denoted here as F_P.

The total number of equimomental lines in this truss is

$$n_l = \frac{n(n-1)}{2} = \frac{9(9-1)}{2} = 36 \tag{26}$$

The number of known (or primary) equimomental lines can be written as

$$n_K = n_R + n_E + n_S \tag{27a}$$

where the number of rods $n_R=13$, the number of external forces acting on the truss $n_E=1$, and the number of mobile supports $n_S=1$. Therefore, the number of primary equimomental lines is

$$n_K = 13 + 1 + 1 = 15 \tag{27b}$$

The number of unknown (or secondary) equimomental lines is

$$n_U = 36 - 15 = 21 \tag{28}$$

The procedure to locate the secondary equimomental lines is to draw a circle (referred to as the dual Kennedy circle) and denote the faces of the truss in a clockwise manner on the circumference of this circle, see Fig. 6(*a*). Then represent the primary equimomental lines as solid lines and the secondary equimomental lines as dashed lines in this circle. For example, the secondary equimomental lines m_{PG}, m_{PC}, and m_{CO} are indicated by the dashed lines on Figs. 6(*b*)–6(*d*), respectively.

Note that the dashed line PG creates the two triangles POG and PAG in the quadrangle OPAG, see Fig. 6(*b*). Therefore, the equimomental line m_{PG} must pass through: (i) the point of intersection of equimomental lines m_{PA} and m_{AG}, namely, between the lines of rods 2 and 4, respectively, and (ii) the point of intersection of the equimomental line m_{PO} (i.e., the line coincident with the line of action of the external force $\vec{P}$) and the equimomental line m_{GO} (i.e., the line coincident with the line of action of the reaction force $\vec{r}$). Similarly, the dashed line PC creates the two triangles CBP and CGP in the quadrangle BCGP, see Fig. 6(*c*). Therefore, the equimomental line m_{PC} must pass through: (i) the point of intersection of equimomental lines m_{PB} and m_{BC}, namely, between the lines of rods 5 and 6, respectively, and (ii) the point of intersection of the equimomental line m_{PG} and the equimomental line m_{CG} which is coincident with rod 7. Finally, the dashed line OC creates the two triangles OPC and OGC in the quadrangle OPCG, see Fig. 6(*d*). Therefore, the equimomental line m_{CO} must pass through: (i) the point of intersection of the equimomental line m_{PO} and the equimomental line m_{PC}, and (ii) the point of intersection of equimomental lines m_{CG} and m_{GO}. The equimomental lines m_{PG}, m_{PC}, and m_{CO} are shown in Fig. 7. The figure also shows the equimomental lines m_{PO}, m_{GO}, and m_{CG}. For the convenience of the reader the procedure to obtain the secondary equimomental lines in a truss is documented in Table 7.

The internal reaction force in rod 7 can be written from Eq. (23) as

$$\vec{F}_7 = \vec{F}_G - \vec{F}_C \tag{29a}$$

where $\vec{F}_C$ and $\vec{F}_G$ are the face forces of the two adjacent faces C and G on the left and right sides, respectively. The two adjacent face forces are directed along the absolute equimomental lines m_{CO} and m_{GO} and can be written from Eq. (25), see Table 7, as

$$F_C = \pm \frac{m_{PC}m_{PO}}{m_{PC}m_{CO}} F_P \quad \text{and} \quad F_G = \pm \frac{m_{PG}m_{PO}}{m_{PG}m_{GO}} F_P \tag{29b}$$

where $m_{PC}m_{PO}$ and $m_{PC}m_{CO}$ are the perpendicular distances from an arbitrary point on the relative equimomental line m_{PC} to the absolute equimomental lines m_{PO} and m_{CO}, respectively, and $m_{PG}m_{PO}$ and $m_{PG}m_{GO}$ are the perpendicular distances from an arbitrary point on the relative equimomental line m_{PG} to the absolute equimomental lines m_{PO} and m_{GO}. Choosing the arbitrary points U and V, see Fig. 8(*a*), on the equimomental lines m_{PC} and m_{PG}, respectively, the four perpendicular distances can be measured from a scaled drawing of the truss.

