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Graph theoretical duality perspective on conjugate structures and its applications

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Abstract

The paper shows that representing structures, beams and frames by mathematical models based on graph theory enables to provide new perspective on known conjugate structure theorems in mechanics. It is shown that the latter theorems can be derived from the graph theoretical duality principle applied upon the graph representations of the structures. The results reported indicate upon theoretical value of the approach, as the established mathematical foundation can be employed in a variety of mechanical disciplines.

Keywords: Conjugate structures; Duality; Graph representations; Structures; Graph theory

1. Introduction

The approach adopted in this paper uses discrete mathematical models, called graph representations (Shai, 2001), to reflect the structure, behavior and properties of the engineering systems from various engineering disciplines. The structure of an engineering system is transformed into a graph augmented by additional mathematical properties, thus enabling to substitute the process of reasoning over the engineering system by reasoning over the corresponding graph representation. Different types of graph representations have been studied and developed upon a thorough investigation of different graph theory applications to engineering that were reported in the literature. The main engineering domains that were examined are: machine theory, where graph representations have been used for analysis and enumeration of mechanisms (Freudenstein and Maki, 1979), in dynamics for analysis of multidimensional dynamic systems (McPhee, 1996), and in structural mechanics where the pioneers are Fenves and Branin (1963).

Current paper takes advantage of the relation between graph representations and skeletal structures that has been established in Fenves (1966), Kaveh (1991), Wang and Bjorke (1991), Ta'aseh and Shai (2002). The duality relation is applied here to the graph representation of the skeletal structure, resulting in a systematic derivation of the conjugate theorems.

The term duality that is employed in this paper is widely used in the literature for defining several different issues. Engineering and science account for a number of known duality principles (Girvin, 1996) that may intuitively seem similar to the

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Nomenclature

asterisk*	duality transformation
A	angular
dL	segment of infinitesimal length
Δ	potential difference/belonging to potential graph
$\vec{\mathbf{\Delta}}(G)$	vector of potential differences in graph G
E	modulus of elasticity
F	flow/belonging to flow graph
$\vec{\mathbf{F}}(G)$	vector of flows in graph G
FS	dependent flow source edge
G^{F}	Flow Graph Representation (FGR)
G^{Δ}	Potential Graph Representation (PGR)
GR	Graph Representation
$\mathbf{H}$	hybrid relation matrix
I	moment of inertia
L	linear
P	external force
PS	dependent potential source edge
$\boldsymbol{\Pi}$	potential measuring edge
$\mathbf{R}$	resistance relation matrix
R	resistance edge

duality underlying this paper, but actually possess different mathematical basis. In order to avoid confusion, a special notion for this term is provided in this paper. Two such principles should be mentioned here, as they are sometimes mistakenly considered to coincide with the approach adopted in this paper. One of the widely known duality principles in engineering practice is the statical-kinematical duality (McGuire and Gallagher, 1979). This principle finds its origin in the virtual work theorems which result in the orthogonality between the kinematic and static variables underlying the behavior of the same structure. The mathematical basis in this case is linear algebra, and the relation can be traced through similarities in the corresponding matrices. Rigorous discussion of the role of the duality principle in mathematics in general and discrete mathematics in particular has been conducted in Recski (1989), Iri and Recski (1982).

The principle of duality is also widely used in screw theory (Phillips, 1984), and in the aspect of mechanical networks and screw theory (Davies, 1983). The duality relation is also used between serial and parallel manipulators (Waldron and Hunt, 1991; Gosselin and Lallemand, 2001). These works show that the equations underlying the kinematics and statics of these two types of manipulators are the same, which makes these two systems to be dual to each other. An in-depth study of this relation contributed to a better understanding of the properties of parallel and serial manipulators (Duffy, 1996). The term duality is also widely used in projective geometry where a line corresponds to a point (Pedoe, 1963) and was employed to prove the duality between plane trusses and grillages (Tarnai, 1989).

2. Graph Representations and relations between them

The representations were developed after a thorough investigation on the use of linear graph theory in engineering and combining the results of several central works done in the field. In order to avoid confusion, it should be noted that the term 'graph representation' refers in this paper only to the models developed on the basis of the linear graph theory, so the claims made here should not be automatically projected upon other fields employing this term, such as flowgraphs (Ward, 1968) where a graph represents a system of equations (Lorens, 1964). The graphs used in the paper comprise the information on the behavior of the engineering systems in additions to their structure thus the equations are derived directly from the graph opposed to other types of graphs appearing in the literature, including block diagrams (Cha et al., 2000) and bond graphs (Paynter, 1961).

The use of graph theory in engineering is well known, and this paper employs some of the results developed in the various engineering disciplines, such as: electricity – the works of Kron (1963), Seshu and Reed (1961) and others for representing and analysis of electrical circuits; In machine theory – the graphs were used to represent linkages and gear trains for analysis or enumeration, such as: Freudenstein (Dobrjanskyj and Freudenstein, 1967; Freudenstein and Maki, 1979), Tsai (Tsai et al., 1998); in dynamics the graphs that were used to represent multidimensional dynamic systems upon which to perform the analysis (Andrews, 1971; McPhee, 1996); in structural mechanics – network graphs were studied in order to develop computer software for analysis (Fenves and Branin, 1963) who implemented the first computer program from the network topology and graph algorithms (Fenves et al., 1965).

Representing different engineering systems with bond graphs (Paynter, 1961) also yielded a variety of practical applications including simulation (Wilson, 1992) and design (Finger and Rinderle, 2002). Bond graph representation has not yet been incorporated within the work introduced in this paper, as till now it has focused on employing linear graph theory.

