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THE MULTIDISCIPLINARY COMBINATORIAL APPROACH AND ITS APPLICATIONS IN ENGINEERING

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ABSTRACT.

The current paper describes the Multidisciplinary Combinatorial Approach (MCA), the idea of which is to develop discrete mathematical representations, called "Combinatorial Representations" (CR) and to represent with them various engineering systems. During the research, the properties of each representation and the connections between them were investigated thoroughly, after that they were associated with various engineering systems in order to solve related engineering problems. The CR developed up until now are based on graph theory, matroid theory and discrete linear programming. The approach opens up new ways of working with representations, reasoning and design, some of which are reported in the paper, as follows:

- Multidisciplinary representation – systems which contain elements from different fields are represented by the same CR. Consequently, a uniform analysis process is performed on the representation, and thus on the whole system, without concerning about the specific disciplines the elements belong to.
- Deriving known methods and theorems – new proofs to known methods and theorems are derived in a new way, this time on the basis of the combinatorial theorems embedded in CR. This enables to develop a meta representation of engineering as a whole, through which the engineering reasoning will become convenient. In the current paper this issue is illustrated on structural analysis.
- Deriving novel connections between remote fields – new connections are derived on the basis of the relations between different combinatorial representations. An innovative connection between mechanisms and trusses, shown in the paper, has been derived on the basis of the mutual dualism between their corresponding CR. This new connection alone has opened new avenues of research, since knowledge and algorithms from machine theory are now available for usage in structural analysis and vice versa. Furthermore, it opened opportunities for developing new design methods, in which structures with special properties are developed based on known mechanisms with special properties and vice versa, as demonstrated in the paper.

1 INTRODUCTION.

This paper presents an overview on a general approach, called Multidisciplinary Combinatorial Approach (MCA). During the research conducted by this approach, first the representations based on discrete mathematics, called Combinatorial Representations (CR) were developed. At this stage, the properties, theorems and methods embedded in each of the CR were investigated thoroughly and the connections between individual CR were established. At the next stage, the CR have been applied to represent engineering systems and then to solve engineering problems from different fields.
From the results already achieved, it appears that the approach contributes to both practical and theoretical wings of engineering. In the current paper, few of these results are provided, while preserving its main objective to give a comprehensive perspective on MCA as a whole. This objective is achieved through 6 sections, each presenting a different aspect of the approach, as follows:

Section 2 starts with providing the mathematical foundation to the main two types of CR used in MCA : graph and matroid theories. The latter is an advanced topic in discrete mathematics that is not familiar to the engineering community and therefore graph theory terminology was employed in its explanation in order to make it as apprehensible as possible. In the current paper four graph representations are introduced, which are: Flow Graph Representation (FGR) which is applied to represent static systems; Potential Graph Representation (PGR) employed to represent mechanisms; Resistance Graph Representation (RGR) with its two embedded methods, employed to represent electrical, dynamical, hydraulic systems and indeterminate trusses; and the Line Graph Representation (LGR) that represents planetary gear systems. One matroid representation called Resistance Matroid Representation (RMR) is introduced, and is shown to be a generalization of the RGR.

Section 3 introduces one of the contributions of MCA, which is deriving new connections between remote engineering fields. In this section, based on the duality connection between FGR and PGR a novel connection between trusses and mechanisms is derived. This innovation opens up new avenues in research and practical applications, some being reviewed in the following sections.

Section 4 gives a brief introduction on the contribution of MCA to the theoretical research. It postulates, that the theorems embedded in the CR can be considered to be metatheorems, from which known theorems and methods in engineering can be derived. This issue is demonstrated by proving that known theorems and methods in structural mechanics are derived from theorem embedded in RGR, called: Tellegen's theorem. This enables a new way of research in which novel theorems and methods will be developed on the basis of the knowledge embedded in the CR.

Section 5 highlights the contribution of MCA to dealing with multidisciplinary systems. It is based on the fact that different engineering fields can be represented by the same combinatorial representation, in this case RGR. This opens up a possibility of applying a unified method to deal with the elements from different fields assembling one multidisciplinary system. This section presents an example of a system composed of elements from dynamics, statics and electricity that influence each other. The graph representation of that system, on the other hand, does not distinguish between those different types of elements.

Section 6 introduces a further application of MCA that enables to check the validity of the engineering problem before applying the analysis methods to solve it. One can notice the similarity to the first representations used in AI - logic representation , where the logic formulas should satisfy syntax rules to decide whether they are well-formed formulas or not (Genesereth and Nilsson 1987). MCA deals differently with this issue: its checking rules are based on the knowledge embedded in the CR. Demonstration of this ability is done in section 6.5, where a problem of checking the rigidity of a truss which was found to be difficult even for experts, is easily solved using MCA. This issue enhances Simon's postulate: "solving a problem simply means representing it so as to make the solution **transparent**. If the problem solving could actually be organized in these terms, the issue of representation would indeed become central" (Simon 1981).

Section 7 introduces a possibility of developing new design techniques by using properties of MCA. Here, the idea for the design is derived from knowledge and ready designs from other fields. This idea is carried out in this section by using the new connection between mechanisms and trusses introduced in section 3

The use of graph theory in engineering and AI is widely accepted and many related works were reported in the literature, some of which are listed below. In structural analysis, the first work was done by Kron (Kron 1963), who used the analogy between electrical networks and elastic structures. Fenves was the first to develop a software program "STRESS" (Fenves, Logcher and Mauc 1965) which used a method based on graphs and networks for the formulation of the structural problems. Structure analysis and optimization using graph theory has been performed by Kaveh (1991,1997). In machine theory, the first study of graph theory as a representation of mechanisms was conducted by Freudenstein and Dobrjanskyj (1964) . In dynamics, Andrews (1971) associated vector algebra with graph theory, and called it the "vector-network model". Computer programs based on this formulation have been reported, for example: VECENT (Andrews and Kesavan 1975). An approach, which uses graph theory in a more general perspective, was published by Bjorke (Wang and Bjorke 1989). Bjorke found out that network theory is probably the best foundation to establishing of a unified theory to represent a manufacturing system.

In like manner, many works published in Artificial Intelligence used graphs for knowledge representation. One of the first applications was to represent the state-space by graphs, in which vertices corresponded to states, and edges to the operators causing the change of the states (Nilsson 1971). Additionally, other special graphs, such as the conceptual graphs (Sowa 1984) and others.

The usage of matroid theory to represent engineering systems is less known in the literature. In structural mechanics, the known works are due to Kaveh who used matroid theory to represent structures (Kaveh 1997). From a more theoretical side Rescki used matroids to represent electrical networks and to check the rigidity of trusses (Recski 1989). An extensive list of matroid theory applications can be found in (Iri 1983).

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The approach adopted in this paper is different from the works reported above. In this approach, first the research focused entirely on developing the Combinatorial Representations and investigating their properties and interrelations. Only then, the CR were applied to represent engineering systems and solve the related engineering problems. This conception provided general engineering knowledge representations, which enabled to obtain the results reported in this paper.

2 COMBINATORIAL REPRESENTATIONS.

Combinatorial Representations (CR) are special representations based on discrete mathematics used in MCA to represent various engineering systems. Combinatorial representations are based on graph theory, matroid theory and discrete linear programming. Table 1 lists combinatorial representations used in the paper and the engineering systems to which they are applied.

The Combinatorial Representation	Have been applied to	Section in the current paper
Flow Graph Representation (FGR)	determinate trusses	2.2
Potential Graph Representation (PGR)	mechanisms, planetary gear systems.	2.3
Resistance Graph Representation (RGR)	mass-spring-damper dynamic systems, electric circuits, hydraulic systems, multidimensional indeterminate trusses	2.4
Resistance Matroid Representation (RMR)	indeterminate trusses	2.5
Line Graph Representation (LGR)	planetary gear systems.	2.6

Table 1: Combinatorial representations, their applications and the corresponding sections in current paper.

2.1 Mathematical foundation of the Combinatorial Representations.

Current section gives a brief introduction to the mathematicaltopics, on which the combinatorial representations developed in this paper are based. These mathematical topics are: network graphs and matroid theory. Network graphs are used in four graph representations: Flow Graph (FGR), Potential Graph (PGR), Resistance Graph (RGR) and Line-Graph (LGR) representations. The matroid theory is used in the Resistance Matroid Representation (RMR).

2.1.1 NETWORK GRAPHS.

The current section provides the reader with a brief survey on the graph theory terminology. More details can be found in (Shai 1997, Shai and Preiss 1999b) or books on graph theory, such as (Swamy and Thulasiraman 1981).

A graph is defined by the ordered pair G=<V,E>, where V is the vertex set and E is the edge set, and every edge is defined by its two end vertices. If each edge in the graph is assigned a direction, then the graph is known as a directed graph. The directed graph is a network graph, if each edge and vertex has properties of flow and potential, respectively.

For convenience, this paper uses a linetype attributes, which are:

a solid line – represents an edge with unknown value of flow or potential difference.

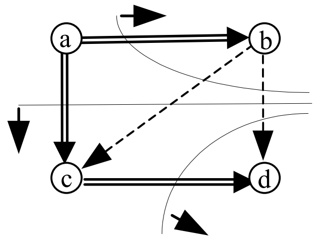
a bold line – represents an edge for which the flow or potential difference is known.

a dashed line – represents a chord, which is an edge not included in the spanning tree. If the value of flow in the chord is known, then it is both dashed and bold.

a double line – represents a branch of a spanning tree.

In order to deal with the graph representations used in the paper, one should first define cutset and circuit matrices in their vector and scalar forms. Given a connected network graph, choosing a spanning tree within it, defines its branches and chords.

A *cutset* in a connected graph is a minimal set of edges whose removal results in a disconnected graph. It can be proved (Swamy and Thulasiraman, 1981) that a cutset separates the graph into two components (maximal connected subgraphs). When a cutset includes only one branch of the spanning tree it is called a “fundamental cutset”. This paper deals only with fundamental cutsets, hence for brevity they will be called cutsets. Each cutset is defined by the corresponding branch and is labeled with its branch index. The direction of the cutset is defined by the direction of its branch , as shown in Figure 1a ,



(a)

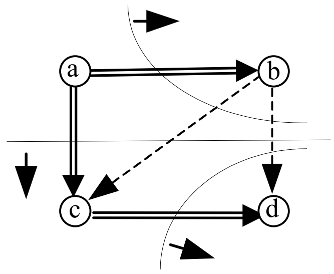
$$\vec{\mathbf{Q}} = \begin{array}{c} \\ ab \\ ac \\ cd \end{array} \begin{array}{c} \begin{array}{ccccc} ab & ac & cd & bc & bd \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

(b)

Figure 1. Example of a vector cutset matrix (a) The cutsets of a graph (b) The vector cutset matrix.

The *vector cutset matrix* $\vec{\mathbf{Q}}$ is a matrix that describes all the cutsets, but contains only topological information. The matrix has e(G) columns (corresponding to the edges of the graph) and v(G)-1 rows (corresponding to the cutsets or branches that define them). The value of the matrix element $[\vec{\mathbf{Q}}]_{ij}$ may be +1, 0, or -1. It is +1 if edge 'j' is included in the cutset that is defined by branch 'i' and has the same orientation as the cutset, -1 if it has the opposite orientation, and 0 if it is not included in the cutset. The vector cutset matrix of the graph of Figure 1a. is shown in Figure 1b.

The *scalar cutset matrix* **Q** is obtained from the vector cutset matrix $\vec{\mathbf{Q}}$ by multiplying each column with a unit vector in the direction of the edge to which it corresponds. For example, the scalar cutset matrix of the graph of Figure 2a is given in Figure 2b.



(a)

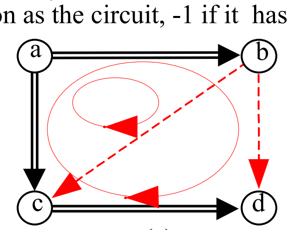
$$\mathbf{Q} = \begin{array}{c} \\ ab \\ ac \\ cd \end{array} \begin{array}{c} \begin{array}{ccccc} ab & ac & cd & bc & bd \end{array} \\ \begin{pmatrix} \hat{r}(ab) & 0 & 0 & -\hat{r}(bc) & -\hat{r}(bd) \\ 0 & \hat{r}(ac) & 0 & \hat{r}(bc) & \hat{r}(bd) \\ 0 & 0 & \hat{r}(cd) & 0 & \hat{r}(bd) \end{pmatrix} \end{array}$$

(b)

Figure 2. Example of scalar cutset matrix (a) The cutsets of the graph. (b) The scalar cutset matrix.

A *circuit* is a set of edges that form a closed path. A circuit is called a fundamental circuit if it includes exactly one chord. This paper deals only with fundamental circuits, and for brevity they will be called circuits. Each circuit will be labeled with the name of the chord that defines it. The direction of the circuit is defined by the direction of its chord, as shown in Figure 3a.

The *vector circuit matrix* $\vec{\mathbf{B}}$, demonstrated in Figure 3b, has e(G) columns as for the vector cutset matrix and e(G)-v(G)+1 rows corresponding to the circuits. Each circuit is defined by a chord, therefore the number of rows is equal to the number of chords defined by the spanning tree. The element $[\vec{\mathbf{B}}]_{ij}$= +1 if edge 'j' is included in the circuit defined by chord 'i', and has the same orientation as the circuit, -1 if it has opposite orientation, 0 otherwise.



(a)

$$\vec{\mathbf{B}} = \begin{array}{c} \\ bc \\ bd \end{array} \begin{array}{c} \begin{array}{ccccc} ab & ac & cd & bc & bd \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

(b)

Figure 3. Example of a vector circuit matrix. (a) The circuits of a graph. (b) The corresponding vector circuit matrix.

Every edge is assigned a vector called the *flow* and designated by $\vec{F}(e)$ and can correspond to a force, flow of liquid, money, goods or the like[1].

Every vertex is assigned a value called the *potential*[2] and designated by $\vec{\pi}(v)$. The potential may represent a physical quantity such as displacement, pressure or voltage, but it can also be used for other attributes. For instance in the shortest path algorithm it represents the lower bound of the distance (or the combined weights of the edges) from the current vertex to the target vertex (Shai 1997). The potentials of the vertices of edge $e=<v_1,v_2>$ define the potential difference of the edge, as follows:

$$\vec{\Delta}(e) = \vec{\pi}(v_2) - \vec{\pi}(v_1) \qquad (1)$$

2.1.2 BASICS OF MATROID THEORY.

Matroid is a discrete mathematical representation that possesses various important properties while the main one is its generality: among other things, it can be considered to be a generalization of graph theory. To simplify the explanation, matroid theory is introduced in this paper using terminology of graph theory, although some of the cases covered are those, in which matroid theory can be used to represent engineering systems while graph theory cannot. Matroid representation is used in MCA to represent various engineering systems, whereas in this paper it is used to represent and analyze indeterminate trusses.

Definition: If we denote S to be a finite set and **F** to be a collection of certain subsets of S, then the pair M=<S,**F**> is called a matroid if the following properties are satisfied:

1) $\varnothing \in F$
2) If $X \in F$ and $Y \subseteq X$ then $Y \in F$ must also hold.
3) If $X \in F$ and $Y \in F$ and $|X| > |Y|$ then there exists an element $x \in X - Y$, so that $Y \cup \{x\} \in F$

S is said to be *the underlying set of matroid M.* The subsets of S which belong to **F** are said to be *independent subsets,* otherwise they are called *dependent* subsets.

Maximal independent sets of M, i.e. independent sets which are not contained in any other independent set of M are called *bases of M*. For every base of M there is a corresponding *cobase* which is the complement of the base to S. It can be proved (Recski 1989) that the sizes of all the bases of a matroid are equal. In graph theory terminology a base is a spanning tree over the matroid.Thus, every matroid can be described by the collection of all its bases **T**, instead of the collection of all its independent sets **F**.

Minimal dependent sets of M, i.e. dependent sets which do not contain other dependent sets are called *circuits of M*. The collection of all the circuits of M is denoted by **C**. It also can be used to describe the matroid instead of **F**.

Definition of a matroid cutset: The subset $X \subseteq S$ is called a *cutset of M* if and only if it satisfies the following conditions:

a) $X \neq \varnothing$
b) $|X \cap Y| \neq 1$ for every $Y \in C$
c) X is minimal with respect to these properties.

Since a base is a maximal possible set of independent elements, adding an additional element to the base turns it into a dependent set, i.e. a set that contains a circuit. Therefore every cobase element defines exactly one circuit which contains the element itself and all the other elements are from the base only. Such a circuit is called a fundamental circuit.

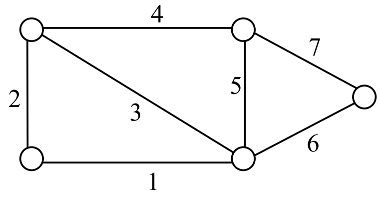
It can also be shown that every base element defines a unique cutset that contains the element itself and all the other elements are cobase elements. Such a cutset is called *fundamental cutset.*

2.1.2.1 Representing graph as a matroid.

Consider the graph of **Figure 4**a and its corresponding vector cutset matrix in **Figure 4**b.

1 In control theory, this is called the “through variable”, but the word “flow” is more suitable for the work reported here.

2 The potential difference between the vertices defining an edge is known in control theory as the “across variable”.



(a)

$$\vec{\mathbf{Q}} = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 4 & 6 & 3 & 5 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(b)

Figure 4. Graph to be represented by matroid.

One can define a matroid associated with this matrix as follows: $M_Q = \langle S_Q, F_Q \rangle$, where the underlying set S_Q is equal to the set of columns of $\vec{\mathbf{Q}}$ and the family of independent set $\mathbf{F}_Q$ is the collection of all sets of columns which are linearly independent. From this definition, every set of graph edges that corresponds to a minimal dependent set of columns forms a circuit in the graph. By matroid properties such a set is also a circuit in the matroid. Thus, the circuits in M_Q completely correspond to the circuits in the graph. Moreover, this claim is true for the bases of M_Q and spanning trees of the graph, cutset in M_Q and cutset in the graph, etc.

One can see for example, that columns 1,2 and 3 of $\vec{\mathbf{Q}}$ are linearly dependent, whereas edges 1,2 and 3 form a circuit in the graph. On the other hand, columns 1,2,4 and 6 of the matrix form maximum possible idependent set of columns, i.e. the base of M_Q, whereas edges 1,2,4 and 6 in the graph form a spanning tree.

2.1.2.2 Representing matrix as a matroid.

Previously, it was explained that every vector cutset matrix of a graph can be considered as a matroid and that such a matroid actually represents the graph. In the current section, this issue is expanded and it is shown that every matrix corresponds to a matroid. This time it can occur that this matroid does not have a corresponding graph.

Let Q be a m x n matrix. The matroid $M_Q=\{S_Q,\mathbf{F}_Q\}$ can be defined as follows:

1) The underlying set S_Q is the set of n column vectors of Q.
2) Every subset of linearly independent columns of Q belongs to $\mathbf{F}_Q$.

Consider for example matrix of Figure 5

$$Q = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

Figure 5. Matrix to be represented by the matroid.

The underlying set S_Q of the matroid representing the matrix of Figure 5 is:

$$S_Q = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

Some of the independent subsets of FQ are:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

the first two of which are also the bases of MQ, since any additional column from SQ will cause a linear dependence.

And some of the circuits (elements of CQ) are:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

2.2 The Flow Graph Representation (FGR).

Definition of the Flow Graph Representation (FGR): A network graph G is a flow graph, designated by G_F, if the flows in the edges are independent of the potential differences and satisfy the Flow Law, stated as follows: the vector sum of the flows in every cutset of G is equal to zero.

The flow law may be recognized as a generalization of the well-known Kirchhoff's Current Law (KCL). Note that KCL is restricted only to one dimension which is appropriate for electrical circuits, while the flow law can be multidimensional, thus it can be used also for structures and other engineering systems, that require two or three dimensions.