The sign convention in Eqs. (29*b*) will determine the direction of the face forces $\vec{F}_C$ and $\vec{F}_G$. Note that the face force $\vec{F}_P$ is directed from the left half-plane, defined by the relative equimomental line m_{PC}, to the right half-plane, see Fig. 8(*b*). Therefore, in order to ensure that $\vec{F}_C$ produces the same moment at every point on the equimomental line m_{PC} as $\vec{F}_P$ then $\vec{F}_C$ must also be directed from the left half-plane to the right half-plane as shown in the figure. Similarly, in order to ensure that $\vec{F}_G$ produces the same moment at every point on the relative equimomental line m_{PG} as $\vec{F}_P$ then $\vec{F}_G$ must also be directed from the left half-plane to the right half-plane. Therefore, the positive signs must be used in both equations presented in Eqs. (29).

Substituting the four perpendicular distances that were measured in Fig. 8(*a*) into Eqs. (29*b*) and the results into the vector equation given by Eq. (29*a*) will determine the internal reaction force in rod 7. The sign convention for an internal reaction force is as follows: The force is taken to be positive if it is in the same direction as the edge and the force is taken to be negative if it is in the opposite direction to the edge [15]. This sign convention implies that when the answer for an internal reaction force is positive then the rod is in compression and when the answer for an internal reaction force is negative then the rod is in tension.

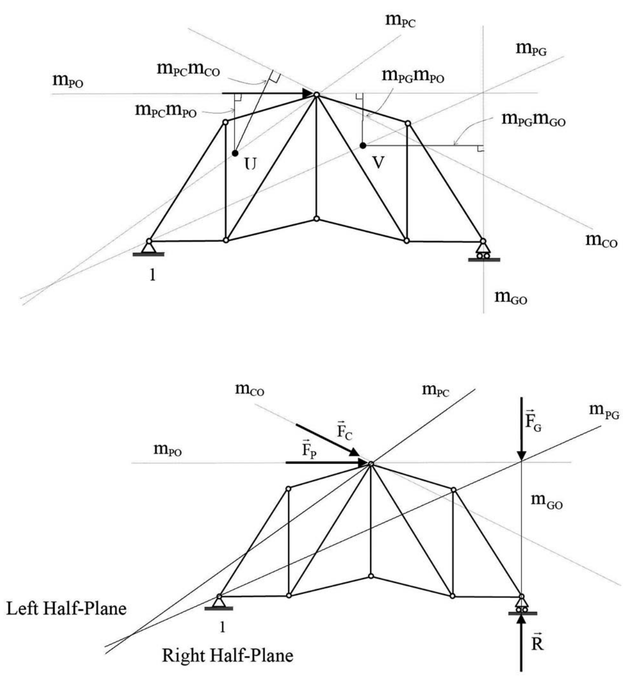


Fig. 8 (*a*) The perpendicular distances to the equimomental lines. (*b*) The direction of the face forces $\vec{F}_C$ and $\vec{F}_G$.

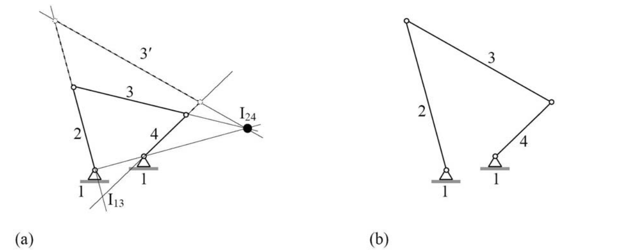


Fig. 9 Kinematic synthesis of a four-bar linkage. (*a*) Original geometry of the four-bar linkage. (*b*) Modified geometry of the linkage.

Synthesis Example. To demonstrate how instant centers can be used in the kinematic synthesis of a linkage consider the planar four-bar linkage denoted as links 1, 2, 3, and 4 (where link 1 is the ground link), see Fig. 9(*a*). The problem is to change the location of the coupler link 3 without changing the transmission ratio between the input link 2 and the output link 4. The transmission ratio for the linkage depends on the location of the instant center I_{24}. From the Aronhold-Kennedy theorem, this instant center is located at the point of intersection of the line connecting the instant centers I_{23} and I_{34} and the line connecting the instant centers I_{12} and I_{14}, see Fig. 9(*a*). Therefore, the geometry of the coupler link can only be changed such that the instant center I_{24} remains in the same location. The change in the geometry, stemming from a simple graphical construction, is shown by the dashed lines in Fig. 9(*a*). The transformed version of the linkage is shown in Fig. 9(*b*).