Graph Representation is basically a graph augmented by special mathematical properties and by relations to other graph representations. Each type of graph representation can be used to represent a number of engineering domains, to which this representation is related through special construction rules (Shai, 2001). When applied to represent an engineering system, the elements and properties of the system are mapped, in an isomorphic manner, into the elements and the properties of the graph representation. Graph Representations are classified into several types, in accordance to their augmented properties.

Current paper focuses on only three types of graph representations: Potential Graph Representation (PGR), Flow Graph Representation (FGR) and the Resistance Graph Representation (RGR). In previous publications (Shai, 2001; Ta'aseh and Shai, 2002) these representations were shown to be applicable to represent various forms and aspects of structures.

It was shown and proved (Shai, 2001) that the three representations possess strong mathematical relations between one another:

Relation 1. RGR contains properties of both FGR and PGR. In other words, PGR and FGR reflect two different aspects comprising RGR. This property will be extensively elaborated in Section 4.

Relation 2. Graph duality property (Swamy and Thulasiraman, 1981) yields a duality relation between PGR and FGR. The rule underlying this relation can be stated as follows: for each graph G^{Δ} of type PGR, there exists a dual graph G^{F} of type FGR, so that the potential differences of the edges of the former are equal to the flows in the corresponding edges of the latter. Formally, this is expressed as:

$$(G^{\mathrm{F}})^* = G^{\Delta},$$
$$\vec{\mathbf{F}}(G^{\mathrm{F}}) = \vec{\mathbf{\Delta}}(G^{\Delta})$$

where asterisk superscript stands for the duality transformation, $\vec{\mathbf{F}}(G^{\mathrm{F}})$ is the vector of flows in G^{F}, and $\vec{\mathbf{\Delta}}(G^{\Delta})$ – vector of potential differences in G^{Δ}.

The properties of the duality relation, as were established in Shai (2001), can be summarized as follows: for each edge in PGR there exists a corresponding edge in its dual FGR; for each vertex in one graph representation there is a corresponding face in the dual representation; the flow through an edge in FGR is equal to the potential difference of the corresponding edge in the dual PGR; for each cutset in one representation there exists a dual circuit in the dual representation.

General description of a structure usually requires to list all the mathematical equations underlying both the statics and the kinematics of the system, thus it can fully be represented by means of RGR. Through Relation 1 the graph representation of the structure can be reduced to either PGR and FGR. The possibility of constructing the dual representations of these FGR and PGR and then interpreting them as representations of some other engineering systems is examined throughout this paper.

3. Representing beams with graph representations and deriving the conjugate beam theorem

The aim of this section is to develop an engineering system that is dual to a general beam. As it was reported in Ta'aseh and Shai (2002), the representation suitable for a beam is the Resistance Graph Representation (RGR). Using Relation 1, the RGR of a beam can be reduced to a PGR. Then, the duality Relation 2 can be applied to build the FGR dual to this PGR. The obtained FGR would be interpreted as an engineering system – the dual system of the original beam.

In this paper the original structure (a beam or a frame), is represented by PGR which is then transformed into the dual FGR of the corresponding dual system.

3.1. Resistance graph representation of a beam

The approach in this section is applied to the beam shown in Fig. 1. Throughout the paper it will be referred to as *the original beam*.

Firstly, the RGR of the beam is shortly explained on the basis of the previous work done by the authors (Ta'aseh and Shai, 2002). The representation will first be developed for a single infinitesimal segment of the engineering system. The representation of the whole system will then be obtained by interconnecting the representations of the segments.

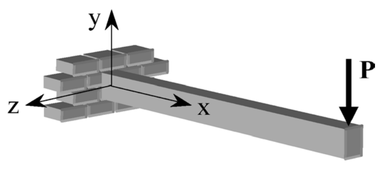


Fig. 1. The original beam to which the approach is applied.

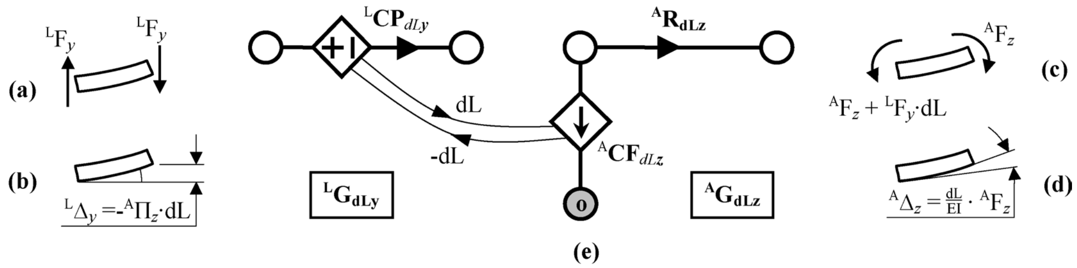


Fig. 2. Resistance graph of an infinitesimal beam segment. (a) Linear force. (b) Linear displacement. (c) Angular force. (d) Angular displacement. (e) The corresponding resistance graph.

A horizontal beam segment of an infinitesimal length, $\mathrm{d}L$, shown in Fig. 2(a)–(d) with the corresponding variables is represented by the RGR shown in Fig. 2(e). The graph, related to the segment $\mathrm{d}L$, is composed of two parts: a linear graph, ${}^{\mathrm{L}}\mathbf{G}_{\mathrm{d}L}$, and an angular graph, ${}^{\mathrm{A}}\mathbf{G}_{\mathrm{d}L}$ (throughout the paper, the rear-superscript 'L' will stand for linear, or translational aspects, while 'A' will stand for angular, or rotational aspects). The linear graph deals with representing the translational deformations and the linear forces of the segment. It should be noted that in this paper the term 'linear graph' differs in its meaning from the accepted meaning in graph theory, for example in Deo (1974). The angular graph, in contrast to the linear graph, represents the angular deformation and the bending moment (angular force) of the segment.