The matrix form of the Flow Law is:

$$\vec{\mathbf{Q}} \cdot \vec{\mathbf{F}} = \mathbf{0} \qquad (2)$$

where $\vec{\mathbf{F}}$ is the vector of the flows, or *Flow Vector*.

The FGR can be used to represent various engineering systems, such as: simple electrical circuits, mass-cable systems in force equilibrium etc.

The important property of the flow graph is that it should not contain cutsets consisting entirely of the flow sources. For if such cutsets of sources exist, then, by the flow law, there is a linear dependence between the flows in these sources, violating the definition of the flow sources. To assure that it won't happen, the spanning tree of the flow graph should be chosen in such a way that it does not include flow sources.

2.2.1 USING FLOW GRAPH REPRESENTATION FOR MASS-CABLE SYSTEMS.

Figure 6 shows an example of a system of masses connected by cables which is known to be in static equilibrium. The objective it to find the gravitation force acting on the mass B.

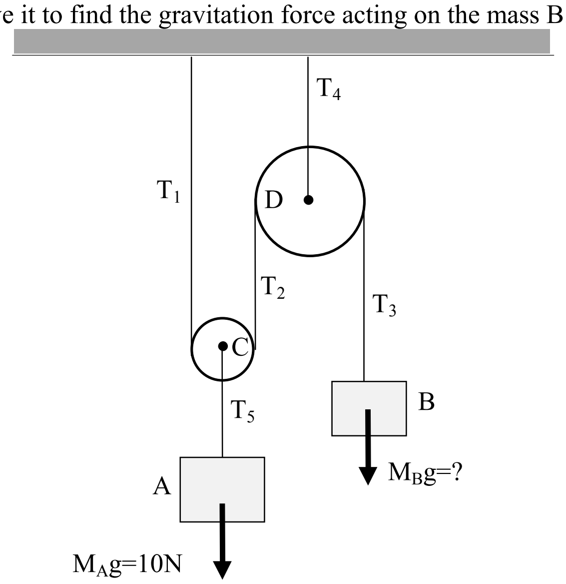


Figure 6. System of cables and masses in static equilibrium.

Since the system is in static equilibrium, it is obvious, that it should be solved using statical equations relating the forces acting in the system. The most convenient representation for this purpose is FGR.

Each edge in FGR corresponds to an acting force, no matter whether it is tension in the cable or an external force. Each vertex will correspond to a point or a body on which number of forces is acting, like mass or a pulley.

The flow graph representing the system of Figure 6 is shown on Figure 7.

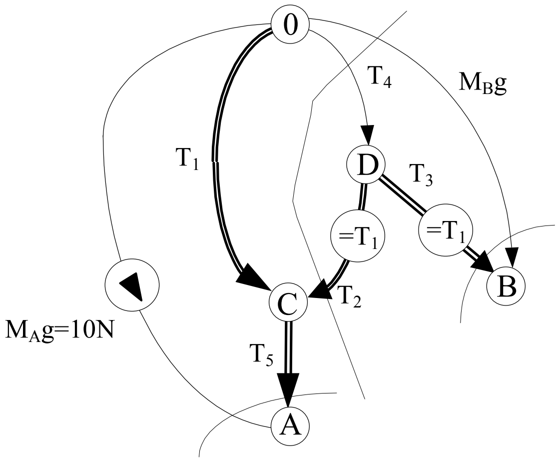


Figure 7. FGR representing the system of Figure 7.

The analysis equations for this graph are as follows:

$$\vec{\mathbf{Q}}\cdot\vec{\mathbf{F}}=\mathbf{0}\Rightarrow\begin{pmatrix}1&0&0&0&1&1&1\\0&1&0&0&-1&0&-1\\0&0&1&0&0&0&1\\0&0&0&1&0&1&0\end{pmatrix}\cdot\begin{pmatrix}T_1\\T_2\\T_3\\T_5\\T_4\\M_Ag\\M_Bg\end{pmatrix}=0,\ T_2=T_3=T_1\Rightarrow\begin{pmatrix}1&0&1&1\\1&0&-1&-1\\1&0&0&1\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}T_1\\T_5\\T_4\\M_Bg\end{pmatrix}=-\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot M_Ag$$

Note, that since the engineering system of Figure 6 is one-dimensional, its vector and scalar cutset matrices are identical.

2.2.2 EMPLOYING FLOW GRAPH REPRESENTATION (FGR) IN REPRESENTING DETERMINATE TRUSSES.

The conventional procedure used to analyze determinate trusses is based on building the force equilibrium equation for each joint of the truss and for each coordinate axis. This strongly correlates with the flow law, according to which, sum of the flows at each vertex of the graph is equal to zero. Accordingly, one can represent a determinate truss by d(G) one-dimensional FGRs, each corresponding to a different coordinate axis. For example, a plane determinate truss of Figure 8a is represented by two flow graphs – one corresponding to the X coordinate (Figure 8b) and the other to Y coordinate (Figure 8c).

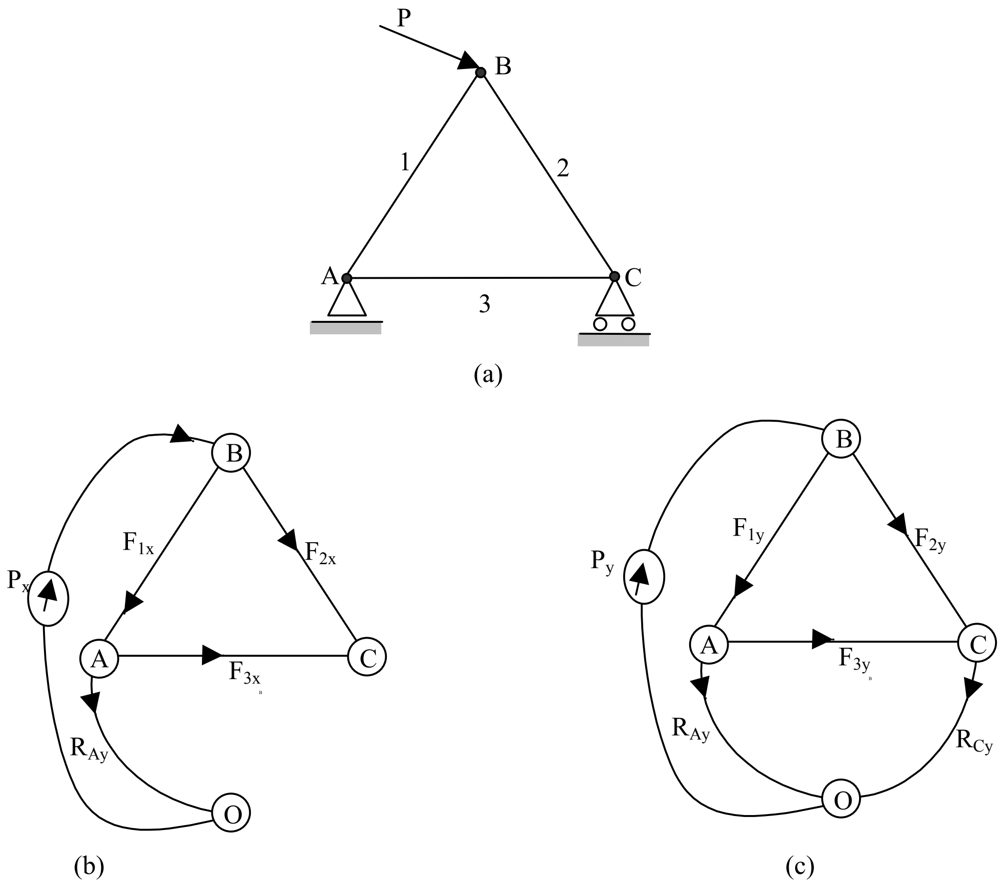


Figure 8. Plane determinate truss and two one-dimensional FGRs.
(a) Plane determinate truss (b) (c)FGR corresponding to the X (Y) coordinate

The graphs presented on Figure 8b and Figure 8c are of the same topology as the represented truss. One can think of FGR, as if the flow comes out from vertex O through the flow source (external force), flows through the edges (rods) and returns back to the vertex O through the reaction edges. Vertex O is called the "*reference vertex"* since it assures that the sum of all the external forces acting on the truss is also equal to zero. The reference vertex can be considered a generalization of the "earth" in electrical circuits.

One can see, that if the flows in the edges of the corresponding truss are equated to the correct forces in the truss, they would be legal, i.e. would satisfy the flow law.

In order to perform the statical analysis of the truss there is a need to use the angles of the rods and external forces. The latter knowledge affects the ratio between the flows in the two FGR's corresponding to the X and Y coordinates. The most efficient way to store such information is by merging the two one-dimensional graphs into one two-dimensional and setting the angle of some of the flows to be constant. In other words, instead of representing the truss by two FGR's with one dimensional flow in each, one FGR with multidimensional flow is used, as explained below.

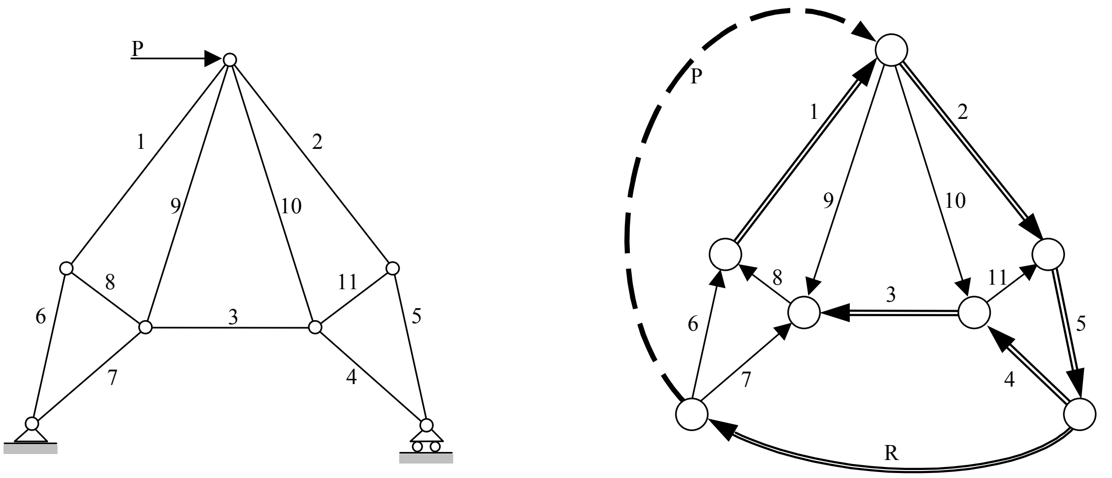
The steps for representing the truss by a multi-dimensional flow graph are:

1. Create a vertex in the graph for every pinned joint in the truss.
2. For every rod create an edge in the graph, called a "truss edge"; its end vertices correspond to the joints that connect the corresponding rod to the truss. Assign an arbitrary orientation to each truss edge and a unit vector $\hat{r}(e)$ directed from the tail joint to the head joint. The engineering meaning of the flow in the edge corresponding to a rod is the force applied on the head vertex (joint) by the rod in the direction of the unit vector $\hat{r}(e)$, which is, of course, equal to the force that the tail vertex (joint) applies on the rod. One can see, that if the flow in the edge is positive, then the rod is in the state of compression, otherwise it is in a state of tension, as is explained in detail in (Shai 2000a).
3. Create a reference vertex or choose one of the vertices corresponding to the joint connected to the hinged support reactions to be the reference vertex of the graph.
4. For each external force and reaction, add an edge as follows:

For each external applied force a "***flow source edge***" is added. Its tail vertex is the reference vertex and the head vertex is the vertex corresponding to the joint upon which the external force acts. As explained earlier , these edges should always be chosen to be chords. Since flow source edges are chords, the flows for which are known, they appear in the graph as bold dashed lines.

For each mobile support reaction a "***reaction edge***" is added. Its tail vertex is the vertex corresponding to the joint upon which the reaction acts and the head vertex is the reference vertex. The reaction edge is assigned an angle equal to the inclination angle of the plane. For each hinged support (except for the one corresponding to the reference vertex), two "reaction edges" are added, the first having the corresponding angle equal to 180^0 and second 270^o.

In statically determinate trusses the sum of forces at every joint is equal to zero. In the terminology of the flow graph representation, this means that in the graph corresponding to the truss, the flow law is satisfied. Thus the force analysis process of the truss is transformed into searching for flows that satisfy the flow law in the corresponding graph.
Example of a truss, its corresponding graph and the equations written according to (2) is given in Figure 9.



(a) (b)

$$\begin{array}{cc} & \begin{array}{cccccccccccc} R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{array}{c} R_x \\ R_y \\ 1_x \\ 1_y \\ 2_x \\ 2_y \\ 3_x \\ 3_y \\ 4_x \\ 4_y \\ 5_x \\ 5_y \end{array} & \begin{pmatrix} \text{Cos}(270) & 0 & 0 & 0 & 0 & 0 & -\text{Cos}(77) & -\text{Cos}(45) & 0 & 0 & 0 & 0 \\ \text{Sin}(270) & 0 & 0 & 0 & 0 & 0 & -\text{Sin}(77) & -\text{Sin}(45) & 0 & 0 & 0 & 0 \\ 0 & \text{Cos}(53) & 0 & 0 & 0 & 0 & -\text{Cos}(77) & 0 & -\text{Cos}(150) & 0 & 0 & 0 \\ 0 & \text{Sin}(53) & 0 & 0 & 0 & 0 & -\text{Sin}(77) & 0 & -\text{Sin}(150) & 0 & 0 & 0 \\ 0 & 0 & \text{Cos}(233) & 0 & 0 & 0 & -\text{Cos}(77) & 0 & -\text{Cos}(150) & \text{Cos}(250) & \text{Cos}(340) & 0 \\ 0 & 0 & \text{Sin}(233) & 0 & 0 & 0 & -\text{Sin}(77) & 0 & -\text{Sin}(150) & \text{Sin}(250) & \text{Sin}(340) & 0 \\ 0 & 0 & 0 & \text{Cos}(180) & 0 & 0 & 0 & \text{Cos}(45) & -\text{Cos}(150) & \text{Cos}(250) & 0 & 0 \\ 0 & 0 & 0 & \text{Sin}(180) & 0 & 0 & 0 & \text{Sin}(45) & -\text{Sin}(150) & \text{Sin}(250) & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{Cos}(135) & 0 & 0 & \text{Cos}(45) & -\text{Cos}(150) & \text{Cos}(250) & \text{Cos}(340) & -\text{Cos}(40) \\ 0 & 0 & 0 & 0 & \text{Sin}(135) & 0 & 0 & \text{Sin}(45) & -\text{Sin}(150) & \text{Sin}(250) & \text{Sin}(340) & -\text{Sin}(40) \\ 0 & 0 & 0 & 0 & 0 & \text{Cos}(257) & -\text{Cos}(77) & 0 & -\text{Cos}(150) & \text{Cos}(250) & \text{Cos}(340) & -\text{Cos}(40) \\ 0 & 0 & 0 & 0 & 0 & \text{Sin}(257) & -\text{Sin}(77) & 0 & -\text{Sin}(150) & \text{Sin}(250) & \text{Sin}(340) & -\text{Sin}(40) \end{pmatrix} \end{array} \bullet \begin{pmatrix} F_R \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \end{pmatrix} = \begin{pmatrix} \text{Cos}(0) \\ \text{Sin}(0) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \bullet P$$

(c)

Figure 9. Example for analysis of determinate truss using the flow graph representation. (a) Statically determinate plane truss. (b) Corresponding flow graph. (c) Force analysis equations in the scalar cutset matrix form.

2.3 The Potential Graph Representation (PGR).

Definition of a Potential Graph Representation (PGR)- Let G_Δ be a network graph the potential difference on the edges of which is independent onthe value of the flow in that edge. In addition, every circuit satisfies the Potential Law, which states:

The Potential Law - For every circuit in the graph, the sum of the potential differences of the circuit edges of the circuit is equal to zero. In matrix representation this is written:

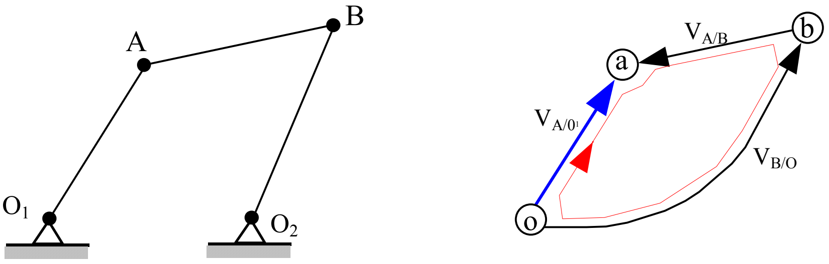
$$\vec{\mathbf{B}} \cdot \vec{\Delta} = \mathbf{0} \qquad (3)$$

where $\vec{\Delta}$ is the vector of potential differences, or *Potential Difference Vector*.

Proof of the potential law: In the summation of potential differences of all circuit edges, each potential appears twice - once for the head vertex and once for the tail vertex, with opposite signs. Therefore this summation is equal to zero.
This law is a vectorial generalization to several dimensions of KVL (Kirchhoff's Voltage Law) which is stated for a one dimensional or scalar system.
The important property of PGR is that there can be no circuits consisting of only potential difference sources. For if such circuits of sources exist, then, by the potential law, there is a linear dependence between the potential differences in these sources, thus violating the definition of potential difference sources. To assure that it won't happen, the spanning tree of PGR should be chosen so that it includes the potential difference sources.

2.3.1 THE POTENTIAL GRAPH REPRESENTATION OF A MECHANISM.

The main property of mechanisms is that the vector sum of the kinematic relative velocities is equal to zero in every circuit formed by its links. This property suffices for the analysis, so it is reasonable to represent it as a PGR. In this representation, the potential difference of the edges will correspond to the relative velocities of the links in the mechanism. Note that this is different from the graph representation that is commonly used for mechanisms
The steps for representing a mechanism by PGR:
1. For every joint of the mechanism having individual velocity create a corresponding vertex in the graph. The potential of vertex 'i', designated $\pi(i)$, is equal to the linear velocity of the corresponding joint. The velocity of all the fixed joints in the mechanism is zero, thus all these joints are represented by the same vertex.
2. For every link of the mechanism create a corresponding edge in the graph; its end vertices correspond to the joints that connect the link to the mechanism. The potential difference of this edge, designated $\vec{\Delta}(e)$ is equal to the relative velocity of the corresponding link, and can be written: $\vec{\Delta}(e) = \vec{V}(e) = V(e) \cdot \hat{v}(e)$, where V(e) is the magnitude of the relative linear velocity and $\hat{v}(e)$ is a unit vector in the direction of the relative linear velocity of the link.
3. Label the edge corresponding to the driving link with a bold line since its potential difference (corresponding to the relative linear velocity between its end vertices) is known. This edge is called the "source edge".
4. The relative velocity of a link is the velocity of the head vertex minus the velocity of the tail vertex.
The property for analyzing the velocities in mechanisms is that in each circuit of links, the sum of the relative velocities of each link is equal to zero. Since the relative velocity of a link is represented by the potential difference of the corresponding edge, this property is the implementation of the Potential Law.
Figure 10 shows an example of a mechanism and its corresponding PGR.



(a) (b)

Figure 10. Example of representing a mechanism by Potential Graph Representation. (a) The mechanism. (b) The PGR.

The Analysis Algorithm

The algorithm is based on the principle that every fundamental circuit must satisfy the potential law. Every circuit is defined by a chord, as is described in the following steps:
Step 1- Find a spanning tree, and label each branch with a double line. Every chord defines a circuit.
Step 2 - For this spanning tree, write the vector circuit matrix $\vec{\mathbf{B}}$, as defined in section 2.1.1.