This kinematic synthesis technique can be transformed to obtain a method for synthesis in statics. From the principle of duality, the approach in statics remains the same as in kinematics with the important difference that the equimomental lines are now used and not instant centers. Consider the simple truss comprised of rods 2, 3, 4, and 5 and subjected to the external force $\vec{P}$ as shown in Fig. 10(*a*). The problem is to reposition rod 4, as indicated by the dashed line and denoted as 4′, without affecting the ratio of force $\vec{P}$ to the internal reaction force in rod 5. Such a requirement could arise during the construction of a truss, if it is discovered that the current position of the rods is incompatible with the environmental constraints and must be relocated without affecting other properties of the truss.

In an effort to be consistent with Fig. 4(*a*), the four faces of this truss are denoted as P, O, Q, and T, as shown in Fig. 10(*a*). Note that the internal reaction force in rod 5 can be determined from the face force $\vec{F}_{\mathrm{Q}}$ which is defined by the absolute equimomental line m_{QO} and the relative equimomental line m_{PQ}. The absolute equimomental line m_{QO} coincides with rod 5 and the relative equimomental line m_{PQ} is the line connecting the point of intersection of force $\vec{P}$ and rod 5, and the point of intersection of rods 2 and 4, as shown in Fig. 10(*b*). Since the synthesis condition is to maintain the same internal reaction force in rod 5 then the geometry of the truss must be modified. The changes, however, must ensure that the equimomental lines m_{QO} and m_{PQ} remain in the same positions. One possible solution is to change the orientation of rod 2, denoted as 2′, such that the point of intersection of rod 2′ with rod 4′ will remain on the equimomental line m_{PQ}, see Fig. 10(*b*). The geometry of the modified truss, denoted by rods 2′, 3, 4′, and 5, is shown in Fig. 10(*c*).

The following section will illustrate the theorems and properties of a truss and the dual linkage, presented in the paper, to investigate a special configuration of a determinate truss and a dead-center position of a linkage with kinematic indeterminacy; namely, the double flier eight-bar linkage. The section emphasizes

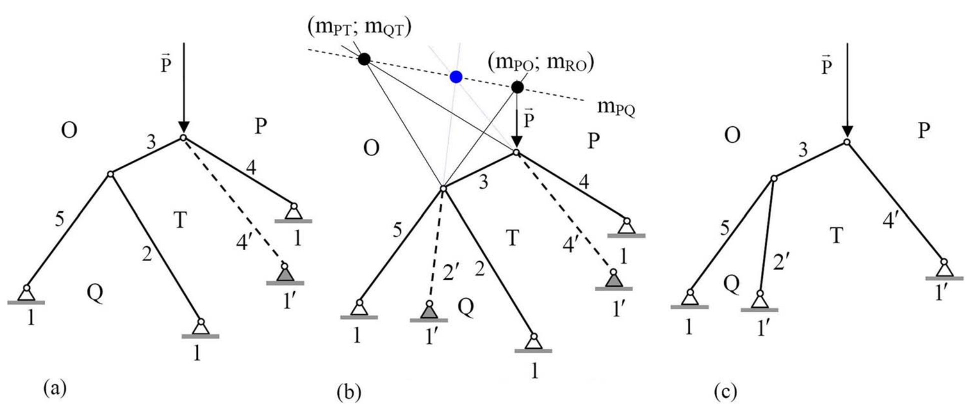


Fig. 10 The synthesis technique applied to a simple truss. (*a*) Original truss and the new position of rod 4. (*b*) Equimomental lines of the original truss. (*c*) Geometry of the modified truss.