In the beam segment of Fig. 2, linear forces and displacements are directed in parallel to the y axis, while angular forces and displacements are directed in parallel to the z axis. Therefore, each of the two graphs is one-dimensional and will be denoted by its relevant dimension in the subscript, ${}^{\mathrm{L}}\mathbf{G}_{\mathrm{d}L\boldsymbol{y}}$ and ${}^{\mathrm{A}}\mathbf{G}_{\mathrm{d}L\boldsymbol{z}}$ as is shown in Fig. 2(e).

Both graphs contain an edge representing the segment, the end vertices of which correspond to the interconnections of the segment to its neighbor segments. The two sides of the segment with which the segment interfaces to the rest of the beam are arbitrary assigned to be the tail side (or joint) and the head side (or joint) of the segment. The potential difference and the flow of the segment edge in the angular graph, ${}^{\mathrm{A}}\mathbf{G}_{\mathrm{d}L\boldsymbol{z}}$, correspond to relative angular deflection and the bending moment, respectively. The potential difference and the flow of the segment edge in the linear graph, ${}^{\mathrm{L}}\mathbf{G}_{\mathrm{d}L\boldsymbol{y}}$, correspond to the relative linear deflection and the shear force, respectively. Additionally, the angular graph, ${}^{\mathrm{A}}\mathbf{G}_{\mathrm{d}L\boldsymbol{z}}$, contains an auxiliary edge connecting one of the end vertices of the segment edge to the reference vertex denoted by '**o**'. The purpose of the auxiliary edge is to account for the moment produced by the shear force of the segment.

As can be seen from Fig. 2, the two graphs are related through dependent sources designated by CP and CF. These sources force the following relations between the variables of the two graphs:

Potential-Controlled Potential Source (PCPS, or CP in short):

$$ {}^{\mathrm{L}}\boldsymbol{\Delta}^{\mathrm{CP}}_{\mathrm{d}Ly} = \mathbf{H}^{\Delta}_{\mathrm{d}Lyz} \cdot {}^{\mathrm{A}}\boldsymbol{\Delta}^{\mathrm{CF}}_{\mathrm{d}Lz} \quad \text{with } \mathbf{H}^{\Delta}_{\mathrm{d}Lyz} = -\mathrm{d}L_x \tag{1} $$

where ${}^{\mathrm{L}}\boldsymbol{\Delta}^{\mathrm{CP}}_{\mathrm{d}Ly}$ is the potential difference in the dependent potential difference source in the linear graph, and ${}^{\mathrm{A}}\boldsymbol{\Delta}^{\mathrm{CF}}_{\mathrm{d}Lz}$ is the potential difference in the dependent flow source edge in the angular graph.

Flow-Controlled Flow Source (FCFS, or CF in short):

$$ {}^{\mathrm{A}}\mathbf{F}^{\mathrm{CF}}_{\mathrm{d}Lz} = \mathbf{H}^{\mathrm{F}}_{\mathrm{d}Lzy} \cdot {}^{\mathrm{L}}\mathbf{F}^{\mathrm{CP}}_{\mathrm{d}Ly} \quad \text{with } \mathbf{H}^{\mathrm{F}}_{\mathrm{d}Lzy} = \mathrm{d}L_x. \tag{2} $$

Where ${}^{\mathrm{A}}\mathbf{F}^{\mathrm{CF}}_{\mathrm{d}Lz}$ is the flow in the dependent flow source edge in the angular graph, and ${}^{\mathrm{L}}\mathbf{F}^{\mathrm{CP}}_{\mathrm{d}Ly}$ is the flow in the dependent potential difference edge in the linear graph.

The elastic properties of the segment are expressed through the resistance relation between the corresponding variables in the segment edge in the angular graph. Since only the angular segment edges possess the resistance relation in the graph representation of beams, in the paper these edges will be referred as 'resistance' edges and designated by superscript R.

$$ {}^{\mathrm{A}}\boldsymbol{\Delta}^{\mathrm{R}}_{\mathrm{d}Lz} = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{R}}_{\mathrm{d}Lz} \cdot {}^{\mathrm{A}}\mathbf{F}^{\mathrm{R}}_{\mathrm{d}Lz}, \; {}^{\mathrm{A}}\mathbf{R}^{\mathrm{R}}_{\mathrm{d}Lz} = \frac{\mathrm{d}L}{EI_z}. \tag{3} $$

Where ${}^{\mathrm{A}}\boldsymbol{\Delta}^{\mathrm{R}}_{\mathrm{d}Lz}$ is the potential difference of the resistance edge in the angular graph, and ${}^{\mathrm{A}}\mathbf{F}^{\mathrm{R}}_{\mathrm{d}Lz}$ is the flow in the resistance edge in the angular graph. ${}^{\mathrm{A}}\mathbf{R}^{\mathrm{R}}_{\mathrm{d}Lz}$ is the resistance of the resistance edge in the angular graph – a constant whose value is defined in (3).

Due to the properties of the infinitesimal beam segment (McGuire and Gallagher, 1979), there is no resistance relation between the linear deformation and the shear force of the segment, thus there are no resistance edges in the linear graph.

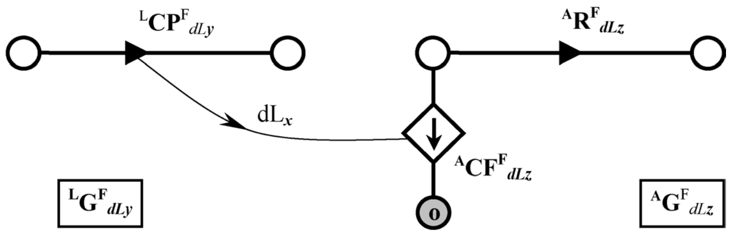


Fig. 3. Linear and angular FGR of a beam segment.