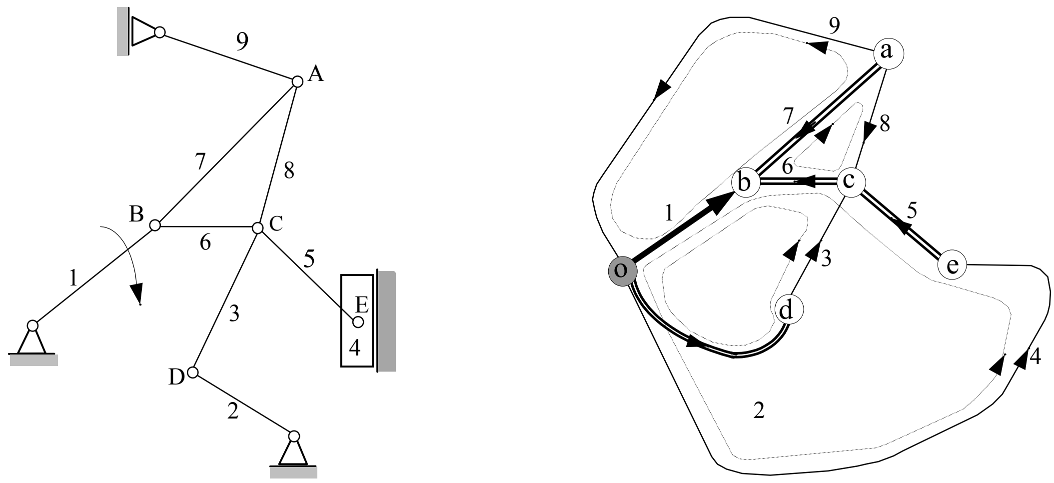
Step 3 – On base of equation (3), write the equations:

$$(\vec{\mathbf{B}}\ \vec{\mathbf{B}}_\Delta)\cdot\begin{pmatrix}\vec{\Delta}\\ \vec{\Delta}_\Delta\end{pmatrix}=\mathbf{0}\ \rightarrow\ \vec{\mathbf{B}}\cdot\vec{\Delta}\ =\ -\vec{\mathbf{B}}_\Delta\cdot\vec{\Delta}_\Delta \qquad (4)$$

where $\vec{\mathbf{B}}_\Delta$ is the part of the vector circuit matrix corresponding to the potential difference sources and $\vec{\Delta}_\Delta$ is the vector of potential differences in the potential difference sources (driving links).

Step 4 - Solve the $2*(e(G)-(v(G)-dr(G)-1))$ equations obtained in step 3.

An example of using this algorithm is shown in Figure 11.



(a) (b)

$$\begin{array}{c} \begin{array}{cccccccc} 3 & 4 & 8 & 9 & 2 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} \cos\alpha_3 & 0 & 0 & 0 & \cos\alpha_2 & 0 & \cos\alpha_6 & 0 \\ \sin\alpha_3 & 0 & 0 & 0 & \sin\alpha_2 & 0 & \sin\alpha_6 & 0 \\ 0 & \cos\alpha_4 & 0 & 0 & 0 & \cos\alpha_5 & \cos\alpha_6 & 0 \\ 0 & \sin\alpha_4 & 0 & 0 & 0 & \sin\alpha_5 & \sin\alpha_6 & 0 \\ 0 & 0 & \cos\alpha_8 & 0 & 0 & 0 & \cos\alpha_6 & -\cos\alpha_7 \\ 0 & 0 & \sin\alpha_8 & 0 & 0 & 0 & \sin\alpha_6 & -\sin\alpha_7 \\ 0 & 0 & 0 & \cos\alpha_9 & 0 & 0 & 0 & -\cos\alpha_7 \\ 0 & 0 & 0 & \sin\alpha_9 & 0 & 0 & 0 & -\sin\alpha_7 \end{pmatrix} \end{array} \cdot \begin{pmatrix} V_3 \\ V_4 \\ V_8 \\ V_9 \\ V_2 \\ V_5 \\ V_6 \\ V_7 \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 \\ \sin\alpha_1 \\ \cos\alpha_1 \\ \sin\alpha_1 \\ 0 \\ 0 \\ -\cos\alpha_1 \\ -\sin\alpha_1 \end{pmatrix} \cdot V_1$$

(c)

Figure 11. Kinematic analysis of the mechanism using PGR.
(a) A mechanism. (b) the corresponding potential graph. (c) the set of equations for its analysis.

2.4 Resistance Graph Representation (RGR) .

2.4.1 DESCRIPTION OF THE REPRESENTATION.

The Resistance Graph Representation (RGR) is a generalization of FGR and PGR. RGR is a network graph, where there are edges with a dependence between the flow and the potential difference. Such a dependence is characterized by either a scalar or a matrix. The scalar is used, if there is an explicit dependence between the vector magnitudes of the flow and potential difference, otherwise the matrix is used. For both scalar and matrix possibilities there are two presentations – resistance presentation (designated by R(e) and **R**(e) respectively) and conductance presentation (designated by G and **G** respectively), as follows:

$$\left|\vec{\Delta}(e)\right| = R(e)\cdot\left|\vec{F}(e)\right|; \qquad \left|\vec{F}(e)\right| = G(e)\cdot\left|\vec{\Delta}(e)\right| \tag{5}$$

$$\vec{\Delta}(e) = \mathbf{R}(e)\cdot\vec{F}(e); \qquad \vec{F}(e) = \mathbf{G}(e)\cdot\vec{\Delta}(e) \tag{6}$$

where $\vec{\Delta}(e)$ is the potential difference in edge 'e' and $\vec{F}(e)$ is the flow.

Flows and potential differences of the resistance graph must satisfy the flow and potential laws respectively.

When dealing with resistance graph representation an important theorem from the graph theory, called the orthogonality principle becomes essential. The orthogonality principle states that vectorial cutset and circuit matrices are orthogonal:

$$\vec{\mathbf{B}}\cdot\vec{\mathbf{Q}}^{t} = \mathbf{0} \tag{7}$$

As it is shown in (Swamy and Thulasiraman, 1981), from this principle the following equations can be established:

$$\vec{\Delta} = \mathbf{Q}^{t}\cdot\vec{\Delta}_{T} \tag{8}$$

$$\vec{\mathbf{F}} = \mathbf{B}^{t}\cdot\vec{\mathbf{F}}_{C} \tag{9}$$

where $\vec{\Delta}_{T}$ is the vector of potential differences in the branches of the spanning tree 'T' and $\vec{\mathbf{F}}_{C}$ is the vector of flows in the chords of the graph.

The edges in the resistance graph are divided into three principal groups: flow sources, potential difference sources and resistance edges. Flow sources, denoted by bold dashed lines, are edges for which the value of the flow is known and is independent of the potential difference. Potential difference sources, denoted by bold solid lines, are the edges in which the potential difference is known and is independent of the flow in that edge. Resistance edges, denoted by black solid lines, are the edges at which there is a dependence between the flow and the potential difference in the edge.

2.4.2 CONDUCTANCE CUTSET METHOD (CCM) FOR ANALYZING THE RGR.

The analysis problem for the resistance graph is: given the flows in the flow sources, the potential differences in the potential difference sources and the resistances (or conductances) of the resistance edges, to find the flows and potential differences in all the e(G) edges of the graph.

The obvious method for solving the resistance graphs is to write all the equations based on equations 2, 3, 5 and 6 and then solve them simultaneously. This method has a high computational complexity, and in the next two sections efficient methods based on graph theory theorems will be shown, whereas the method called the 'Conductance Cutset Method' (CCM) will be explained first.

The first step in solving the resistance graph is to find a suitable spanning tree, which for the reasons explained in sections 2.2 and 2.3, contains all the potential difference sources and does not contain any flow sources.

Then using equation 10, derived from (2), (3), (5) and (6) in (Shai 1999), a set of linear equations is obtained:

$$(\vec{\mathbf{Q}}_{T'R}\cdot\mathbf{G}_{R}\cdot\vec{\mathbf{Q}}^{t}_{T'R})\cdot\vec{\Delta}_{T'} = -(\vec{\mathbf{Q}}_{T'R}\cdot\mathbf{G}_{R}\cdot\vec{\mathbf{Q}}^{t}_{\Delta R})\cdot\vec{\Delta}_{\Delta} - \vec{\mathbf{Q}}_{T'F}\cdot\vec{\mathbf{F}}_{P} \tag{10}$$

where Δ and P are the edges corresponding to the potential difference and flow sources respectively, and R are all the rest edges of the graph - the edges with resistance. T' are those branches of the spanning tree which are not sources.

For convenience, the matrix $(\vec{\mathbf{Q}}_{T'R}\cdot\mathbf{G}_{R}\cdot\vec{\mathbf{Q}}^{t}_{T'R})$ is designated as $\mathbf{G}_{T'}$ and is termed the "conductance matrix of the spanning tree T' ". Matrix $(\vec{\mathbf{Q}}_{T'R}\cdot\mathbf{G}_{R}\cdot\vec{\mathbf{Q}}^{t}_{\Delta R})$ is designated as $\mathbf{G}_{\Delta}$ and is called 'the conductance matrix of the potential sources'. These are shown in equation 11.

$$(\mathbf{G}_{T'})\cdot\vec{\Delta}_{T'} = -(\mathbf{G}_{\Delta})\cdot\vec{\Delta}_{\Delta} - \vec{\mathbf{Q}}\cdot\vec{\mathbf{F}}_{P} \tag{11}$$

The elements of the conductance matrices can be derived based on linear algebra considerations, as follows:

$[\mathbf{G}_{T'}]_{ij}$ is the sum of conductances of the edges which belong to both cutsets i and j which are defined by branches with resistance; the sign of the conductance is taken positive if it is directed similarly relative to both cutsets, negative otherwise.

$[\mathbf{G}_{\Delta}]_{ij}$ is also the sum of conductances of the edges that belong to both cutsets i and j but this time j is a branch which is potential difference source, while 'i' is as before a branch with resistance.
After solving equation 10 or 11, all the potential differences in the branches are known. All the potential differences in the graph are obtained by using equation 8 and after that all the flows in the graph are obtained by equation 5 or 6.

2.4.3 Resistance Circuit Method (RCM) for Solving the RGR.

It is well-known in the literature that each RGR (Balabanian 1961) has its dual graph. It properties are as those of the regular dual graph in graph theory, except for the fact that the resistance of the edge in the dual graph is equal to the conductance of the corresponding edge in the original graph. Thus one can develop a new method for analysis of resistance graph by just rewriting the CCM method in the terminology of the dual resistance graph. This method is called Resistance Circuit Method (RCM) method and its equation is (Shai 1999):

$$(\vec{\mathbf{B}}_{C'R} \cdot \mathbf{R}_R \cdot \vec{\mathbf{B}}^t_{C'R}) * \vec{\mathbf{F}}_{C'} = -(\vec{\mathbf{B}}_{C'R} \cdot \mathbf{R}_R \cdot \vec{\mathbf{B}}^t_{PR}) \cdot \vec{\mathbf{F}}_P - \vec{\mathbf{B}}_{C'P} \cdot \vec{\Delta}_\Delta \qquad (12)$$

where $\mathbf{R}_R$ is a square diagonal matrix the components of which correspond to the resistances in the resistance edges. For convenience, the matrix $(\vec{\mathbf{B}}_{C'R} \cdot \mathbf{R}_R \cdot \vec{\mathbf{B}}^t_{C'R})$ is designated as $\mathbf{R}_{C'}$ and is termed the resistance matrix of the set of chords C'. Matrix $(\vec{\mathbf{B}}_{C'R} \cdot \mathbf{R}_R \cdot \vec{\mathbf{B}}^t_{PR})$ is designated as $\mathbf{R}_P$ and is called the resistance matrix of the flow sources. Using those notations we rewrite equation 12 in the following way:

$$(\mathbf{R}_{C'}) \cdot \vec{\mathbf{F}}_{C'} = -(\mathbf{R}_P) \cdot \vec{\mathbf{F}}_P - \vec{\mathbf{B}}_{C'\Delta} \cdot \vec{\Delta}_\Delta \qquad (13)$$

The elements of the resistance matrices can be derived based on linear algebra considerations, and they are as follows:
$[\mathbf{R}_{C'}]_{ij}$ is the sum of the resistances of the edges which belong to both the circuits i and j, defined by the chords with resistance. The sign of the resistance is positive if the corresponding edge is directed similarly relative to both circuits, negative otherwise.
$[\mathbf{R}_P]_{ij}$ is calculated the same way, except that the circuit j is defined by the chord which is the flow source.
Equation 13 is actually the set of linear equations, the unknowns of which are the flows in the resistance chords of the graph. After solving it, all the flows in the graph are obtained by using equation 9 and after that all the potential differences in the graph are obtained by equation 5 or 6.
From the way the RCM equation has been developed, it follows that CCM and RCM are mutually dual methods. That means that CCM method applied to resistance graph G is equivalent to the RCM method applied to the resistance graph G^* which is dual graph to G.

2.4.4 Representing One-Dimensional Engineering Systems with Resistance Graph Representation (RGR).

The current section shows how to represent various engineering systems with the resistance graph. The fact that the same representation has been applied to represent systems that belong to remote engineering fields opens two far going possibilities: first to use methods developed in one field for the other field, and second – to solve combined multidisciplinary engineering systems. Both issues are described later in the paper.
The systems that are described in this section are: electrical systems, dynamic systems, statical systems and hydraulic systems. More detaild discussion on their representation can be found in (Shearer et al. 1971).
The first step in representing the engineering system by the resistance graph is to define the engineering interpretation of the graph edges and vertices and their flows and potentials (or potential differences) These interpretations of the graphs variables are presented in Table 2, whereas the interpretations of the vertices and edges are given in Table 3.

	Flow	Potential	Potential difference
Electrical circuit	Electrical current in the circuit element	The electric potential (voltage) in the circuit junction	The voltage on the circuit element.
Dynamical system	internal force in dynamic system elements	The velocity of junction	Relative velocity between the end junctions of dynamic element.
Hydraulic system	Fluid flow rate in the element.	Pressure at specific system junction.	Pressure drop in the element.
Statical system	force in the element	displacement of a joint.	relative displacement of the element end joints.

Table 2 Engineering interpretation of the resistance graph variables.

	Edge	Vertex	End vertices of the edge
Electrical circuit	Corresponds to electrical elements: resistor, condenser, coil, current or voltage source.	Junction in the circuit.	Vertices that correspond to the junctions connected to the element.
Dynamical system	Corresponds to dynamical elements: mass, dashpot, string, external force, initial tension or velocity.	Junction having independent velocity.	For strings, dashpots – vertices that connect the element to the system.
Hydraulic system	Corresponds to components of the hydraulic system: pipes or containers.	Junction having independent pressure	Junctions adjacent to the component.
Statical system	Correpsonds to a system element with an internal force.	Joint connecting system elements.	Joints connected to the element.

Table 3. Edges and vertices in the resistance graph.

Table 4 lists the different types of resistance edges corresponding to elements of engineering systems. For each such edge the terminal equation is given, i.e. the equation describing the relation between the flow and potential difference that edge.

Engineering system	Name of the element	Constant	Terminal equation
Electrical circuit	Resistor	resistance R_i	$\Delta_i = F_i R_i$
	Condenser	capacity C_i	$I_i = C_i \frac{d\Delta_i}{dt} = C_i s \Delta_i$
	Coil	inductance L_i	$\Delta_i = L_i \frac{dF_i}{dt} = sL_i F_i$
Dynamical system	Mass	mass m_i	$\Delta_i = \frac{F_i}{m_i s}$
	Spring	stiffness k_i	$\Delta_i = \frac{sF_i}{k_i}$
	Dashpot	damping b_i	$\Delta_i = \frac{F_i}{m_i s}$
Hydraulic system	Fluid resistance (narrowed pipe)	Fluid Resistance R	$\Delta_i = R_i \cdot F_i$
	Fluid capacitance (container)	Fluid capacitance C	$F_i = C_i \frac{d\Delta_i}{dt} = C_i s \Delta_i$
	Fluid internance	Fluid internance L	$\Delta_i = L_i \frac{dF_i}{dt} = sL_i F_i$
Statical system	Stiff rod/cable	Element stiffness: $G = \frac{AE}{L}$	$F_i = G \cdot \Delta_i$

Table 4 Terminal equations of graph edges.

2.4.4.1 Applying RGR to analysis of electrical systems.

The current section shows an example of how a one dimensional engineering system - an electrical circuit is represented by RGR and how it is analysed using the algorithms embedded in the representation. Figure 12 shows an example electrical DC circuit.

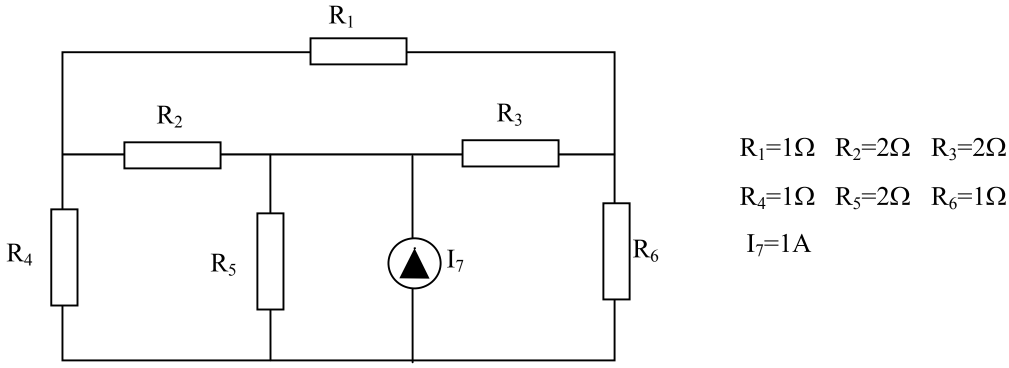


Figure 12. Example of an electrical system.

The resistance graph representing the system appears in Figure 13.

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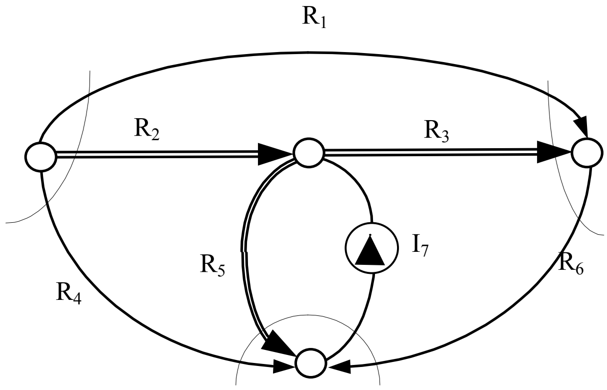


Figure 13. The RGR representing the electrical system of Figure 12.

The solution equations derived by applying the CCM to the graph of Figure 13 are as follows:

$$Q_{T'T'} = \begin{array}{c} \\ -2 \\ -3 \\ -5 \end{array}\begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_4} & \frac{1}{R_1} & \frac{1}{R_4} \\ \frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_6} & -\frac{1}{R_6} \\ \frac{1}{R_4} & -\frac{1}{R_6} & \frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_6} \end{pmatrix} \end{array} = \begin{array}{c} \\ 2 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{pmatrix} 2.5 & 1 & 1 \\ 1 & 2.5 & -1 \\ 1 & -1 & 2.5 \end{pmatrix} \end{array} \qquad Q_F = \begin{array}{c} \\ 2 \\ 3 \\ 5 \end{array}\begin{array}{c} 7 \\ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 2.5 & 1 & 1 \\ 1 & 2.5 & -1 \\ 1 & -1 & 2.5 \end{pmatrix} \cdot \begin{pmatrix} V_2 \\ V_3 \\ V_5 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot I_7$$

2.4.5 USING RESISTANCE GRAPH REPRESENTATION FOR ANALYSIS OF MULTIDIMENSIONAL TRUSSES.

Let d(G) be dimension of the engineering system, i.e., the number of dimensions of potential and flow vectors. The explanation provided is for two dimensions, but the approach is valid for three dimensions ,either.