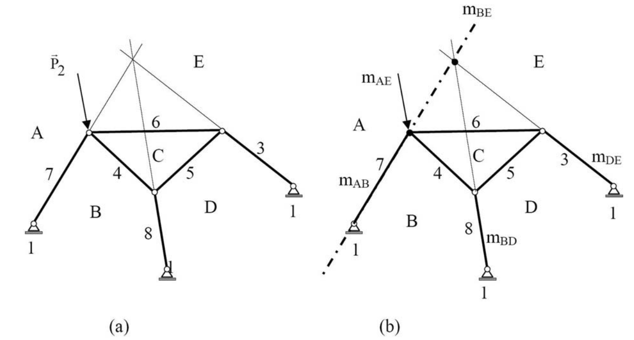


Fig. 11 The rigidity of the truss. (*a*) The truss. (*b*) The equimomental line m_{BE}.

the correspondence between (i) a nonrigid truss and the dual linkage; and (ii) the equimomental lines in the double flier eight-bar linkage in an arbitrary configuration and a dead-center configuration.

6 Special Configurations

The problem of finding unstable configurations of a truss is an important problem in statics. Similarly, the problem of finding dead-center positions of a mechanism is an important problem in kinematics [20,21,32]. An algorithm for finding the dead-center positions for the given topology of a planar linkage was recently developed by Shai and Polansky [33]. Also, a study of dead-center positions of single-degree-of-freedom planar linkages using Assur kinematic chains was presented by Pennock and Kamthe [34]. The examples presented in this section will use the concepts developed in this paper to identify dead-center positions of a mechanism. This procedure will not only determine if a truss is unstable or a mechanism is in a dead-center position, but will also identify the dependence between the locations of elements in the system (truss or mechanism) which causes the system to be in such a configuration. It is commonly accepted that obtaining solutions to this latter problem is not a straight-forward task.

6.1 A Determinate Truss. Consider the truss shown in Fig. 11(*a*) which is subjected to a known external force $\vec{P}_2$ acting at the pin connecting rods 4, 6, and 7. Note that this truss is in a special configuration; i.e., rods 3, 7, and 8 (or the extension of the three rods) intersect at a single point. The five faces of this truss will be denoted as A, B, C, D, and E. Faces B and D are adjacent faces separated by rod 8, therefore, the equimomental line m_{BD} is coincident with rod 8, as shown in Fig. 11(*b*). Similarly, the equimomental line m_{DE} is coincident with rod 3, the equimomental line m_{AB} is coincident with rod 7, and the equimomental line m_{AE} is coincident with the line of action of the external force $\vec{P}_2$.

According to the dual statics theorem, the equimomental line m_{BE} (for the two face forces B and E) is obtained by connecting the points of intersection of: (i) the equimomental lines m_{BD} and m_{DE}; and (ii) the equimomental lines m_{AB} and m_{AE}. Therefore, the equimomental line m_{BE} is the line connecting the point of intersection of rods 3, 7, and 8 (or the rods extended) and the pin connecting rods 4, 6, and 7. In other words, the equimomental line m_{BE} is coincident with rod 7 (or the equimomental line m_{AB}), see Fig. 5(*b*). The conclusion is that the truss is unstable in this configuration (see row 4 of Table 6).

The dual linkage, superimposed on the truss in Fig. 12(*a*) and shown separately in Fig. 12(*b*), is a Stephenson-III six-bar linkage in a special configuration; i.e., links 3, 4, and 6 (or the extensions of the links) intersect at a single point. According to the Aronhold-Kennedy theorem this unique point is the instant center for the coupler link 5 (i.e., I_{15}), see Fig. 12(*c*). Recall that the instant center I_{13} is defined as the point of intersection of the line passing