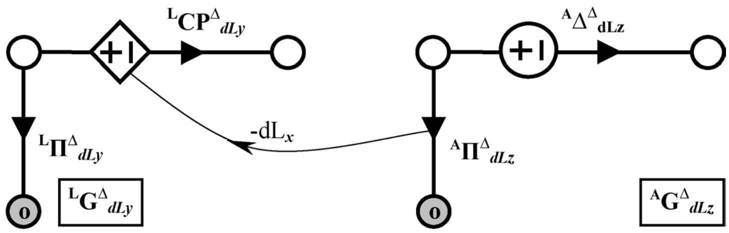


Fig. 4. Linear and angular PGR of a beam segment.

3.2. Reduction of the beam resistance graph to potential and flow graphs

According to Relation 1, RGR can be seen as a combination of two fundamental components: Potential Graph Representation (PGR) and Flow Graph Representation (FGR). Current section uses this relation to reduce the RGR of the beam segment shown above into these two fundamental graph representations. The duality relation (Relation 2) will then be applied to these graphs in order to enable finding the engineering system dual to the beam.

The FGR corresponding to the beam segment is the graph shown in Fig. 3, resulting from eliminating properties related to potential-differences from the resistance graph appearing in Fig. 2. This way, the potential difference source edge, CP, will turn into a simple flow-conducting edge. The resistance edge, R, in the angular graph also becomes a simple flow-conducting edge. Consequently, Eq. (2) is the only relation associated with the Flow Graph Representation of the beam segment.

Similarly, the PGR corresponding to the beam is obtained by eliminating flow properties from the resistance graph of Fig. 2, as shown in Fig. 4. The flow in the **CF** edge is of no relevance, and since this edge is connected to the reference vertex, thus its potential difference is equal to the angular potential of the tail vertex of the resistance edge, and therefore will be denoted by ${}^{\mathrm{A}}\boldsymbol{\Pi}$. In sake of symmetry, a similar edge, ${}^{\mathrm{L}}\boldsymbol{\Pi}$, is added to the linear graph to "measure" the linear potential of the tail vertex of the segment edge. The potential difference in the resistance edge of the angular graph was determined in the resistance graph by the flow through that edge. Since in PGR the flows through the edges are disregarded, the potential differences in these edges, as far as PGR is concerned, are considered as constant potential sources, and are calculated upon building the graph by means of Eq. (4):

$$ {}^{\mathrm{A}}\boldsymbol{\Delta}^{\Delta}_{\mathrm{d}Lz} = {}^{\mathrm{A}}\mathbf{R}^{\mathrm{R}}_{\mathrm{d}Lz} \cdot {}^{\mathrm{A}}\mathbf{F}^{\mathrm{R}}_{\mathrm{d}Lz}. \tag{4} $$

Where ${}^{\mathrm{A}}\mathbf{F}^{\mathrm{R}}_{\mathrm{d}Lz}$ are the flows obtained in the corresponding edge upon solution of the flow graph representation of the beam. Additional relation relevant to the potential graph representation of the segment is the relation ruling the behavior of the dependent potential difference source, given in Eq. (1).

As was mentioned above, the duality application is sought for through the PGR of the original beam, which is obtained by connecting the graphs of all its segments. The PGR of the beam appearing in Fig. 1 is shown in Fig. 5. Following issues should be noted with regard to this graph:

(1) Although the beam is composed of infinite number of segments and so is its representation, only 4 segments are shown in the figure for clarity.
(2) The left-most edge corresponds to the left segment of the beam that is connected to the fixed support. Therefore, its angular and linear displacements are equal to zero, thus the corresponding edge is connected directly to the general reference (grey) vertex.
(3) The external load, **P**, applied to the right end of the beam is not expressed explicitly in the graph, although its influence can be traced through the angular potential difference sources, whose values are determined through the flow graph.

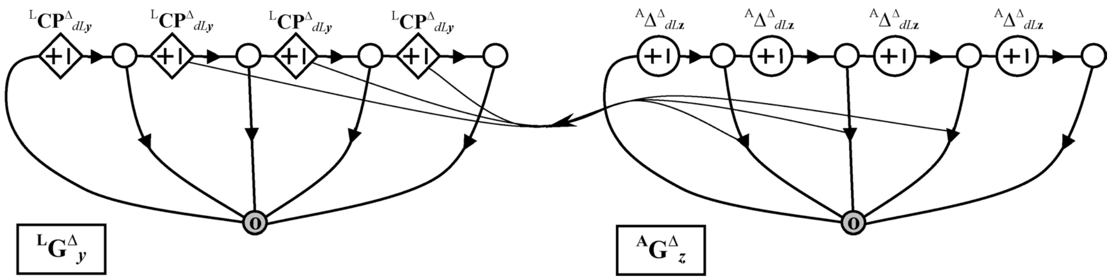


Fig. 5. Linear and angular potential graph representations of the beam.