Equation 6 can be rewritten :

$$\vec{\mathbf{F}} = \mathbf{G} \cdot \vec{\Delta}$$

where **G** is built from the conductivity matrices of the resistance edges, each being a square matrix of size d(G) x d(G)

Figure 14 shows the initial and deformed states of a rod:

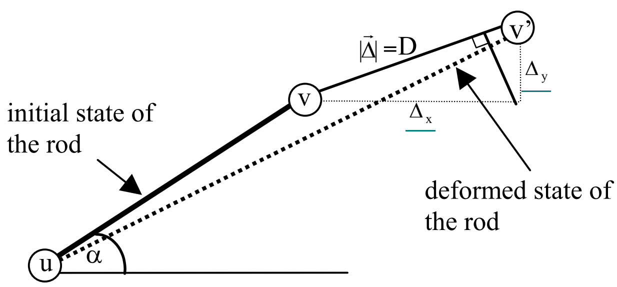


Figure 14. Rod deformation

Let $\vec{\Delta}_i(e)$ correspond to the potential difference between the two end vertices of edge 'e' in coordinate direction 'i'. As one can see from Figure 14, under the small deflection assumption, the following equation describes the scalar magnitude of the potential difference as a combination of its coordinate components:

$$|\vec{\Delta}(e)| = \Delta_x(e) \cdot \cos\alpha + \Delta_y(e) \cdot \sin\alpha \qquad (14)$$

where α is the angle of the element.

Combining (6) and (14) we obtain:

$$\vec{F}(e) = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = G(e) \cdot \begin{pmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{pmatrix} \cdot \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} = \mathbf{G(e)} \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix} \qquad (15)$$

where the square matrix **G(e)** is the 'conductance matrix' of the element.

This two dimensional conductance matrix of the graph edges (designated by **G(e))** is the product of the constant conductivity G(e) and the transformation matrix. For edges corresponding to hinged support reactions, the constant should be taken as 0, since there is no dependence between the displacement of the support and the reaction force, hence both conductance and resistance of the reaction edge are zero.

In indeterminate trusses the forces in the bars cannot be determined by the laws of statics alone, and one must also consider compatibility conditions. In the terminology of graph representation, this means that the corresponding graph of the indeterminate truss should be analyzed by using the flow and potential laws simultaneously.

The components of the potential vector correspond to the displacements of the joints in the directions of the coordinate axis. The flow in an edge corresponds to the internal force in the corresponding rod.

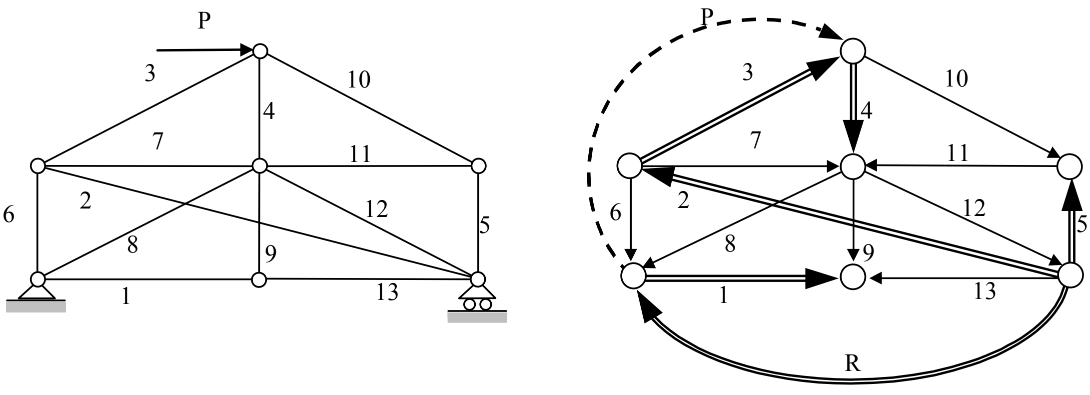
Thus the process of building of the RGR corresponding to an indeterminate truss can be summarized as follows:
Build a graph following the same steps as were explained in section 2.2.2 for building the FGR of a determinate truss. The FGR becomes the RGR when we assign resistances (or conductances) to all its edges that are not sources, as shownin Table 5.

	Type of edge	The conductance of the edge
a.	Truss rod – Resistance edge with finite conductance	$\frac{A(e)\cdot E(e)}{L(e)} \cdot \begin{pmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{pmatrix}$
b.	Fixed and mobile supports-on inclined surface.	zero
c.	Applied force on the truss – Flow source edge	The force is independent of the displacement difference between the joints which correspond to the end vertices of the flow source edge.

Table 5. Types of Resistance Edges in the Graph Representation of an Indeterminate Truss.

The analysis process is based on applying the CCM to the RGR of the indeterminate truss. The first step is choosing a suitable spanning tree. Since the potential differences in the reaction edges are known to be equal to zero, these edges are somewhat similar to the potential difference sources. Thus, the spanning tree must include the reaction edges whereas it should not include the flow source edges.

An example of an indeterminate truss, its corresponding graph, the spanning tree and the cutset conductance matrix is given in Figure 15.



(a) (b)

$$
\begin{array}{c c}
 & \begin{array}{cccccc} R & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} R \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{pmatrix}
G_R+G_6+G_8+G_9+G_{13} & G_9+G_{13} & -G_6-G_8-G_9 & -G_8-G_9 & -G_8-G_9 & 0 \\
G_9+G_{13} & G_1+G_9+G_{13} & -G_9 & -G_9 & -G_9 & 0 \\
-G_6-G_8-G_9 & -G_9 & G_2+G_6+G_8+G_9+G_{10}+G_{11}+G_{12} & G_8+G_9+G_{10}+G_{11}+G_{12} & G_8+G_9+G_{11}+G_{12} & -G_{10}-G_{11} \\
-G_8-G_9 & -G_9 & G_8+G_9+G_{10}+G_{11}+G_{12} & G_3+G_7+G_8+G_9+G_{10}+G_{11}+G_{12} & G_7+G_8+G_9+G_{11}+G_{12} & -G_{10}-G_{11} \\
-G_8-G_9 & -G_9 & G_8+G_9+G_{11}+G_{12} & G_7+G_8+G_9+G_{11}+G_{12} & G_4+G_7+G_8+G_9+G_{11}+G_{12} & -G_{11} \\
0 & 0 & -G_{10}-G_{11} & -G_{10}-G_{11} & -G_{11} & G_5+G_{10}+G_{11}
\end{pmatrix}
\end{array}
\bullet
\begin{bmatrix} \Delta_{Rx} \\ \Delta_{Ry} \\ \Delta_{1x} \\ \Delta_{1y} \\ \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \\ \Delta_{4x} \\ \Delta_{4y} \\ \Delta_{5x} \\ \Delta_{5y} \end{bmatrix}
= -
\begin{bmatrix} -\mathrm{Cos}(0) \\ -\mathrm{Sin}(0) \\ 0 \\ 0 \\ \mathrm{Cos}(0) \\ \mathrm{Sin}(0) \\ \mathrm{Cos}(0) \\ \mathrm{Sin}(0) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\bullet P
$$

(c)

Figure 15. Statically indeterminate truss and its resistance graph. (a) The truss. (b) Corresponding RGR. (c) CCM analysis equations.

2.5 Resistance Matroid Representation (RMR).

The graph theory was used above to derive from CCM in RGR the method for analysis of indeterminate trusses. However, this approach has been shown to have its limitations, one of which is the fact that RCM is not applicable to trusses, i.e. the method dual to CCM (Shai 1999). This is due to the fact that the conductance matrices of truss edges are singular. Thus, it does not have an inverse matrix, hence the rod edge in RGR cannot be assigned a resistance matrix.

It is known from the literature, that matroid theory, whose definitions and properties are given in section 2.1.2, is a generalization of graph theory. Therefore, representing engineering systems by matroid theory enables obtaining a more general perspective. Such a generalization is demonstrated in this section by representing indeterminate trusses by Resistance Matroid Representation (RMR). The direct consequence of such a generalization is the fact that RCM in RMR is shown to be applicable to indeterminate trusses.

2.5.1 Matroid Representation for Indeterminate Trusses.

The first step of representing an indeterminate truss by a matroid is to represent it by a resistance graph (section 2.4.5). Let G_R be the resistance graph representation of the indeterminate truss, and **Q**(G) its scalar cutset matrix. The scalar cutset matrix defines the matroid M_Q=<S,***F***> where S is the set of columns of **Q**(G) and ***F*** is a family of all linearly independent subsets of S. The subscript Q in M_Q is used to emphasize that the matroid corresponds to the scalar cutset matrix **Q**. Each element of M_Q is a scalar cutset matrix column that in its turn corresponds to a truss element, which is one of the following: rod, external reaction or external force.

An example of a truss with its corresponding matroid is given in Figure 16a and Figure 16d respectively.

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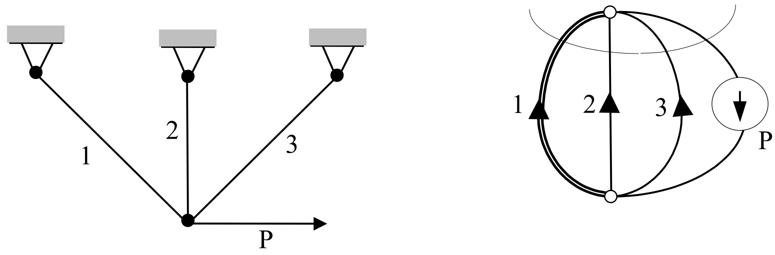

(a) (b)

$$Q(G)=\begin{array}{c} \\ 1x \\ 1y \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & P \end{array} \\ \begin{pmatrix} \cos(135^0) & \cos(90^0) & \cos(45^0) & -\cos(0^0) \\ \sin(135^0) & \sin(90^0) & \sin(45^0) & -\sin(0^0) \end{pmatrix} \end{array}$$

$$=\begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & -1 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \end{pmatrix}$$

M=<S,***F***>
S={1,2,3,P}
F **=** {{1}, {2},{3},{P},{1,2}, {1,3},{1,P}, {2,3},{2,P},{3,P}}.

(c) (d)

Figure 16. Example of a truss and its corresponding matroid.
(a) The truss (b) The graph G (c) The scalar cutset matrix Q(G) (d) The matroid.

2.5.2 STRUCTURAL INTERPRETATION OF MATROID COMPONENTS.

2.5.2.1 *Dependent sets of M_Q.*

The flow law for RGR is given by

$$\mathbf{Q}(G)\cdot\mathbf{F}=\mathbf{0} \quad (16)$$

where **F** is the vector of force scalar values acting in the truss elements.
Therefore the non-zero entries of the vector **F** define a set of linearly dependent columns of the scalar cutset matrix. By definition, such a set is also the set of dependent elements in the matroid M_Q. Thus, a dependent set in M_Q corresponds to a set of truss elements in which internal forces can act simultaneously, i.e. the truss elements that have nonzero internal forces during some state of self-stress. Such a set forms an indeterminate subset of truss rods (a *subtruss*).

2.5.2.2 *Circuits of M_Q.*

A circuit of the matroid is a minimal dependent set, i.e. removing even one of its elements results in an independent set. Therefore in the terminology of structures, a circuit in M_Q corresponds to a minimal indeterminate subtruss, which is a rigid subtruss indeterminate to the first degree. Such a subtruss has the properties of a circuit, since removing any of the rods from such a truss will create a determinate truss or even a mechanism.

2.5.2.3 *Base of M_Q.*

The base of a matroid – is the maximal independent subset of S, i.e. adding any element to the base results in a dependent set. Thus, the base in M_Q corresponds to a determinate subtruss that contains all the pinned joints of the truss. It is well known, that adding a rod to a determinate truss, without adding a pinned joint, makes the truss indeterminate.
In the sake of consistency with the graph theory, the base of the matroid representing the truss, is chosen so, that it doesn't contain any external forces (flow sources) acting on the truss.

2.5.2.4 *Cobase of M_Q.*

The cobase of M, i.e. the set of elements, which are not in the base, is the set of external forces and redundant rods of the truss.
The notation that is used in this paper for graphs (section 2.1.1), is applied also to matroids. For this reason, the base elements (the determinate subtruss elements) are represented by double lines, the cobase elements (redundant truss elements and external forces) by dashed lines, whereas the cobase elements which correspond to the external forces are both dashed and bold.

Figure 17 shows the truss from Figure 16, with highlighted base and cobase elements (a) and the two fundamental circuits (circuits containing only one redundant rod or only one external force) (b) and (c).

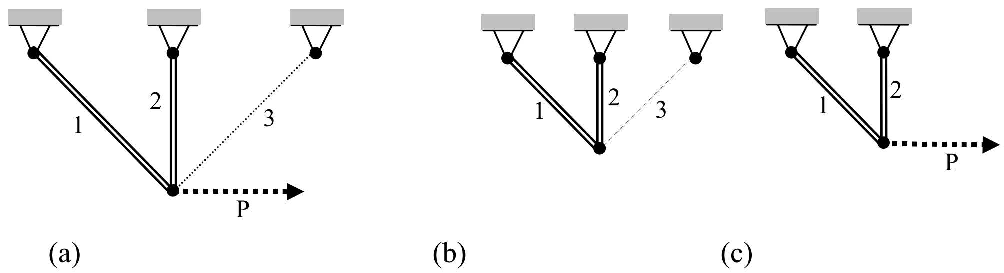

Figure 17. Example of fundamental circuits in the matroid of a truss.
(a) The base and cobase of M. (b) A fundamental circuit defined by redundant 3. (c) A fundamental circuit defined by external force P.

2.5.2.5 Circuit Matrix of M_Q.

By definition of circuit in matroid, each fundamental circuit in M_Q corresponds to a set of linearly dependent columns in Q. In other words for each fundamental circuit C_i, it can be written:

$$\sum_{j \in c_i} \lambda_{ij} \mathbf{Q}_{\downarrow j} = \mathbf{0} \qquad (17)$$

where $\mathbf{Q}_{\downarrow j}$ - is the jth column of matrix **Q**. In the terminology of trusses λ_{ij} is the force acting in the truss element 'j' while the state of self-stress produced by a force in the redundant truss element 'i'.
The set of fundamental circuits is represented by a special matrix $\mathbf{B}(M_Q)$, called *a circuit matrix of M_Q*. The rows of **B**(M) correspond to the cobase elements of M and the columns to all the elements of M. An entry 'ij' of the matroid circuit matrix - is defined:

$$[\mathbf{B}(M)]_{ij} = \lambda_{ij} \qquad (18)$$

Obviously, equation 23 still holds, when for some i, all λ_{ij} are multiplied by same arbitrary scalar. Therefore it is legitimate to 'normalize' the circuit matrix, i.e. to multiply the rows of **B**(M), so that the matrix is written as follows:

$$\mathbf{B}(M) = (\mathbf{B}(M)_T \mid \mathbf{I}) \qquad (19)$$

where **I** is a unit matrix whose size is equal to the number of cobase elements, and $\mathbf{B}(M)_T$ is a matrix with rows and columns corresponding to the cobase and base elements respectively. In structural mechanics terminology the value of $[\mathbf{B}(M)]_{ij}$ becomes: the force in truss rod or reaction 'i' when a unit force is applied in a redundant element 'j' and in all the other redundant elements the forces are set to zero.
For example, the circuit matrix of matroid M_Q that represents the truss of Figure 16 is developed as follows:
for cobase elements 3 and P, equations based on (17) are written respectively:

$$\lambda_{3,1} \cdot \mathbf{Q}_{\downarrow 1} + \lambda_{3,2} \cdot \mathbf{Q}_{\downarrow 2} + \lambda_{3,3} \cdot \mathbf{Q}_{\downarrow 3} = 1 \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} - 1.414 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

$$\lambda_{P,1} \cdot \mathbf{Q}_{\downarrow 1} + \lambda_{P,2} \cdot \mathbf{Q}_{\downarrow 2} + \lambda_{P,P} \cdot \mathbf{Q}_{\downarrow P} = -1.414 \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Hence the circuit matrix of M_Q is:

$$\mathbf{B}(M) = \begin{array}{c} \\ 3 \\ P \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & P \end{array} \\ \left(\begin{array}{cc|cc} 1 & -1.414 & 1 & 0 \\ -1.414 & 1 & 0 & 1 \end{array}\right) \end{array}$$

Proposition 1: Every admissible force vector **F** is a linear combination of rows of **B**(M).
Proof: Forces in the determinate subtruss (base) are uniquely defined by the forces in the redundant elements. Moreover, each row of **B**(M) corresponds to the forces in the determinate subtruss yielded by a unit force in the corresponding redundant element. Therefore, by the superposition principle, every

admissible force vector is derived by summing over all the rows of **B**(M) each multiplied by the force in the corresponding redundant element.

Proposition 2: The matroid potential law:

$$\mathbf{B}(\mathrm{M})\cdot\mathbf{D} = \mathbf{0} \qquad (20)$$

where **D** is a vector of scalar displacements in truss elements.

Proof: According to the definition of matroid M_Q, each row of **B**(M) corresponds to a state of self stress, which is a vector of admissible flows in G_R. From the other hand, vector **D** corresponds to a vector of admissible scalar potential differences in G_R. Thus, according to the equilibrium between the internal strain energy of the truss and the work done by the external forces (West 1993), multiplication of every row in **B**(M) by vector **D** is equal to zero.

2.5.2.6 The Cutset Matrix of Matroid.

The cutsets of a matroid are represented by a cutset matrix as explained below.

Proposition 3: The matrix $\mathbf{Q}(\mathrm{M})=(\ \mathbf{I}\ \mid -\mathbf{B}_{\mathrm{T}}^{\mathrm{t}})$ is the cutset matrix of matroid M_Q, i.e. each row of **Q**(M) defines a fundamental cutset in the matroid.

Proof: To prove this property we have to prove that every row of **Q**(M) satisfies the conditions of a cutset (as shown in section 2.1.2).

Conditions (a) and (c) of the section 2.1.2 are satisfied since **Q**(M) contains a unit matrix, whose rows are non-empty and do not contain other rows of the matrix.

Condition (b) requires that for any circuit 'i' and any cutset 'j' the number of common elements is not equal to one. This can be easily proved by considering the form of circuit and cutset matrices (Figure 18). The number of common elements in circuit 'i' and cutset 'j' is the number of elements corresponding to the nonzero entries in rows 'i' and 'j' in the circuit and the cutset matrices respectively. From Figure 18 one can see that this number can be either 0 or 2 depending on whether element B_{ij} is equal to zero or not. Thus, the number of common elements in circuit and cutset can never be equal to one.

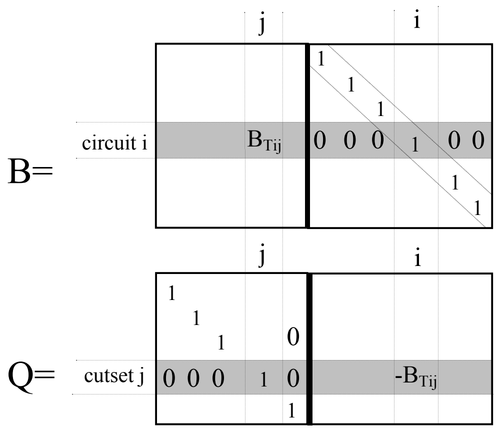

Figure 18. The form of circuit and cutset matrices.