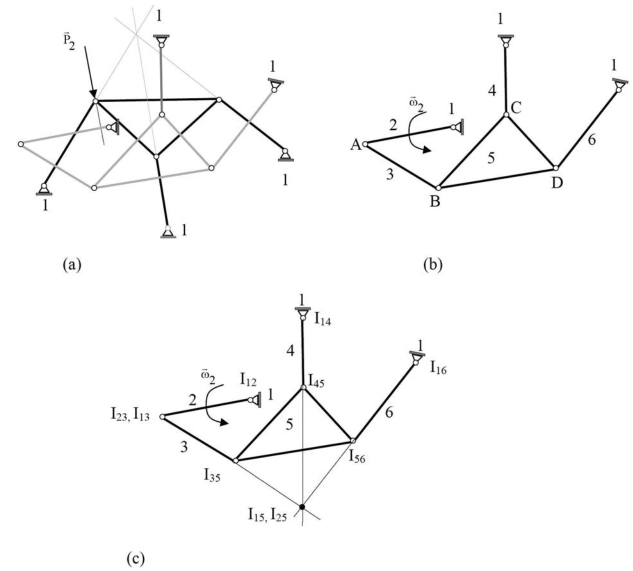


Fig. 12 The dual linkage in a dead-center position. (*a*) The nonrigid truss. (*b*) The dual linkage. (*c*) Instant center I_{13} is coincident with instant center I_{23}.

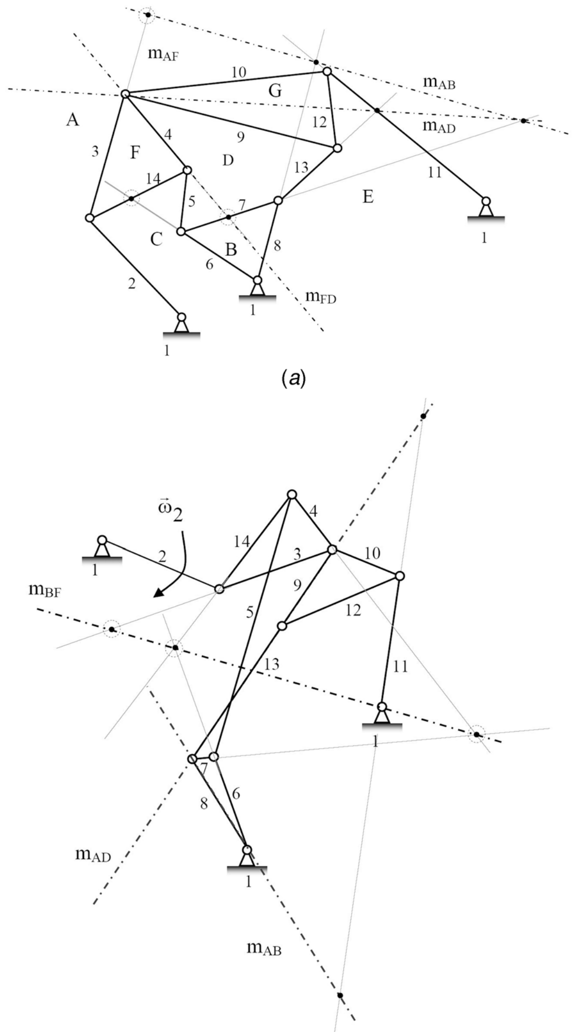
through the instant centers I_{12} and I_{23} and the line passing through the instant centers I_{15} and I_{35}. Therefore, the instant center I_{13} is coincident with the pin that connects links 2 and 3; i.e., I_{13} is coincident with I_{23}, as shown in Fig. 12(*c*). This implies that the dual linkage is instantaneously locked in this configuration (see row 4 of Table 6); i.e., the input angular velocity is zero. The only constraint for the linkage to be in this special configuration, commonly referred to as a dead-center position, is that the instant center I_{15} be located on link 3 (or link 3 extended).

6.2 The Double Flier Eight-Bar Linkage. This single-degree-of-freedom planar linkage is shown in an arbitrary configuration in Fig. 13(*a*). The seven faces of the dual truss will be denoted as A, B, C, D, E, F, and G, and for convenience are shown in Fig. 13(*a*).