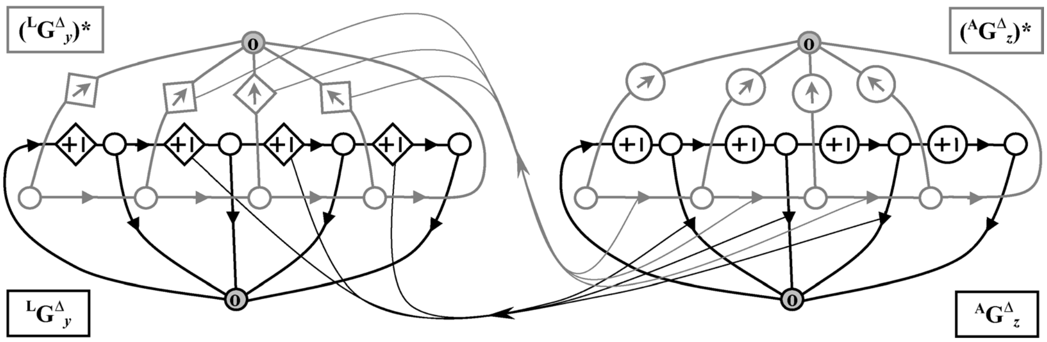


Fig. 6. The dual FGR superimposed over the original PGR of the beam.

3.3. Constructing the dual flow graph from the potential graph of the original beam

We shall now apply Relation 2 (Section 2) to the Potential Graph Representation obtained in the previous section for the original beam and derive its dual Flow Graph Representation. The transformation to the dual graph representation is done through the rules corresponding to Relation 2 described in Section 2. The dual Flow Graph Representation obtained through applying the transformation rules to the Potential Graph representation of Fig. 5 is shown in Fig. 6, where it appears in gray superimposed upon the original potential graph. According to the PGR-FGR duality, the potential difference value associated with each original edge is equal to the flow value associated with the corresponding dual edge. The direction of each dual edge is determined by the direction of the corresponding original edge according to the right hand rule.

3.4. The mechanical interpretation of the dual flow graph representation

Current section examines the FGR obtained in the previous section for correspondence with some known engineering system. It follows from Fig. 6 that the dual flow graph, like the original potential graph, is composed of a series of infinitesimal edges, and each such edge is accompanied with a flow-controlled flow source. The beam segment flow graph reduced from the beam RGR (Fig. 2) also includes an infinitesimal segment edge accompanied with a flow-controlled flow source, and possesses the same topology. This indicates that the dual flow graph is itself an FGR of some other beam which will be referred in the paper as *the dual beam*. In the continuation of this section, the process of deducing this beam from its graph representation is performed.

For convenience, the directions of the flow source edges of the dual graph, $({}^{A}G^{\Delta}_{z})^*$, are altered to match the general form of the Flow Graph Representation of the beam, without loss of consistency with the original system, as shown in Fig. 7.

From comparing the graphs appearing in Fig. 7 with those appearing in Fig. 3, it can be argued that the dual of the linear PGR corresponds to an angular FGR, and the dual of the angular PGR corresponds to a linear FGR. Following this example, one can straightforwardly deduce the relations between the variables of the original and the dual beam representations. Table 1 lists the correspondence between the terms and the variables of the two representations. The notations for the variables of the dual beam representation are the same as for the original one, but are augmented with asterisk in order to enable distinction between the two.

Fig. 8 shows the graph of Fig. 7 with the edges denoted with the notations introduced in Table 1.

Using Table 1, the potential dependence relation (Eq. (1)) associated with the original PGR, can now become a flow dependence relation in the dual FGR, as follows:

$${}^{A}\mathbf{F}^{*\mathrm{CF}}_{\mathrm{d}Ly} = \mathbf{H}^{*\mathrm{F}}_{\mathrm{d}Lyz} \cdot {}^{L}\mathbf{F}^{*\mathrm{CP}}_{\mathrm{d}Lz}. \tag{5}$$

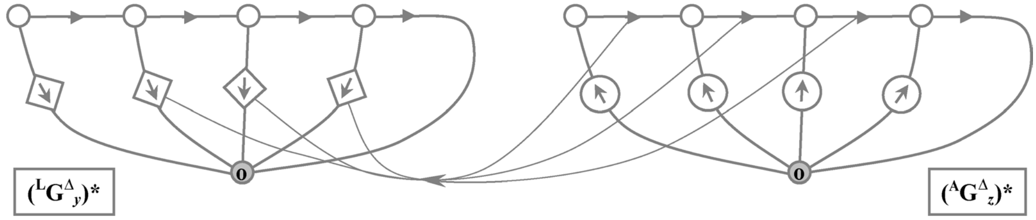


Fig. 7. The graph dual to the PGR of the original beam of Fig. 1.

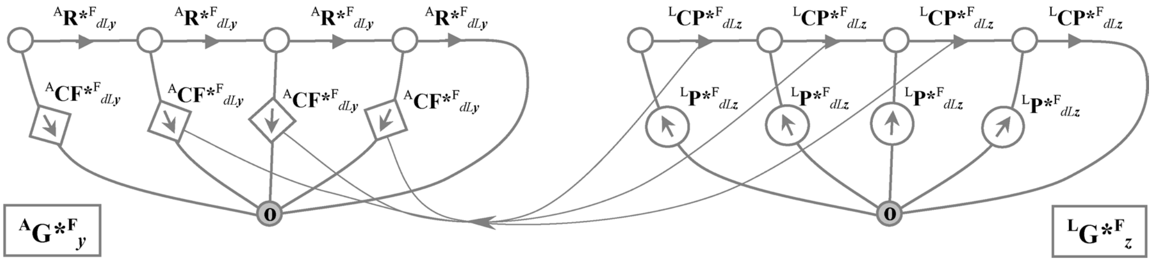


Fig. 8. The FGR of the dual beam.