Proposition 4: The orthogonality principle:

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{B}^{\mathrm{t}}(\mathrm{M})=\mathbf{0} \qquad (21)$$

Proof: By substituting equation 19 into equation 21 we obtain

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{B}^{\mathrm{t}}(\mathrm{M})=\begin{pmatrix}\mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}}\\ \mathbf{I}\end{pmatrix}.\,(\mathbf{I}\mid-\mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}}) \qquad (22)$$

After the multiplication we get: $\mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}} - \mathbf{B}(\mathrm{M})_{\mathrm{T}}^{\mathrm{t}} = \mathbf{0}$.

Proposition 5: Matroid flow law:

$$\mathbf{Q}(\mathrm{M})\cdot\mathbf{F} = \mathbf{0} \qquad (23)$$

Proof: By proposition 1, every admissible force vector **F** is a linear combination of **B**(M) rows. According to proposition 4, **Q**(M) is orthogonal to **B**(M), hence it is orthogonal to every linear combination of its rows, i.e. **F**.

Because of the validity of propositions 1 to 5, the flow, potential and orthogonality laws are all valid for matroid M. Such a matroid is called a Resistance Matroid. Since equation 13 was derived using only these properties of the resistance graph, it is also valid for the matroid M_Q. Substituting to 13 the matrices corresponding to M_Q instead of those corresponding to G, we obtain:

$$(\mathbf{B}(\mathrm{M})_{\mathrm{C'R}} \cdot \mathbf{R}_{\mathrm{R}} \cdot \mathbf{B}(\mathrm{M})^{\mathrm{t}}_{\mathrm{C'R}}) \cdot \mathbf{F}_{\mathrm{C'}} = -(\mathbf{B}(\mathrm{M})_{\mathrm{C'R}} \cdot \mathbf{R}_{\mathrm{R}} \cdot \mathbf{B}(\mathrm{M})^{\mathrm{t}}_{\mathrm{PR}}) \cdot \mathbf{F}_{\mathrm{P}} - \mathbf{B}_{\mathrm{C'P}} \cdot \mathbf{D}_{\mathrm{D}} \quad (24)$$

2.5.3 EXAMPLE OF APPLICATION OF THE RCM IN RMR TO AN INDETERMINATE TRUSS.

The method derived above is demonstrated in the following example. First RGR representing the truss of Figure 19a is built (Figure 19b). At the next stage, the cutset matrix of the RGR is found - Figure 19c. Finally the circuit matrix of the RMR is build from the cutset matrix of the RGR - Figure 19d. The elements of the circuit matrix were substituted into (24) and the analysis equations were obtained (Figure 19e).

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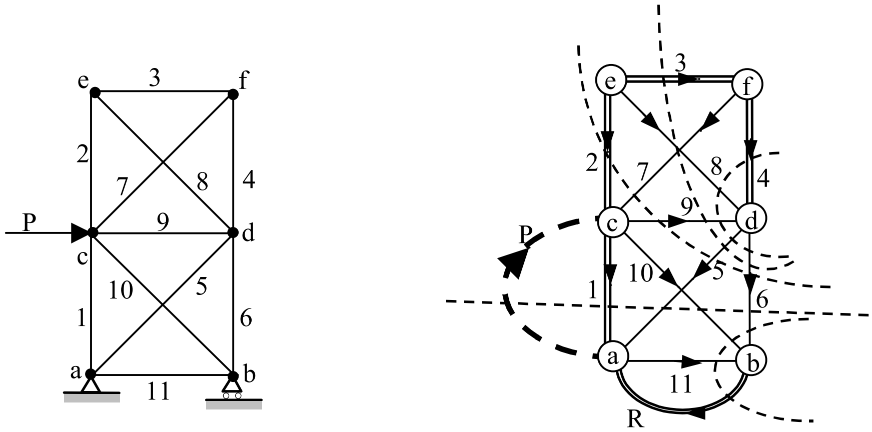


(a) (b)

$$\mathbf{Q}=\begin{pmatrix}
\cos(270^0) & 0 & 0 & 0 & 0 & 0 & -\cos(270^0) & 0 & 0 & 0 & -\cos(315^0) & -\cos(0^0) & 0\\
\sin(270^0) & 0 & 0 & 0 & 0 & 0 & -\sin(270^0) & 0 & 0 & 0 & -\sin(315^0) & -\sin(0^0) & 0\\
0 & \cos(270^0) & 0 & 0 & 0 & \cos(225^0) & \cos(270^0) & 0 & 0 & 0 & \cos(315^0) & 0 & \cos(0^0)\\
0 & \sin(270^0) & 0 & 0 & 0 & \sin(225^0) & \sin(270^0) & 0 & 0 & 0 & \sin(315^0) & 0 & \sin(0^0)\\
0 & 0 & \cos(270^0) & 0 & 0 & \cos(225^0) & \cos(270^0) & \cos(225^0) & 0 & -\cos(0^0) & 0 & 0 & 0\\
0 & 0 & \sin(270^0) & 0 & 0 & \sin(225^0) & \sin(270^0) & \sin(225^0) & 0 & -\sin(0^0) & 0 & 0 & 0\\
0 & 0 & 0 & \cos(0^0) & 0 & -\cos(225^0) & -\cos(270^0) & -\cos(225^0) & \cos(315^0) & \cos(0^0) & 0 & 0 & 0\\
0 & 0 & 0 & \sin(0^0) & 0 & -\sin(225^0) & -\sin(270^0) & -\sin(225^0) & \sin(315^0) & \sin(0^0) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cos(270^0) & -\cos(225^0) & -\cos(270^0) & 0 & \cos(315^0) & \cos(0^0) & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sin(270^0) & -\sin(225^0) & -\sin(270^0) & 0 & \sin(315^0) & \sin(0^0) & 0 & 0 & 0
\end{pmatrix}$$

(c)

$$\mathbf{B}_M=\begin{array}{c} \\ 7\\ 10\\ P\end{array}\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 & P\end{array}\\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 & 0\\
-0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1.414 & -1 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}\right)\end{array}$$

(d)

$$\mathbf{B}_{C'R}=\begin{array}{c} \\ 7\\ 10\end{array}\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10\end{array}\\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0\\
-0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1
\end{array}\right)\end{array}$$

(e)

$$\mathbf{B}_{PR}=\begin{array}{c} \\ P\end{array}\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10\end{array}\\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
0 & 0 & 0 & 0 & -1.414 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right)\end{array}$$

(f)

$$\begin{pmatrix} 4.823 & 0.5\\ 0.5 & 4.823\end{pmatrix}\begin{pmatrix} F_7\\ F_{10}\end{pmatrix}=-\begin{pmatrix} 0.707\\ 3.411\end{pmatrix}\cdot P$$

(g)

Figure 19. Example of analysis of indeterminate truss using RMR.
(a). Indeterminate truss, (b). RGR of the truss, (c) scalar cutset matrix of RGR of the truss, (d,e,f) circuit matrix of RMR of the truss and its components (g) analysis equations based on RCM in RMR.

A base (statically indeterminate subtruss) is obtained by removing from the truss the redundant rods 7 and 10. Hence the cobase elements of the resistance matroid representing the truss are 7, 10 and P, whereas the latter is the flow source. The circuit matrix Figure 19d is now built by calculating three self-stresses each having a unit force in one of the cobase elements. Then the parts of the circuit matrix are substituted into (24) and the analysis equations are obtained (Figure 19g). After solving the equations of Figure 19g, the flows in all the cobase elements are known and by (9) the flows in all the rest elements of the matroid are obtained. Then using the resistance relations, the potential differences are obtained as well.

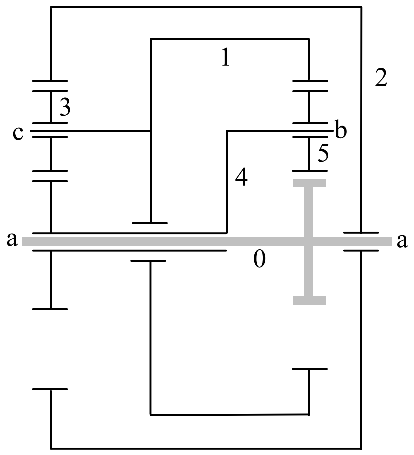
2.6 Line Graph Representation (LGR).

Line Graph Representation (LGR) – is the only graph representation dealt with in this paper, which has no knowledge embedded in it. LGR is a regular graph, the main property of which comprises in the way it is used to represent engineering systems. In contrast to FGR, PGR and RGR, the elements of the engineering system are represented in LGR not by edges, but by vertices. This enables to use the edges of LGR in order to describe the connections between the elements. LGR was used to represent planetary gear systems, traffic control problem (Shai 1997), and various network optimization problems (Shai 1997). Current paper uses LGR to represent the planetary gear systems.

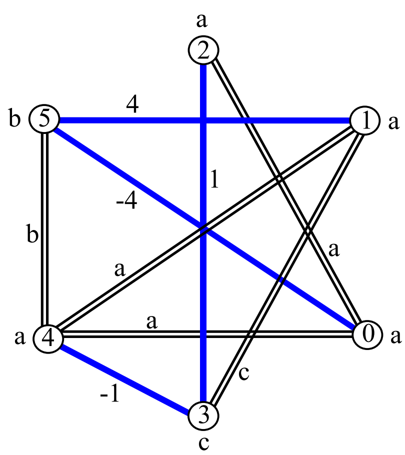
2.6.1 LGR FOR PLANETARY GEAR SYSTEMS.

All the links of a planetary gear systems are represented by vertices in LGR, whereas the connections between them are represented by the edges connecting the corresponding vertices. There are two types of connections, so there are two types of edges, marked as bold and double as explained below.

a. Bold edge (bold line) – by knowing the ratio of the connected gear wheels one can calculate the ratio between the angular velocities (potentials) of the gear wheels. In the terminology of this paper the latter edge is a dependent potential source and for this reason it appears in the graph as a bold line.
b. Regular edge (double line) - an edge which represents a turning connection. It will be shown below that the turning edges form a spanning tree.
Other information about the labeled edges and the vertices is added to the representation as follows:
c. Labeled double line - every double line (turning edge) has a label which represents the level, being the location of the rotating connection.
d. Reference vertices - the distance between each pair of connected gear wheels must be constant all the time, being maintained by a link or planet carrier. This is called in the literature (Freudenstein 1971) a 'transfer vertex'. In the terminology of this paper, the name 'local reference vertex' is more suitable. In this representation, all the turning edges on one side of the local reference vertex are at the same level, and those on the opposite side of the local reference vertex are at a different level.
e. Labeled bold line – every bold line (gear edge) has a label which represents the planet carrier (local reference vertex) that maintains the distance between the two gear wheels which correspond to the end vertices of the gear edge. In addition, the bold line has a sign, where the 'plus' (or minus) sign means that the transmission of movement between the two gear wheels is internal (or external).
f. Labeled gear wheel vertex – every vertex that corresponds to a gear wheel has a label that represents its center level.



(a)



(b)

Figure 20. A planetary mechanism (a) and its line graph representation (b).

Note, that Figure 20a is a standard representation in engineering drawing for a gear system.

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3 NEW CONNECTIONS BETWEEN ENGINEERING FIELDS.

Current section shows the application of MCA to obtaining novel relationships between engineering fields. These were obtained through the mathematical relationships between the corresponding CR. More specifically, on the basis of the dualism between FGR and PGR, which will be developed in section 3.1.1, a new mutual relationship between determinate trusses and mechanisms is established in 3.1.2. This relation postulates that for every determinate truss there exists a specific mechanism, called the "dual mechanism". In this mechanism there is a link for every rod of the truss, such that the force in the rod is equal to the relative velocity of the corresponding link.

This innovation opens up a new avenue of research and practical application, by creating favorable conditions for cooperation between structural and mechanical engineers, since information, algorithms and new technologies developed in one field can now be applied to the other. One of the possible applications of that research direction is developing of a new design technique in engineering, shown in Section 7. The potential inherent in the innovations of this section can be appreciated from its applications given in sections 6.5 and 7.

3.1 Duality between trusses and mechanisms based on the duality of their corresponding CR.

3.1.1 Duality between flow and potential graphs

This section introduces the duality connection between the flow and potential graph representations from which the dualism between trusses and mechanisms was derived by applying the following inference rules:

RULE 1: IF G is FGR, i.e $G = G_F$ THEN $\mathbf{Q}(G_F) * \vec{\mathbf{F}}(G_F) = \mathbf{0}$,

RULE 2: IF G is PGR, i.e $G = G_\Delta$ THEN $\mathbf{B}(G_\Delta) * \vec{\Delta}(G_\Delta) = \mathbf{0}$.

FACT1: Q(G) = B(G*) (according to duality between graphs, (Swamy and Thulasiraman 1981))

CONCLUSION 1: FACT1 AND RULE 1 → $\mathbf{B}(G^*) * \vec{\mathbf{F}}(G_F) = \mathbf{0}$

CONCLUSION 2: CONCLUSION 1 AND $G^* = G_\Delta$ → $\vec{\Delta}(G^*) = \vec{\mathbf{F}}(G)$

So a graph, dual to the FGR is the PGR and its potential difference vector is identical to the flow vector of the former.

3.1.2 Duality between trusses and mechanisms.

Based on the dualism connection between FGR and PGR given in the previous section, a new relation between determinate trusses and mechanisms is derived. This 'invention' has been achieved by applying the following rules:

FACT 1: for every flow graph there exists a dual potential graph and vice versa (section 3.1.1).

FACT 2: Determinate trusses are isomorphic to flow graphs (section 2.2.2)

FACT 3: Mechanisms are isomorphic to potential graphs (section 2.3.1)

CONCLUSION: FACT 1 AND FACT 2 AND FACT 3 → for every mechanism there exists a dual determinate truss and vice versa.

This reasoning is outlined in Figure 21.

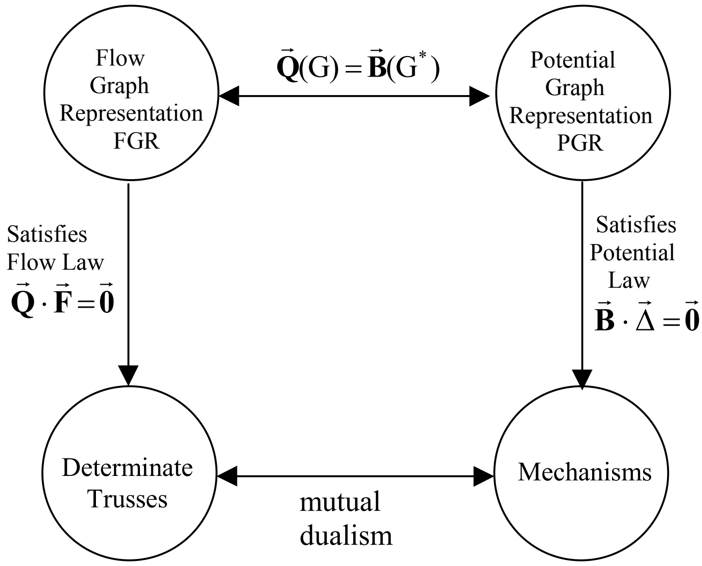


Figure 21. Diagram explaining the duality between determinate trusses and mechanisms.

Description of a dual mechanism – Let T be a statically determinate truss, G(T) its FGR and G*(T) be its dual PGR. Each rod of the truss has a direction unit vector $\hat{r}(e)$ and each link of the mechanism has a unit vector $\hat{v}(e)$ representing the direction of linear relative velocity of the end joints of the link. Mechanism M is called a "dual mechanism to truss T" if G*(T) is its potential graph and for every edge (rod) 'e' in graph G(T) its corresponding edge (link) 'e'' in the dual graph G*(T), the equality (25) is satisfied.

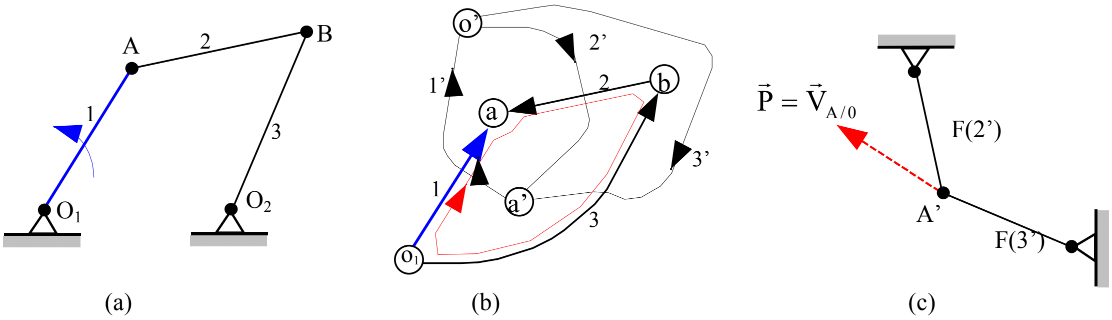
$$\hat{r}(e) = \hat{v}(e') \tag{25}$$

Table 6 summarizes the attributes of the duality between trusses and mechanisms.

IN A MECHANISM LINK	IN A TRUSS ROD
$\vec{\Delta}$ - Relative velocity vector.	$\vec{\mathbf{F}}$ - Force vector.
Circuit.	Cutset.
Potential difference.	Flow.
Potential difference of edge 'e'= relative linear velocity of the corresponding link 'e' = $\vec{\Delta}(e) = V(e) * \hat{v}(e)$.	Flow in edge 'e' = force acting in rod 'e' = $\vec{F}(e) = F(e) * \hat{r}(e)$
$\hat{v}(e)$ -relative linear velocity unit vector.	$\hat{r}(e)$ - unit vector in the rod direction.
$\omega_{i/0}$ - Angular velocity of link i = linear velocity/length.	$\frac{Fi}{L_i}$ - Force per unit length.

Table 6. The Duality Attributes

Figure 22 shows a four-bar mechanism (a) and its dual truss.



$$\vec{V}_{A/0} - \vec{V}_{A/B} - \vec{V}_{B/0} = \vec{0}$$

$$\begin{pmatrix} \hat{v}(1) & -\hat{v}(2) & -\hat{v}(3) \end{pmatrix} \begin{pmatrix} V_{A/0} \\ V_{A/B} \\ V_{B/0} \end{pmatrix} = (0)$$

(d)

$$\vec{P} - \vec{F}(2) - \vec{F}(3) = \mathbf{0}$$

$$\begin{pmatrix} \hat{r}(\vec{P}) - \hat{r}(2') - \hat{r}(3') \end{pmatrix} \cdot \begin{pmatrix} \vec{P} \\ \vec{F}(2) \\ \vec{F}(3) \end{pmatrix} = \mathbf{0}$$

(e)

Figure 22 Example of a mechanism, its dual truss and the corresponding matrix representation. (a) The mechanism. (b) The potential graph and its dual (dashed line). (c) The dual truss. (d,e) The corresponding matrices.

More detail on the duality connection between trusses and mechanisms can be found in (Shai 2000).

4 META LAWS AND THEOREMS .

As it was mentioned in section 2, each combinatorial representation contains combinatorial theorems called "embedded theorems", which have been thoroughly studied and investigated. The embedded theorems are actually metatheorems that provide additional knowledge and enable to derive theorems exclusively from the representation. Therefore, when CR are used to represent an engineering problem, its embedded theorems become available as well. For example, when the resistance graph representation was applied to represent indeterminate trusses (section 2.4.5), its two analysis methods became available and were used. In this section, it is shown how theorems and methods in structural mechanics can be derived from a theorem embedded in RGR, called: Tellegen's Theorem. Moreover, from the dualism law in RMR a new proposition is deduced, that states that displacement and force methods are actually dual methods (Shai 1999).
All these results show the potential inherent in applying MCA to enable in the future the derivation of new theorems and methods from the knowledge embedded in the CR.

4.1 Tellegen's theorem embedded in RGR.

The theorem discussed in this section was developed by Professor B. D. H. Tellegen (Tellegen, 1952) and therefore bears his name. The main use made of this theorem nowadays is in electric circuit theory (Penfield et al., 1970; Chua et al.,1987). Since electric circuits are represented in MCA by RGR, it is derived that this theorem can also be employed in other engineering systems represented with RGR. This is done in this section. According to section 2.4.5, indeterminate trusses are represented by RGR, therefore Tellegen's theorem, which is a metatheorem in RGR is applied, and engineering theorems and methods are derived.