From the equimomental line theorem, see Table 7, the three equimomental lines defined by faces A, F, and D (i.e., m_{AF}, m_{FD}, and m_{AD}) must intersect at a single point. Link 3 separates faces A and F, therefore, the line along link 3 is the equimomental line m_{AF}. Similarly, link 4 separates faces D and F, therefore, the line along link 4 is the equimomental line m_{FD}. Therefore, the equimomental line m_{AD} must pass through the point of intersection of links 3 and 4. From a similar argument, the equimomental line m_{AD} must also pass through the point of intersection of links 11 and 13. Therefore, the equimomental line m_{AD} is the line connecting these two points of intersection, see Fig. 13(*a*). In the same manner, the equimomental line m_{AB} must pass through the point of intersection of links 8 and 11, and the point of intersection of link 7 and the equimomental line m_{AD}. Finally, the equimomental line m_{BF} (if it exists for the double flier eight-bar linkage in this configuration) must pass through the points of intersections of: (i) links 6 and 14, (ii) links 4 and 7, and (iii) the equimomental line m_{AB} and link 3. Note that these three points, marked with circles in Fig. 13(*a*), do not lie on the same straight line. Therefore, the equimomental line m_{BF} does not exist for the linkage in this configuration. The conclusion is that the linkage is instantaneously movable in the given configuration; i.e., the mobility is one.

Now consider the double flier eight-bar linkage in the configuration shown in Fig. 13(*b*).

Note that the three points of intersections of: (i) links 6 and 14, (ii) links 4 and 7, and (iii) the equimomental line m_{AB} and link 3, again marked with circles in Fig. 13(*b*), now lie on the same



(b)

Fig. 13 (*a*) Double flier eight-bar linkage in an arbitrary configuration. (*b*) The equimomental line m_{BF} for a singular configuration.

straight line; i.e., the equimomental line m_{BF}. This indicates that the linkage is in a singular configuration. The linkage can resist externally applied forces and instantaneously constitutes a structure; i.e., the mobility is zero.

7 Conclusions

The paper presents the duality relation between the domains of planar kinematics and statics through two integrated levels: (i) the level of correlation between the basic concepts and theorems underlying these fields, and (ii) the level of duality between specific engineering systems. The paper introduces the concept of an equimomental line which is a unique line where the moments about each point on the line, due to two arbitrary coplanar forces, are equal. The authors believe that equimomental lines are a fundamental concept in statics and are a significant contribution to the literature. Two theorems are then presented based on the duality between equimomental lines and instantaneous centers of zero velocity. The first theorem, referred to as the equimomental line theorem, states that the three equimomental lines defined by three coplanar forces must intersect at a unique point. The second theorem states that the equimomental line for two coplanar forces acting in a truss with two degrees of indeterminacy must pass through a unique point. The paper then uses the concept of a face force to introduce a graphical technique to locate the equimomental lines of a determinate truss. This technique, referred to as the dual Kennedy theorem, is a dual form of the well-known Aronhold-Kennedy theorem in kinematics and is believed to be a significant contribution to the literature.

The practical examples presented in this paper emphasize the duality that exists between kinematics and statics. For instance, fundamental principles in kinematics can be used to check the stability of determinate trusses and principles in statics can be used to check the dead-center positions of linkages. A well-known rule in kinematics for checking if a linkage is in a dead-center position was transformed to statics to provide a new rule for checking the rigidity of a truss. The results of this paper afford engineers from both domains the opportunity to solve common problems using these new concepts. In addition, research groups in kinematics and statics will be able to share their knowledge and expedite their research work. Since the paper operates on the edge between kinematics and statics, the results presented in this paper have great potential for practical problems in kinetostatics. There is reason to believe that the duality relation can be applied to additional types of engineering systems. Examples include deployable structures which have attracted the attention of the aeronautics and astronautics community and tensegrity structures which have attracted the attention of the robotics and biological communities [35].

The duality between determinate trusses and planar linkages will yield a variety of practical and theoretical applications, including new engineering theorems and concepts, and the design of new systems. The authors believe that this paper makes a significant contribution to the theory of duality between planar kinematics and statics and a stronger contribution to the teaching of this theory. The authors hope that this duality will be developed further in order to derive new theorems in both the kinematics of mechanisms and in the statics of a wide variety of structures. The authors continue to explore this possibility and a future paper will present several new concepts in the kinematic analysis and synthesis of both planar and spatial mechanisms and the statics of structures consisting of one-, two-, and three-dimensional components. Future work will also include analytical techniques to complement the graphical techniques that are the primary focus of this paper.

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