Table 1
Correspondence between PGR of a beam and its dual FGR

PGR	Dual FGR
Potential difference – Δ	Flow – F
Linear graph component – L	Angular graph component – A
Angular graph component – A	Linear graph component – L
Potential difference source	Flow source
CP edge	CF edge
${}^{\mathrm{L}}\boldsymbol{\Delta}^{\Pi}_{\mathrm{d}Ly}$	${}^{\mathrm{A}}\mathbf{F}^{*\mathrm{R}}_{\mathrm{d}Ly}$
${}^{\mathrm{L}}\boldsymbol{\Delta}^{\mathrm{CP}}_{\mathrm{d}Ly}$	${}^{\mathrm{A}}\mathbf{F}^{*\mathrm{CF}}_{\mathrm{d}Ly}$
${}^{\mathrm{A}}\boldsymbol{\Delta}^{\Pi}_{\mathrm{d}Lz}$	${}^{\mathrm{L}}\mathbf{F}^{*\mathrm{CP}}_{\mathrm{d}Lz}$
$-{}^{\mathrm{A}}\boldsymbol{\Delta}^{\Delta 1}_{\mathrm{d}Lz}$	${}^{\mathrm{L}}\mathbf{F}^{*\mathrm{P}}_{\mathrm{d}Lz}$
$\mathbf{H}^{\Delta}_{\mathrm{d}Lyz}$	$\mathbf{H}^{*\mathrm{F}}_{\mathrm{d}Lyz}$

Note that the flow dependence coefficient appearing in (5) is the inverse of the coefficient appearing in (2), since this coefficient is anti-symmetric:

$$\mathbf{H}^{*\mathrm{F}}_{\mathrm{d}Lyz} = -\mathbf{H}^{\mathrm{F}}_{\mathrm{d}Lzy}. \tag{6}$$

Since the relation stated in (5) has the same form of the flow dependence relation (Eq. (2)) by which an FGR of a beam is characterized, the argument stated above is valid. Therefore, the flow graph of Fig. 8 can be interpreted as the representation of the force and moment distributions along the beam shown in Fig. 9. This beam is related to as the dual beam, and it can be deduced that the dual beam member is oriented the same as the original one. Yet, by the topology of the dual graph, it is concluded that the dual beam is fixed-supported at the end which was originally free, while its other end which was originally fixed supported is free.

Also, applying Table 1 to the 'external' potential source term (4), each of the dual linear flow sources can be calculated as follows:

$${}^{\mathrm{L}}\mathbf{F}^{*\mathrm{P}}_{\mathrm{d}Lz} = -{}^{\mathrm{A}}\mathbf{R}^{\mathrm{R}}_{\mathrm{d}Lz} \cdot {}^{\mathrm{A}}\mathbf{F}^{\mathrm{R}}_{\mathrm{d}Lz}. \tag{7}$$

So, in order to determine the flow sources in the dual FGR, the graph representation of the original beam should first be analyzed for flows. The followed mechanical interpretation is that the externally applied load on the dual beam is a distributed load that is given by:

$${}^{\mathrm{L}}\mathrm{P}^{*}_{z}(x) = -\frac{\mathrm{d}L}{EI_z} \cdot {}^{\mathrm{A}}\mathrm{F}_z(x) \tag{8}$$

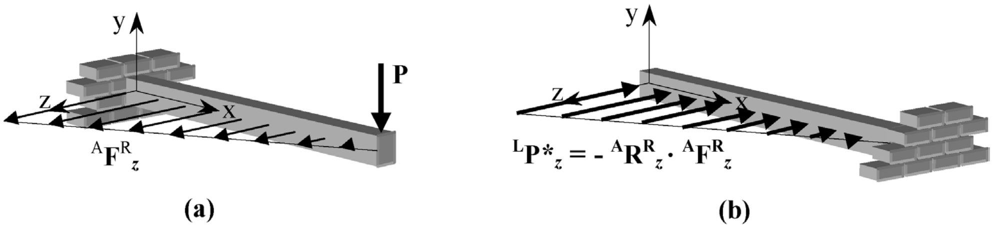


Fig. 9. The original and the dual beams. (a) The original beam with a diagram of the bending moment along the member. (b) The dual beam, externally loaded by linear force which is proportional to the bending moment along the original beam.

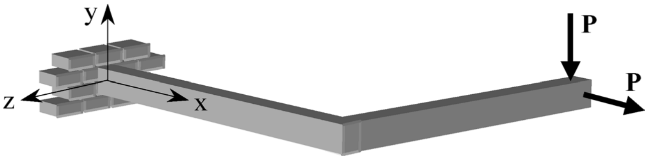


Fig. 10. The original frame.

which means that, in order to obtain this external load, one should first solve the original determinate beam for its internal moments, and then multiply them by the resistance relation to obtain the magnitude of the angular deformation in each segment. When interpreting this magnitude as a linear force it becomes the load exerted on the corresponding section of the dual beam.

3.5. The duality relation between the original and the dual beams

The preceding sections transformed the original beam to the dual beam through graph representations. Current section compares the two beams to establish a direct transformation between them.

The relation between the properties of the two beams can be formulated as follows:

Relation 3.

(1) The z-dimension linear internal force in a section of the dual beam is equal to the z-dimension angular displacement of the corresponding section of the original beam.
(2) The y-dimension angular internal force in a section of the dual beam is equal to the y-dimension linear displacement of the corresponding section of the original beam.

It can be verified that the dual beam is actually the conjugate beam as was first pointed out by Otto C. Mohr in 1860, through his well-known conjugate beam theorem (Hibbeler, 1985). Yet, in contrast to the conjugate beam theorem, the dual beam has resulted in a systematic way by applying the properties embedded in the graph representations.

Due to this direct transformation it is also revealed that the duality between the beams is conserved within a common axis system, i.e., the k-dimension force/moment in the dual beam is equal to the k-dimension slope/deflection in the original beam. Neither the beam nor the axis system needs to be revolved, unlike the conjugate beam method.