Theorem 1. Tellegen's Theorem (combinatorial representations formulation): Let G_F and G_Δ be two isomorphic graphs, whereas the first is flow graph and the second is potential graph, then:

$$\sum_{e}^{\substack{\text{all} \\ \text{edges}}} \vec{F}^{t}_{G_F}(e) \cdot \vec{\Delta}_{G_\Delta}(e) = 0 \tag{26}$$

The theorem deals with two isomorphic graphs, one of which (G_F) satisfies the flow law and the other the potential law (G_Δ). It postulates that the sum of scalar multiplications between the flows in the edges of G_F and potential differences in the corresponding edges of G_Δ is equal to zero.

4.1.1 EXPLANATION OF TELLEGEN'S THEOREM USING THE ELECTRICAL NETWORKS.

In order to facilitate the understanding, Tellegen's theorem is first applied to electrical circuits. Consider two different electric circuits with same topology shown in Figure 23(a,b). Their corresponding graphs appear in Figure 23 (c,d).

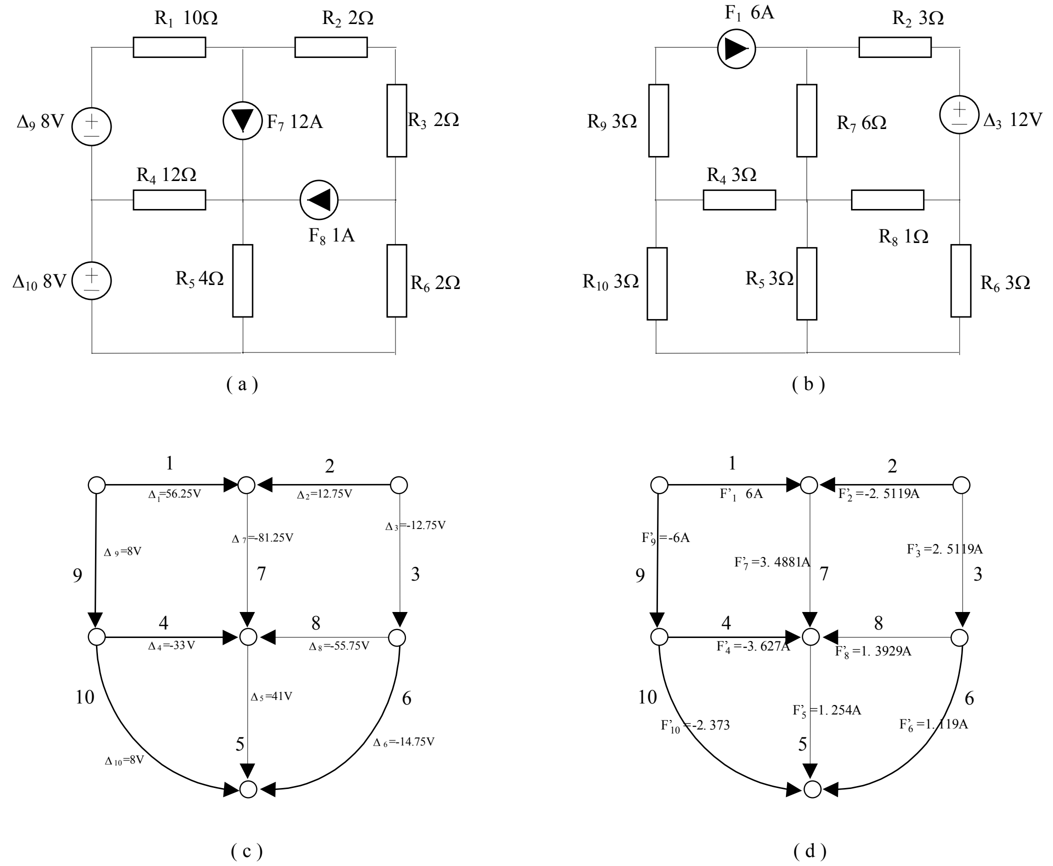


Figure 23. Applying Tellegen's theorem to electrical circuits.
(a),(b) different electrical circuits possessing the same topology. (c) G_Δ – RGR representing the system (a). (d) G_F- RGR representing the system (b).

Analysis of the electrical systems gives the results shown on Figure 23 beside the edges in the corresponding graphs.
We can now choose the graph representing the system of (a) to be G_Δ and the graph representing the system of b to be G_F. These graphs satisfy the requirements of theorem 1. Substituting the results into (26) confirms the Tellegen's theorem:

$$\sum_{i=1}^{10} \Delta_i (G_\Delta) \cdot F'_i (G_F) = 0 \qquad (27)$$

4.1.2 APPLICATION OF TELLEGEN'S THEOREM TO TRUSSES.

The example that appears in the preceding section concerns with one-dimensional systems. However, one can deduce from equation 26, that the theorem can be applied to the multidimensional systems as well.
The multidimensional trusses are represented by RGR, hence the multidimensional Tellegen theorem embedded in the this representation can be employed in their analysis. The formulation of the Tellegen theorem for trusses is as follows: given two trusses with same topology, sum over scalar multiplications of the forces in the first and the potential differences in the second.

When the angles of the rods of the truss are known, the following scalar formulation can be developed (Shai 2000b):

$$\sum_{\substack{\text{rods of the}\\\text{truss}}} F_i(G_F)\cdot D_i(G_\Delta) - \sum_{\substack{\text{external}\\\text{forces}}} P_i(G_F)\cdot D_{Pi}(G_\Delta) = 0\,. \tag{28}$$

This equation will be referred in the paper as the: “Multidimensional Tellegen’s Theorem” for Trusses.

4.1.3 DERIVING THE METHOD FOR ANALYSING JOINT DISPLACEMENT BASED ON TELLEGEN’S THEOREM .

In this section, it is shown that the known equation for analysing the displacement of a joint in a truss is a special case of the Multidimensional Tellegen’s theorem for Trusses.
In order to apply Tellegen’s theorem, PGR and FGR will be used. The steps for building the CR are the same as was explained in section 2.2.2. An extra edge, called “control edge”, is added. Its head vertex is the vertex whose displacement is to be analysed, and the tail vertex is the reference vertex.
The combinatorial representations - the flow and potential graphs are used in two different ways as follows:

For the real potential graph G_Δ^R the flow values in the source edges are the values of the external forces. In the control edge we put a ‘potential difference measurement’, which corresponds to a potential difference measuring device (e.g. voltmeter in electrical circuit) that is located between the end vertices of the corresponding edge. The ‘R’ superscript over G indicates, that the potential differences in it are due to the “real” external forces applied to the structure, whereas the structure should satisfy only the flow law.

For the virtual flow graph G_F^V all the source edges which correspond to the external forces are assigned flow sources with values equal to zero. One can think about it as a disconnection. In the control edge, a unit force is applied in the direction of the displacement that has to be measured. The ‘V’ superscript over G indicates that the flows in the graph are not the real forces in the structure, but the forces due to a virtual external force applied to the structure.

Applying the Multidimensional Tellegen’s Theorem to the two graphs, gives:

$$\sum_{\substack{\text{rods of the}\\\text{truss}}} F_i(G_F^V)\cdot D_i(G_\Delta^R) - \sum_{\substack{\text{external}\\\text{forces}}} 0\cdot D_{Pi}(G_\Delta^R) - 1\cdot D_{\text{control}}(G_\Delta^R) = 0 \tag{29}$$

From here, the well-known equation (West 1993) for analyzing the displacement of a joint is derived :

$$D_{\text{control}}(G_\Delta) = \sum_{\substack{\text{rods of the}\\\text{truss}}} \frac{F_i(G_F)\cdot F_i(G_\Delta)\cdot L_i}{A_i\cdot E_i} \tag{30}$$

An example for applying equation (30), is given in Figure 24, where the horizontal displacement of joint ‘c’ is to be analysed.

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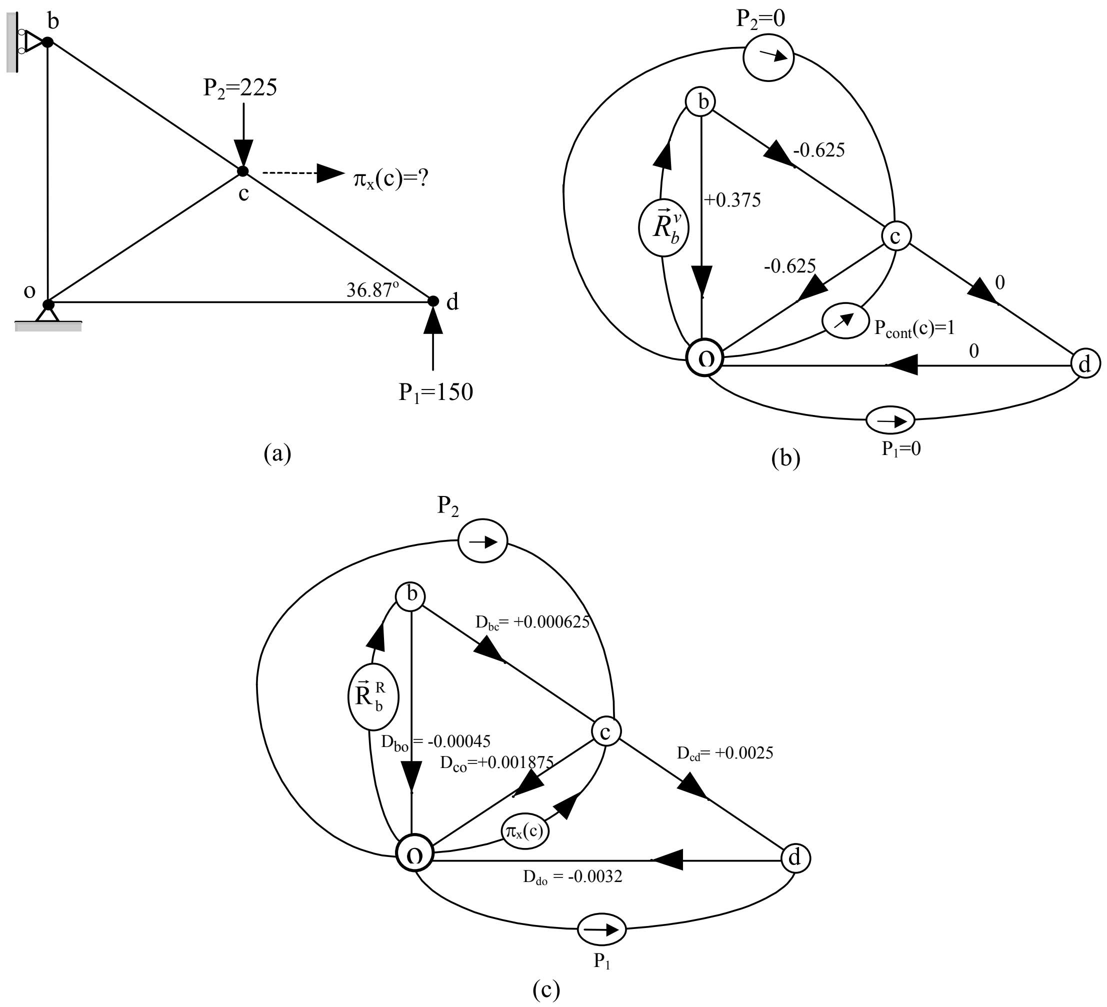


$$\pi_x(c) = F_{ba}(G_F^V)\cdot D_{ba}(G_\Delta^R) + F_{bc}(G_F^V)\cdot D_{bc}(G_\Delta^R) + F_{ca}(G_F^V)\cdot D_{ca}(G_\Delta^R) =$$
$$(0.375\cdot -0.00045) + (-0.625\cdot 0.000625) + (-0.625\cdot 0.001875) =$$
$$-0.001731[m]$$

(d)

Figure 24. Example of analyzing joint displacement by the Multidimensional Tellegen's theorem.
(a) The Truss. (b) The virtual flow graph G_F^V. (c) The real potential graph G_Δ^R. (d) Calculation of the displacement of joint 'c' in the direction of the x axis.

-

4.1.4 Deriving Betti's Law From a Theorem Embedded in RGR.

Previous section showed an example of derivation of a known method in structural analysis from the RGR embedded knowledge. In the current section it is shown that known theorems can also be derived from the embedded knowledge. This is demonstrated by deriving Betti's law from Tellegen's theorem.

Given a truss and two different sets of external loads applied on it. The first set of external loads - $\vec{\mathbf{P}}_1$ causes joint displacements $\vec{\boldsymbol{\pi}}_1$, internal forces $\vec{\mathbf{F}}_1$ and deformations $\vec{\mathbf{D}}_1$. The second set of external loads $\vec{\mathbf{P}}_2$, causes joint displacements $\vec{\boldsymbol{\pi}}_2$, internal forces $\vec{\mathbf{F}}_2$ and deformations $\vec{\mathbf{D}}_2$.

Since both sets of loads act on the same truss, and the forces (potential differences) satisfy the flow (potential) law, then according to Tellegen's Theorem (equation (16)) multiplication of forces from one set by the potential differences from the other set is equal to zero, as follows:

$$\left(\vec{\mathbf{F}}_1^{\mathrm{t}} \quad \vec{\mathbf{P}}_1^{\mathrm{t}}\right)\cdot\begin{pmatrix}\vec{\mathbf{D}}_2\\ -\vec{\pi}_{\mathrm{P2}}\end{pmatrix}=\vec{0} \;\rightarrow\; \vec{\mathrm{F}}_1^{\mathrm{t}} * \vec{\mathbf{D}}_2 = \vec{\mathbf{P}}_1^{\mathrm{t}}\cdot\vec{\pi}_{\mathrm{P2}} \tag{31}$$

$$\vec{\mathbf{P}}_1^{\mathrm{t}}\cdot\vec{\pi}_{\mathrm{P2}} = \vec{\mathbf{F}}_1^{\mathrm{t}}\cdot\vec{\mathbf{D}}_2 \overset{\substack{\text{resistance}\\ \text{relation}}}{=} \vec{\mathbf{F}}_1^{\mathrm{t}}\cdot(\mathbf{R}\cdot\vec{\mathbf{F}}_2) = (\vec{\mathbf{F}}_1^{\mathrm{t}}\cdot\mathbf{R})\cdot\vec{\mathbf{F}}_2 \overset{\substack{\text{since R is}\\ \text{diagonal}}}{=} \vec{\mathbf{D}}_1^{\mathrm{t}}\cdot\vec{\mathbf{F}}_2 \tag{32}$$

Another form of the Tellegen theorem for the two graphs is:

$$\left(\vec{\mathbf{F}}_2^{\mathrm{t}} \quad \vec{\mathbf{P}}_2^{\mathrm{t}}\right)\cdot\begin{pmatrix}\vec{\mathbf{D}}_1\\ -\vec{\pi}_{\mathrm{P1}}\end{pmatrix}=\mathbf{0} \;\rightarrow\; \vec{\mathbf{F}}_2^{\mathrm{t}}\cdot\vec{\mathbf{D}}_1 = \vec{\mathbf{P}}_2^{\mathrm{t}}\cdot\vec{\pi}_{\mathrm{P1}} \tag{33}$$

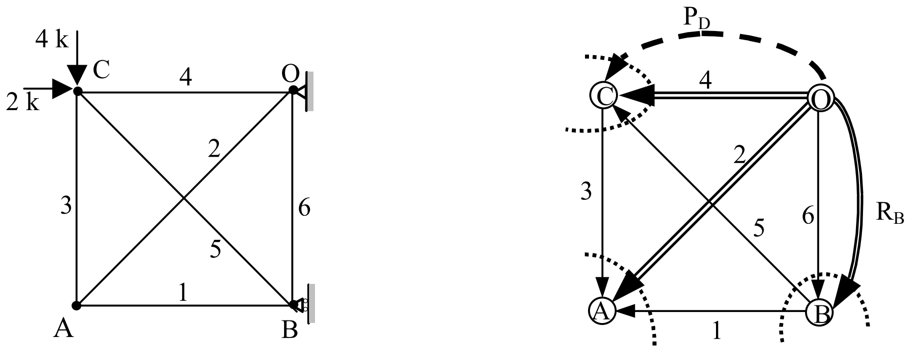
Combining the last two equations gives:

$$\mathbf{P}_1^{\mathrm{t}} * \vec{\pi}_{\mathrm{P2}} = \vec{\mathbf{P}}_2^{\mathrm{t}} * \vec{\pi}_{\mathrm{P1}} \tag{34}$$

This is the reciprocity theorem or Betti's law, which is well-known in the literature (Hibbeler 1984).

4.2 The Relation between the Conductance Cutset Method for Analyzing Indeterminate Trusses and Other Known Methods.

In the current section it is shown that a known method in structural mechanics is derived from CCM. Many methods for analyzing indeterminate trusses have been reported in the literature, most of them based on virtual work and minimum energy. In the displacement method (Hibbeler 1984), for each axis along which the joint is able to move, a variable is designated as the "unknown displacement". In the conductance cutset method, the absolute potential of a vertex is the displacement of the vertex relative to the reference vertex.



(a) **(b)**

$$\begin{pmatrix}\mathbf{G}_{R_B}+\mathbf{G}_1+\mathbf{G}_5+\mathbf{G}_6 & -\mathbf{G}_1 & -\mathbf{G}_5\\ -\mathbf{G}_1 & \mathbf{G}_2+\mathbf{G}_1+\mathbf{G}_3 & -\mathbf{G}_3\\ -\mathbf{G}_5 & -\mathbf{G}_3 & \mathbf{G}_4+\mathbf{G}_3+\mathbf{G}_5\end{pmatrix}\begin{pmatrix}\vec{\Delta}_{RB}\\ \vec{\Delta}_2\\ \vec{\Delta}_4\end{pmatrix} = -\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix} -0.035 & -0.1 & 0 & -0.035 & 0.035\\ 0.135+0\cdot(-0.035) & 0 & 0 & 0.035 & -0.035\\ 0+0\cdot(-0.1) & 0.135 & 0.035 & 0 & 0\\ 0+0\cdot(0) & 0.035 & 0.135 & 0 & -0.1\\ 0.035+0\cdot(-0.035) & 0 & 0 & 0.135 & -0.035\\ -0.035+0\cdot(0.035) & 0 & -0.1 & -0.035 & 0.135\end{pmatrix}\cdot\begin{pmatrix}\Delta_{R_B y}\\ \Delta_{2x}\\ \Delta_{2y}\\ \Delta_{4x}\\ \Delta_{4y}\end{pmatrix} = -\begin{pmatrix}0\\0\\0\\0\\ \mathrm{P_{Dx}}\\ \mathrm{P_{Cy}}\end{pmatrix}$$

(c)

Figure 25. Indeterminate truss analysis.
(a) Indeterminate truss, (b) the corresponding graph and (c) the corresponding equations

In the example of Figure 25, since all the cutsets are such that they contain exactly one vertex in one of the two sides of the cutset, the cutset conductance matrix is equal to the incidence matrix, a well-known in graph theory literature (Fenves and Branin 1963; Deo 1974).
Therefore, the displacement matrix of the displacement method (Hibbler 1984) is same as that obtained from the resistance graph, as shown in Figure 25.
To derive the displacement method, one starts with the incidence matrix (Deo 1974), the rows of which are linearly dependent on the vector cutset matrix. The flow law (equation 2) can be written by using the incidence matrix as follows:

$$\vec{\mathbf{A}}\vec{\mathbf{F}}_R = -\vec{\mathbf{A}}_P\vec{\mathbf{P}} \tag{35}$$

Since any resistance edge e=<u,v> satisfies equation 6, equation 35 becomes :

$$\vec{\mathbf{A}}\mathbf{G}_R\vec{\Delta}_R = -\vec{\mathbf{A}}_P\vec{\mathbf{P}} \tag{36}$$

Since the potential difference of an edge is equal to the difference between the potentials at the end vertices, this can be written in matrix form:

$$\vec{\Delta}_R = \vec{\mathbf{A}}^t\vec{\pi} \tag{37}$$

that gives us:

$$\vec{\mathbf{A}}\mathbf{G}_R\vec{\mathbf{A}}^t\vec{\pi} = -\vec{\mathbf{A}}_P\vec{\mathbf{P}} \tag{38}$$

The matrix $\vec{\mathbf{A}}\mathbf{G}_R\vec{\mathbf{A}}^t$ is actually the 'stiffness matrix', and the element $[\vec{\mathbf{A}}\mathbf{G}_R\vec{\mathbf{A}}^t]_{ij}$ is the sum of the conductances of the rods that meet both joint 'i' and joint 'j' (in the case i=j it equals the sum of conductances of all the rods meeting joint 'i').
It is important to note that a dual method of the CCM, i.e. RCM, for analyzing indeterminate trusses does not exist in the graph representation. This is due to the fact that determinant of the conductance matrix of each rod is equal to zero and it has no inverse. However in section 2.5 it was shown that a dual to CCM method for trusses does exist when the trusses are represented by RMR.

5 A GLOBAL MULTIDISCIPLINARY PERSPECTIVE.

One of the immediate contributions made by MCA is enabling one to obtain a global perspective on various disciplines. The general perspective is achieved by applying the same combinatorial representation to different engineering fields. For example, Figure 26a shows a complex system composed of interacting dynamic, electric and indeterminate truss elements. Even though the system contains different types of engineering systems, the integrated system is represented by one resistance graph as shown in Figure 26b. Therefore all the different elements are dealt with in the same way when an MCA analysis algorithm is applied .

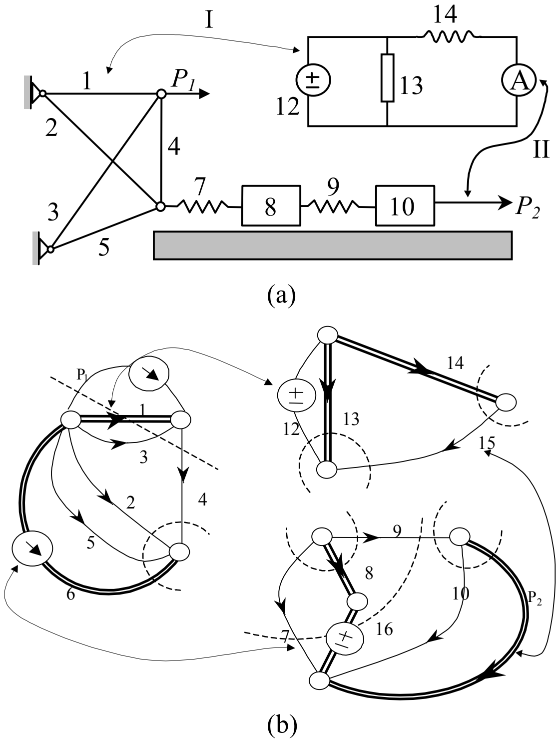


Figure 26. Representing integrated engineering system with RGR.
(a) The integrated system. (b) corresponding RGR.

6 CHECKING THE VALIDITY OF ENGINEERING SYSTEMS ON THE BASIS OF COMBINATORIAL REPRESENTATIONS.

This section shows a further contribution of MCA, which is the ability to check the validity of the engineering systems, before applying to them the analysis process or starting to manufacture the products. The idea behind this issue is the same as the one behind all the rest MCA applications, which employed the knowledge embedded in the CR. In the current section this knowledge is applied to check whether there exists a contradiction between the representation of the engineering system and the rules and theorems embedded in the CR.

The current section is concerned with checking the validity of: truss topology and geometry; planetary gear systems and geometric constrained in CAD systems.

6.1 Checking the Topological Validity of Trusses .

In section 2.2.2 it was explained how to represent trusses by the FGR, and the knowledge embedded in the CR was applied for analyzing. In this section, it is shown how to use the properties of the CR to check the validity of the topological rigidity. This issue contributes to other engineering fields, such as: checking the validity of mechanisms and geometric constraints in CAD.

The checking of the validity of a determinate truss is performed on its corresponding combinatorial representation – the FGR (section 2.2). Whenever, the analysis equations obtained from the flow graph are not soluble, this indicates that the truss represented by it is not stable. The word 'rigidity' is used in the paper when referring to the truss structure without its supports, and 'stability' for the one including the supports. In order to check the validity of truss support as well the following steps are to be performed:

1) Create two extra vertices called 'X' and 'Y' and connect them with an edge.
2) For every pinned support create two edges connecting the vertex corresponding to the support, with the X and Y vertices.
3) For every mobile support create an edge connecting the corresponding vertex with X (or Y) if the support is immobile on the vertical (or horizontal) plane, respectively. If the support is mobile on some

inclined plane, create an additional vertex and connect it using three edges, to the vertices named X and Y and to the vertex corresponding to the support.
Note that edges representing applied loads do not affect the topological consistency of the graph, so they are to be removed from the graph when the validity is checked.
Figure 27b shows the graph that corresponds to the truss of Figure 27a. Since pinned-joint ‘a’ is connected to a fixed support, two edges, one for each coordinate, appear in the graph, and for the mobile support ‘d’ there is only one edge ‘dx’.

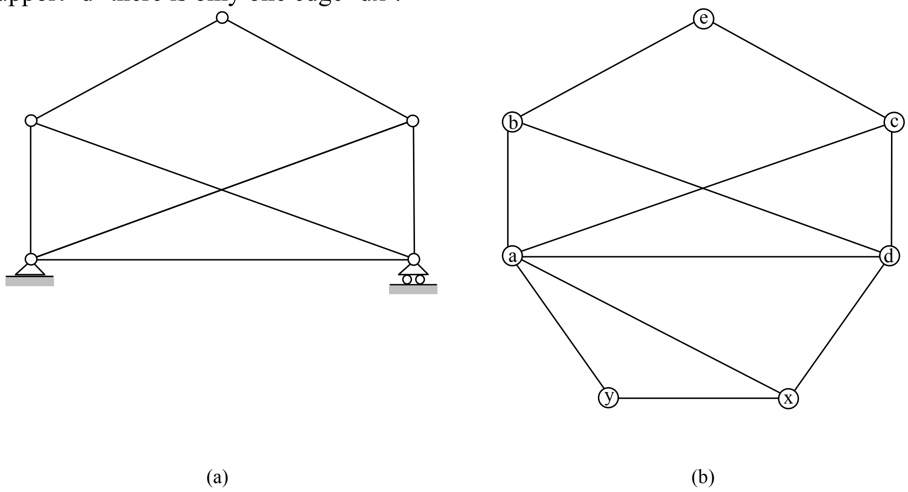


Figure 27. Example for checking the stability of determinate truss using FGR. (a)A determinate truss.(b) the graph that represents it.

6.1.1 RELEVANT THEOREMS EMBEDDED IN THE FGR.

Most of the published literature on the subject of rigidity of trusses deals with determinate trusses (Laman 1970). There then exists a fixed relation between the number of rods e and pinned-joints v, or in the terminology of the graph,

$$e(G) = 2 * v(G) - 3 \qquad (38)$$

Maxwell (1864) proved that if the relation $e(G') \leq 2*v(G') - 3$ holds for every sub-graph G’ of G, then the corresponding determinate truss is rigid. About 100 years later, Laman (1970) proved that this condition is not only necessary, but also sufficient.
The connection between the rigidity of determinate and indeterminate trusses is established by the following theorems.
Theorem. Let G be a graph that corresponds to an indeterminate truss. Then G is rigid if and only if (iff) there exists in G a connected subgraph G’ which includes all the vertices of G.
Proof: If G’ is a determinate truss and is rigid, adding edges (rods) does not affect the property of rigidity. The inverse connection between G and G’ follows directly from the definition of an indeterminate truss. Suppose that in G there are k redundant rods, G then is said to have a redundancy of k. When deleting those k edges from the graph of G, the truss represented by G’ remains determinate.
For a determinate truss, the necessary and sufficient condition for checking whether the truss has a well-formed topology is shown in the following theorem.
Theorem. A determinate truss is rigid if and only if (iff) when doubling each edge in turn in the corresponding graph, all the edges can be covered by two edge disjoint spanning trees.
From this theorem one can derive the algorithm of section 6.1.2 for checking the well-formedness of the topology of determinate trusses.

6.1.2 ALGORITHM FOR CHECKING THE WELL-FORMEDNESS OF THE GRAPH OF A DETERMINATE TRUSS.

1) Build the graph corresponding to the truss as was explained in section 3.1.
2) For every edge in the graph do:

double the edge and search for two edge disjoint spanning trees by using known algorithms (Swamy and Thulasiraman 1981).

3) If step 2 is successful for every edge in the graph, then the graph has a well-formed topology, otherwise not.

For example, Figure 28 shows a truss (a) and its corresponding graph (b). It can be proved to be stable, since when doubling each edge in turn, it has two edge disjoint spanning trees. Figure 28c shows an example of two edge disjoint spanning trees covering the graph when edge '1' is doubled.

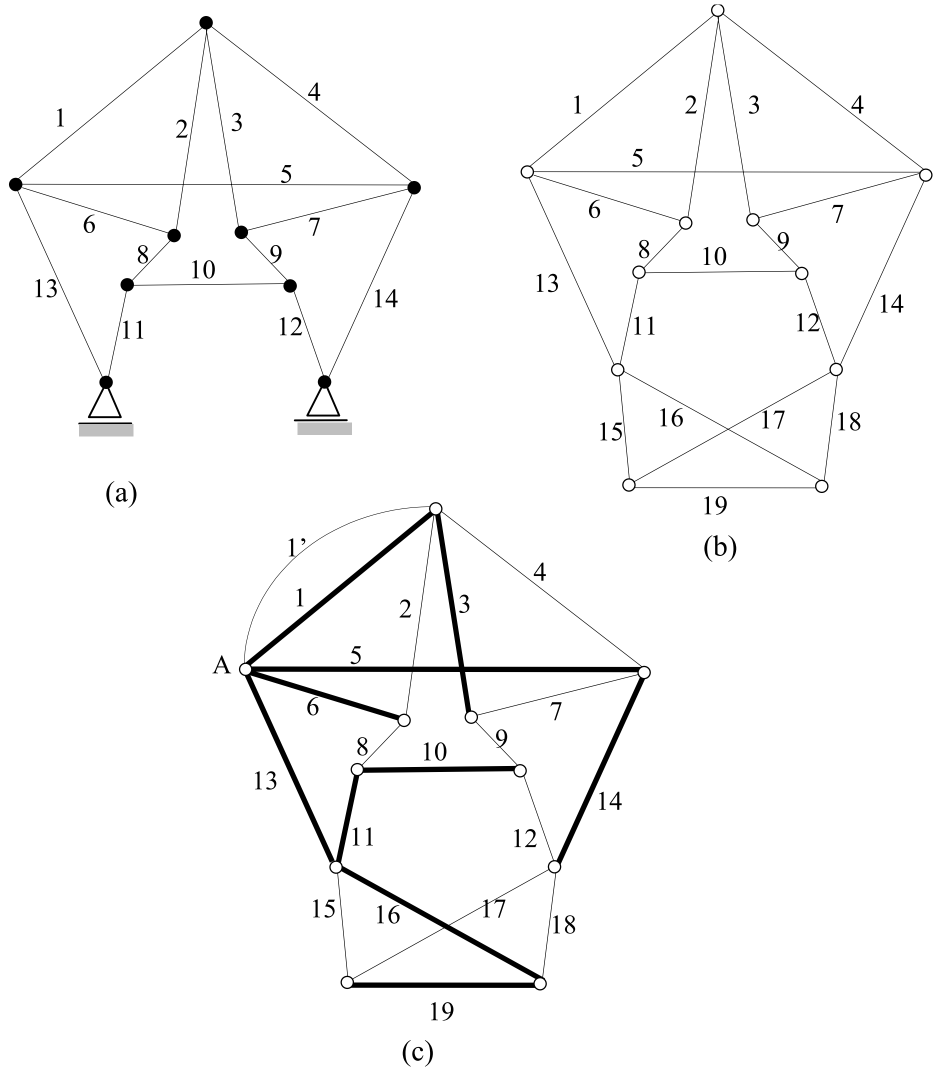


Figure 28. Example of a proof that a determinate truss (a) is stable.
(a). The truss. (b). The corresponding graph. (c). Two edge disjoint spanning trees when doubling edge '1'.

More details on checking the validity of determinate trusses can be found in (Shai and Preiss 1999a).

6.2 Checking the Validity of Dynamic Systems.

A similar process as above for trusses can be applied to a dynamic mass-spring-damper oscillator system. In this section, it will be shown that one can find a contradiction in the topological structure of a dynamic system with given initial conditions, by analyzing the RGR. Given a dynamic system with initial conditions, there can be a solution only if its graph is consistent with the validity rules, i.e. there is no contradiction between the graph topology and the validity rules.

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6.2.1 THE RGR OF THE DYNAMIC SYSTEM.

Using the information given in Table 2, Table 3 and Table 4 , the resistance graph corresponding to a dynamic system can be built.
Here, we shall only expand the representation steps of expressing the initial conditions in the graph representation, as follows:

1) Every spring with initial tension will be represented by two parallel edges. On the basis of the superposition principle, one edge will represent the flow source with the value of the initial tension of the spring and the other will represent the flow (force) change in the spring caused by the changes of the dynamic system.
2) Every mass with initial velocity will be represented by two serial edges. Based on the superposition principle, one edge represents the source of potential difference with a value equal to the initial velocity value of the mass, and the other represents the change in potential (velocity) with time in the dynamic system.

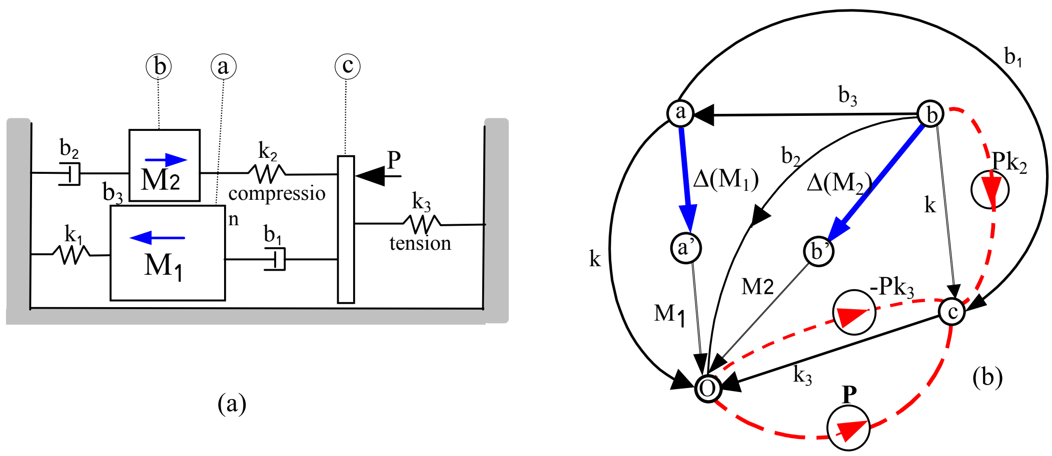


Figure 29. Representing a dynamic system with RGR.
(a) A dynamic system. (b) its corresponding graph.

6.2.2 THE METHOD FOR CHECKING THE VALIDITY OF THE DYNAMIC SYSTEM WITH INITIAL CONDITIONS.

In order that Flow Law and Potential Law will be satisfied, one has to check the following two validity checking rules:
validity rule of cutsets: there should not be a cutset of only flow sources (bold dashed edges).
validity rule of circuits: there should not be circuits consisting only of potential sources (bold solid edges).
The reason for these restrictions is derived from the property of the source edge. For example, if there were a cutset of only flow source edges the sum of flows over a cutset might not be equal to zero, in contradiction with the Flow Law. A similar reason holds for the potential source edges.
An example of a dynamic graph representation that contradicts the cutset validity rule is shown in Figure 30. The contradiction occurs because the graph of Figure 30 has a cutset with only bold dashed lines. For such a graph the initial forces around junction 3 might possibly not satisfy the condition of equilibrium of forces.

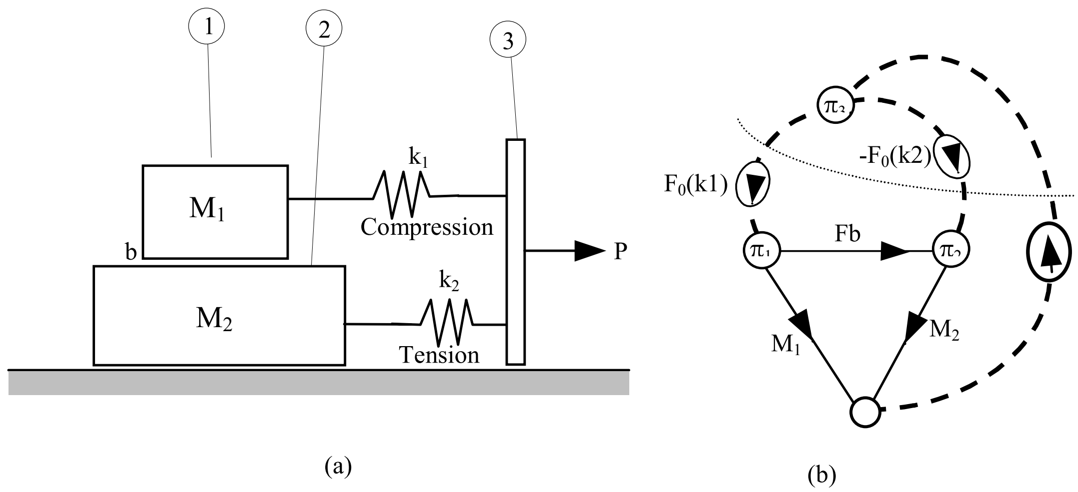


Figure 30. **An example of dynamic system with initial conditions that contradicts the syntax rules of the graph.**

6.3 Checking the Validity of a Planetary Gear System using the LGR.

As it was explained in the introduction, one of the aims of this approach is to derive CR and perform the reasoning on it. One of the purposes of that, is to enable to use the knowledge embedded in the CR. The work reported in this section, employs this possibility, and the domain knowledge consists of the topological validity rules of the graph. Therefore, the process of checking the validity of planetary gear systems becomes a process of checking whether there exists a contradiction between the domain knowledge and the graph representation of the given system.

6.3.1 TOPOLOGICAL VALIDITY RULES OF THE GRAPH.

Part of the embedded properties in the graph representation of the planetary gear system, given below, are based on Erdman (1993), who published a set of necessary conditions which he used for a different purpose: for mechanism synthesis. The paper uses this knowledge to deduce the validity of the system.

Rule 1: Planetary gear system is a kinematic chain → There is no circuit formed exclusively by turning edges.

Rule 2: Circuit of turning edges → locked mechanism OR kinematic chain with degree of freedom greater that 1.

Rule 3: Every link has at list one element around which it → every vertex is incident to at least one turning edge.

Rule 4: Each connected gear pair should operate while a constant distance between their centers is preserved → subgraph of the turning edges forms a connected subgraph

Conclusion 1: Rule 1 AND Rule 2 → the turning edges constitute a spanning tree.

Rule 5: Since each gear pair is located on a different turning edge level AND the distance between the centers should maintain constant → there is one and only one planet carrier in the fundamental circuit defined by this gear pair.