In the next section, the approach is extended and is applied also for a general spatial bending structure.

4. Extending the approach to frame structures

In this section, the same dualism procedure is briefly extended to three dimensional frames, which is introduced and demonstrated through the frame shown in Fig. 10, and referred to as *the original frame*. Here again, the RGR of the structure is developed as a preliminary step, as follows succinctly.

Step 1. *Constructing the* PGR *of the frame*.

One can use the PGR of a segment (Fig. 4) to construct the representation of the original frame results in the PGR shown in Fig. 11. Here, for clarity, only 3 segments are shown in the figure for each one of the frame members, although actually the representation is composed of infinitely large number of segments.

Step 2. *Constructing the dual* FGR *from the* PGR *of the original frame*.

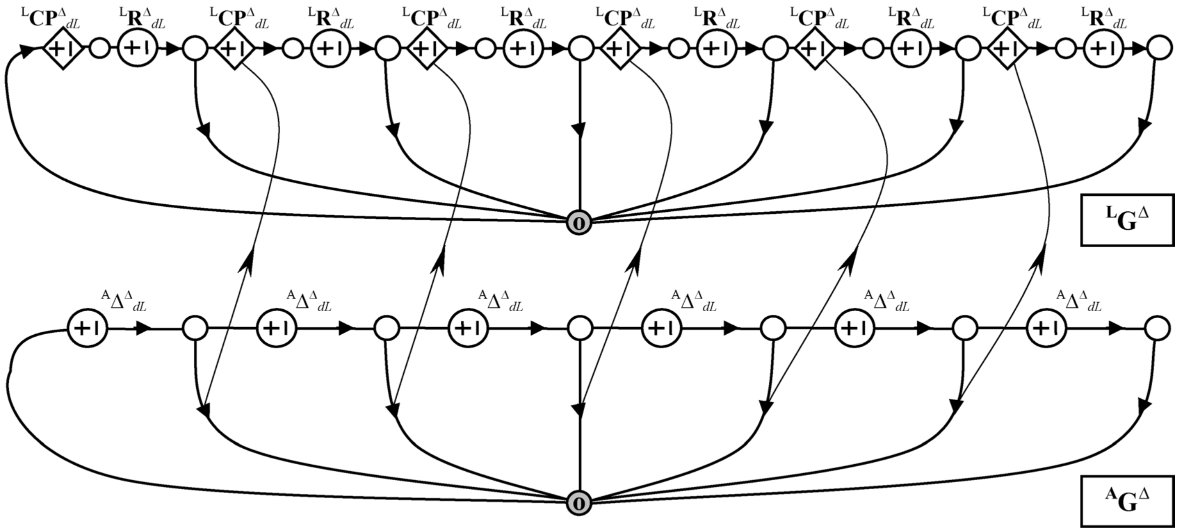


Fig. 11. Linear and angular PGR of the original frame.

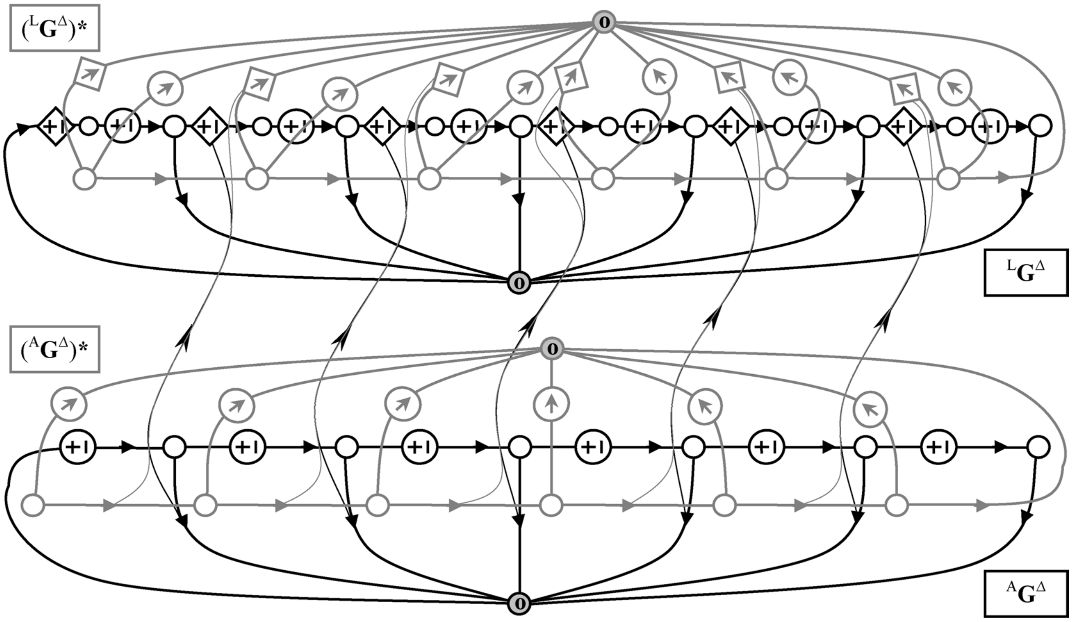


Fig. 12. The dual FGR superimposed over the original PGR of the frame.

The dual flow graph is shown in gray in Fig. 12, superimposed over the original potential graph. The potential difference vector associated with each original edge is equal to the flow vector associated with the corresponding dual edge.

Step 3. *Interpreting the dual graph as a representation of a dual frame*.

The same reasoning developed for the interpretation of the beam's dual FGR is used here to argue that the dual flow graph is itself an FGR of some frame. The dual graph is thus oriented accordingly in a more convenient manner, as in Fig. 13, with opposite directions and values for the flow sources.