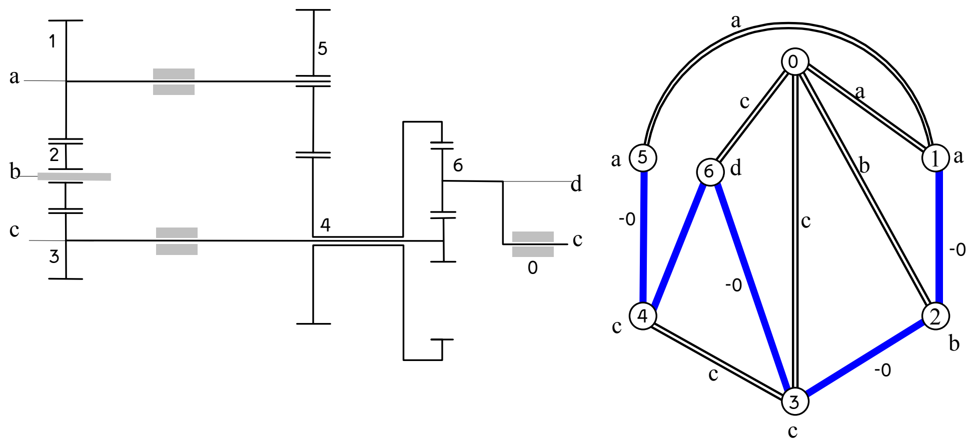
Rule 6: planet carrier = local reference vertex → In each fundamental circuit, there is one and only one local reference vertex .

Rule 7: The geometric center of a gear wheel and its local center of rotation must coincide → In each fundamental circuit, the levels of the vertex representing a gear wheel and the turning edge incident to it must be identical.

With this representation, checking the validity of the system becomes a problem of checking whether there is a contradiction between the domain knowledge (in this case the embedded properties and propositions) and the representation of the given system.

Example of deducing the validity of planetary gear system

The computer program using this representation (Polomodov and Gershon 1995; Preiss and Shai 1996) found that the system in Figure 31 is not valid, and explained why. In addition, it is possible to arrange that the computer program advises the designer what to change in the gear kinematic chain, in order to make it valid.



The system is not valid because there is a contradiction with rule 4. In circuits {6,0,3} and {6,0,3,4} there is no local reference vertex. The level of the turning edge (06) is 'c' while the level of vertex 6 is 'd', which contradicts rule 5.

The explanation to the user is: *The connection between wheels 6 and 3 is not legal because the distance between their centers is zero.*
The same problem occurs with the connection between wheels 6 and 4.

Figure 31. Example of topological analysis of a planetary gear system, with the computer program output shown.

6.4 Checking the validity of constraint systems using the flow graph representation.

One of the main topics in CAD system research, is the problem of checking whether a geometric constraints system is well-constrained. In other words, to find whether the given geometric form is uniquely and validly defined. It was found, according to (Owen 1996; Hoffmann 1995) that such a constraint system can be represented by a special graph. In the terminology adopted in MCA, such a graph is actually FGR.

The steps for representing the geometric constraint system by a flow graph are as following:

each element of the geometric system, like line, point, arc etc. is represented by a vertex in the graph, each constraint represented by an edge in the graph. Therefore, an edge connecting two vertices corresponds to the constraint imposed on the corresponding two elements.

Theorem. Geometric constraint system is valid (well constrained) if and only if its corresponding graph is rigid.

Proof. Each element represented by a vertex has two unknown parameters and is actually substituted by a point in the plane, having two degrees of freedom. Each constraint relating the parameters of the elements makes the number of unknowns in the system to decrease by one. Exactly like a solid line connecting to points in the plane decreases the overall degree of freedom of the system by one. Those claims are valid not only for the whole system but also for each of the subsystems. A more detailed proof can be found in (Owen 1996; Hoffmann 1995).

Hence, the process of checking the validity of a geometric constraint system is performed as follows:

1) build the FGR representing the geometric constrained system.

2) check the rigidity of the graph using methods explained in section 6.1.

Consider for example, the geometric constraints system of Figure 32.

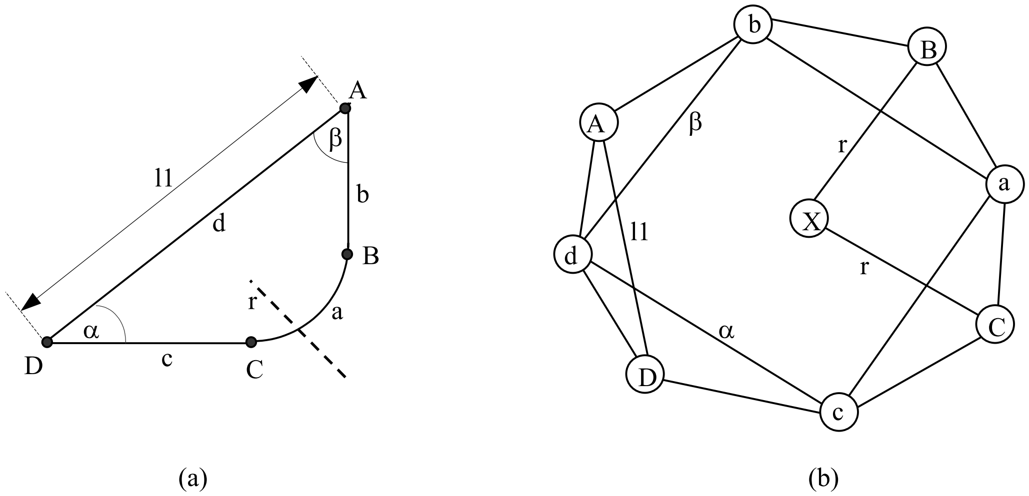


Figure 32. Geometric constraints system and the corresponding graph. (a) Geometric constraints system. (b). Corresponding graph.

In the geometric constraint system presented on Figure 32a there is one arc – ‘a’ and three straight sections ‘b’, ‘c’ and ‘d’. There are 15 constraints: eight for the interconnection between the elements; one for the distance between points ‘A’ and ‘D’ ; one for the angle between sections ‘c’ and ‘d’; one for the angle between sections ‘d’ and ‘b’; the radius of the arc ‘a’ ; two for the constraint requiring the elements ‘c’ and ‘b’ to be tangent to element ‘a’. Futhermore, vertex ‘x’ in th graph corresponds to an element not appearing on the sketch – the center of the arc. Checking whether the given data defines a well constrained geometric system of Figure 32a requires first to build the corresponding flow graph. Each element is represented by a vertex and each constrained by an edge. Such a graph appears on Figure 32b.

It can be checked by one of the methods given in section 6.1 (for instance, the two edge disjoint spanning trees method), that the flow graph presented on Figure 32b is determinate and rigid. Thus by the theorem presented in the beginning of the section, the original system of Figure 32a is valid.

6.5 Employing the connections between the CR in checking the validity of engineering systems.

One of the main contributions of MCA, is the ability that it provides to use knowledge from one field in other fields on the basis of the connections between CR. This property is used in this section to turn the validity checking of the mechanism mobility into checking the rigidity of the its dual truss and vice versa. This new way enables to use knowledge and theorems from machine theory in structural analysis and vice versa.

6.5.1 Using the duality relation to check the rigidity of determinate trusses.

On the basis of the mutual dualism between trusses and mechanisms (section 3.1.2), one can deduce the following rule:

dualism validity rule: determinate truss is valid if and only if its dual mechanism is valid, in other words: determinate truss is rigid if and only if its dual mechanism configuration is mobile.

Hence instead of checking the stability of the truss directly, one can build its dual mechanism and the problem will become a problem of checking the mobility of a mechanism. In many cases checking the instant mobility of mechanisms can be carried out quickly by applying known theorems from mechanism and machine theory.

Consider as an example the truss presented on Figure 33a. While it is not easy to conclude about its rigidity, its dual mechanism Figure 33b is obviously stuck. Therefore, the original truss is not rigid.

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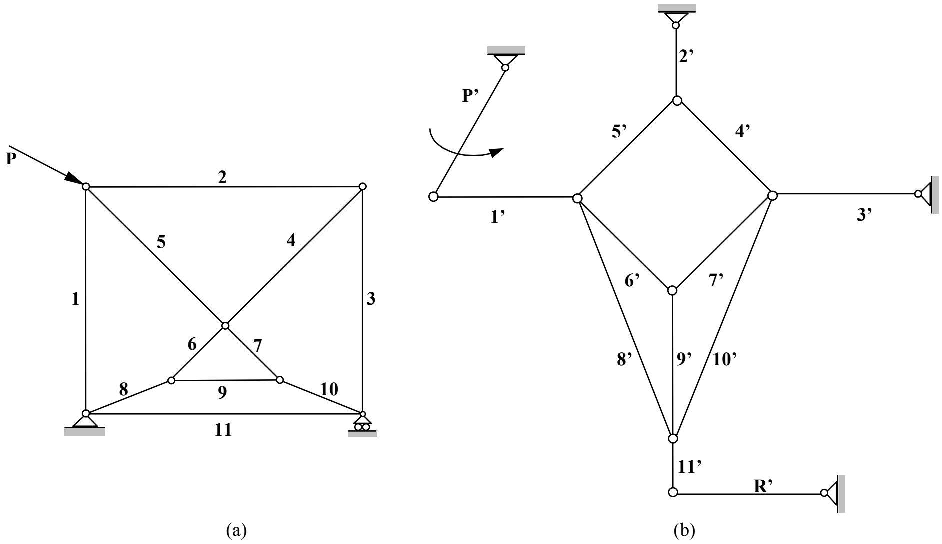

Figure 33. Checking the validity of truss, by checking the validity of its dual mechanism. (a) a truss. (b) its dual mechanism.

6.5.2 CHECKING THE MOBILITY OF MECHANISMS USING STRUCTURAL ANALYSIS.

Previous section has adopted the principle, that a truss is valid if and only if its dual mechanism is also valid. The principle was used in order to check the stability of trusses, by checking the mobility of their dual mechanisms instead. Current subsection demonstrates the dual possibility: it checks the mobility of a mechanism by means of stability of its dual truss:

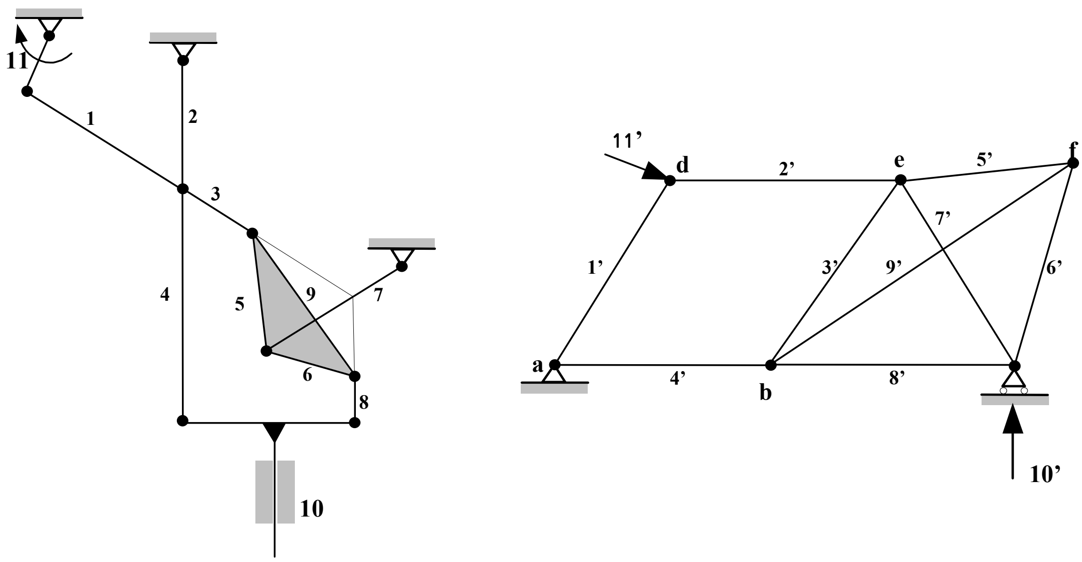

Figure 34. Checking the validity of mechanism by checking the validity of its dual truss. (a) A mechanism. (b) Its dual truss.

Consider for example the mechanism presented on Figure 34a. It is difficult, even for experts in mechanisms, to decide whether the mechanism in Figure 34a is mobile or stuck.
On the other hand, its dual truss presented in Figure 34b, obviously possesses redundancy in its right part, so its left part lacks rods and hence the whole truss is unstable. Thus, the original mechanism of Figure 34a is not valid, i.e. stuck.

7 USING MCA FOR DESIGN.

Design of special trusses, using special mechanisms.One of the main contributions of MCA, is the ability to use knowledge from one field in the other fields. This property is achieved by using the connections between individual CR. This section, shows a new application of this property, which is

developing a new technique in design. The example given here, is based on using the connection between trusses and mechanisms, established in section 3.
The main idea behind this approach lies on the fact that when a mechanism has a special property, its dual truss should also have that property, and vice versa. This idea is demonstrated on the following small example.
Suppose one needs to design a truss, such that when a small force is applied to one of its joints, a magnified force is produced in a specific rod. Applying the approach transforms the problem into a problem in the dual mechanism.
This is done by first finding a known mechanism having similar velocity characteristics. Namely, a mechanism that for a small relative velocity in its driving link produces in its other link a much greater relative velocity. One of many mechanisms satisfying this requirement is presented on Figure 35a. The relative velocity of link ‘1’ of this mechanism is considerably larger than that of the link 5. The truss dual to this known mechanism is presented on Figure 35b. According to the duality property the truss possesses the same force characteristics as the velocity characteristics of the mechanism, i.e. a small external force F causes a much greater force in the rod ‘1’.

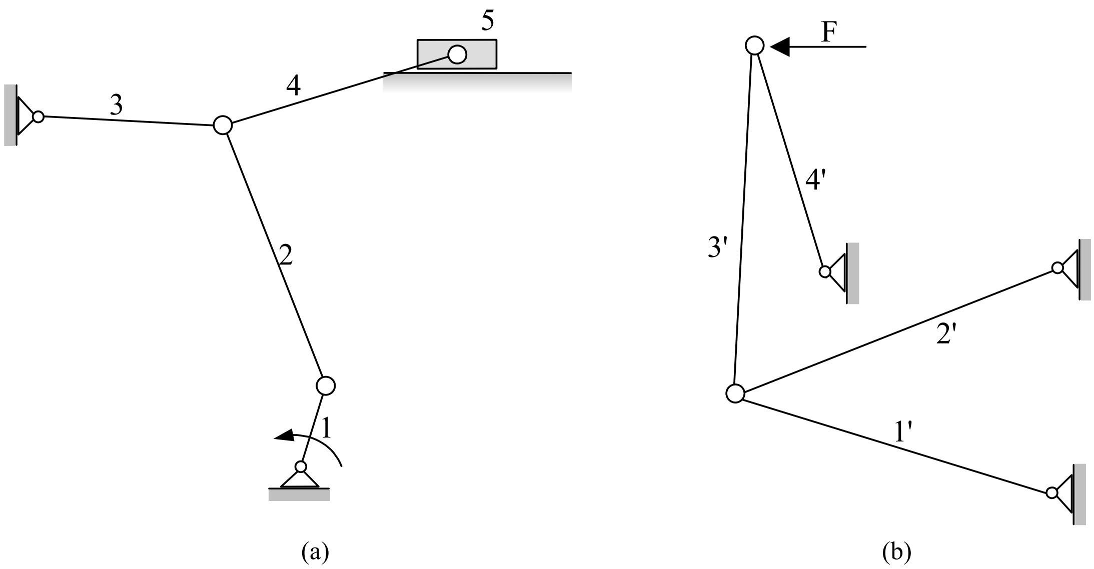

Figure 35. Employing truss-mechanism duality in design.
(a) A known mechanism design. (b) Its dual truss with the same properties.

CONCLUSIONS.

The paper has introduced the idea behind the MCA (Multidisciplinary Combinatorial Approach), which was implemented as follows: first combinatorial representations were developed and the properties of each and their interrelations were thoroughly investigated. Afterwards, these CR were applied to represent engineering problems from different fields, which gave raise to interesting results. Part of these results appeared in the paper.

The paper has shown that representing engineering problems by CR enables to get a general perspective on different engineering fields. Moreover, new relations between engineering fields have been derived. This issue has been demonstrated by introducing a new connection between mechanisms and trusses, which had been derived from the relation between their corresponding CR: PGR and FGR.

The theorems and methods embedded in the CR have been found to be valuable both for theoretical research and practical applications. From the theoretical point of view, they enable to derive theorems and methods in engineering. For instance, known methods, such as the displacement method in structures, have been proved to be special cases of the methods embedded in RGR. On this base, connections between known methods have been derived as well. From the practical point of view, it makes possible to apply knowledge, algorithms and methods from one field to others.

In addition, the paper has obtained a general perspective that enables to represent multidisciplinary systems as one whole and to deal with them in a unified way.

The concept of MCA has reaffirmed the postulate that when encountering a difficult problem, an effective solution strategy is to change its representation so as to make its solution transparent.

NOTATIONS.

$\mathbf{0}$	zero matrix.
$\mathbf{A}$	incidence matrix.
$\mathbf{B}$	scalar circuit matrix.
$\overline{\mathbf{B}}$	vector circuit matrix.
$\mathbf{B}(\mathrm{M})$	circuit matrix of a matroid.
$\mathbf{C}$	set of matroid circuits.
$\mathbf{D}$	vector of scalar displacements in truss elements.
$\dim(\vec{\mathrm{F}})$	dimension of the forces acting in the truss.
E	set of graph edges.
e(G)	number of edges in graph G.
$\vec{\mathbf{F}}$	flow vector.
$\vec{\mathrm{F}}(\mathrm{e})$	flow in edge e.
F(e)	value of the flow in edge e.
$\mathbf{F}$	independent subsets of matroid.
G	graph.
G*	the dual graph of graph G.
G_F	flow graph
G_Δ	potential graph
G(e)	scalar conductance of edge e.
$\mathbf{G}(\mathrm{e})$	matrix conductance of edge e.
G_R	resistance graph.
$\mathbf{G}_R$	square matrix containing the conductances of the resistance edges of a graph
$\mathbf{G}_{T'}$	conductance cutset matrix.
$\mathbf{G}_\Delta$	conductance cutset matrix of the potential difference sources.
M	matroid.
M_Q	matroid defined by matrix **Q**.
$\vec{\mathbf{P}}$	vector of flows in the flow sources.
$\mathbf{Q}$	scalar cutset matrix.
$\vec{\mathbf{Q}}$	vector cutset matrix
$\mathbf{Q}(\mathrm{M})$	cutset matrix of a matroid.
$\hat{\mathrm{r}}(\mathrm{e})$	unit vector in the direction of edge e.
R(e)	scalar resistance of edge e.
$\mathbf{R}(\mathrm{e})$	matrix resistance of edge e.
$\mathbf{R}_{C'}$	resistance circuit matrix.
$\mathbf{R}_P$	resistance circuit matrix of the flow sources.
$\mathbf{R}_R$	square matrix containing the resistances of the resistance edges of a graph.
S	underlying set of a matroid.
T	spanning tree branches.
T'	spanning tree branches, which are not sources.
$\mathbf{T}$	set of matroid bases.
$\vec{\mathrm{V}}_{A/B}$	linear velocity of joint A relative to joint B.
V	set of the graph vertices.
v(G)	number of vertices in graph G.
$\vec{\Delta}$	potential difference vector.
$\vec{\Delta}(\mathrm{e})$	potential difference in edge 'e'
$\pi(\mathrm{i})$	potential of vertex 'i'
$\varnothing$	empty set
CCM	Conductance Cutset Method
CR	Combinatorial Representations
FGR	Flow Graph representation

PGR	Potential Graph Representation
LGR	Line Graph Representation
RCM	Resistance Circuit Method
RGR	Resistance Graph Representation
RMR	Resistance Matroid Representation

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