Step 4. *The duality relation between the original and the dual frames*.

Through the duality between their graph representations, a direct duality between the displacements of the original frame and the forces of the dual frame is established. This duality is demonstrated in Fig. 14, and formulated as follows:

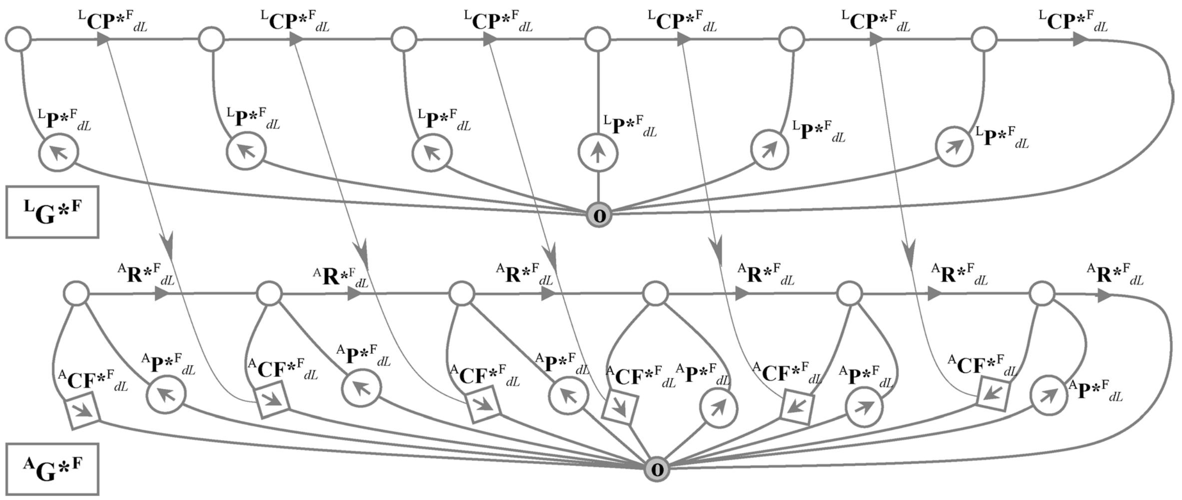


Fig. 13. The FGR of the dual frame.

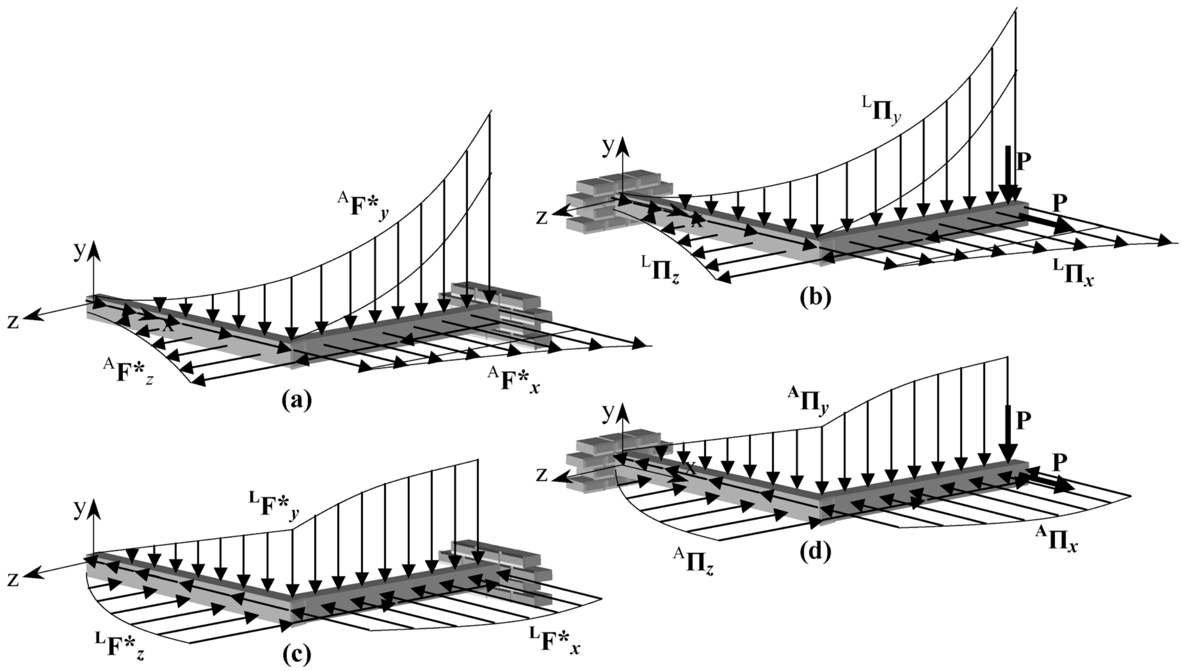


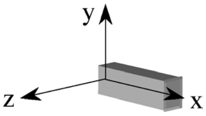
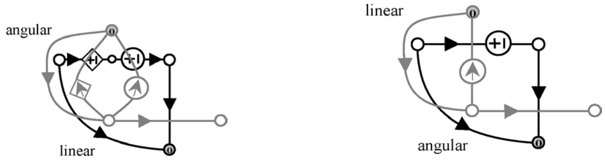
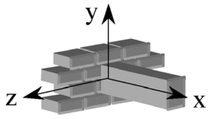
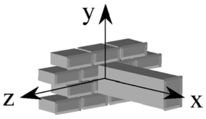
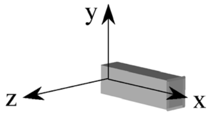
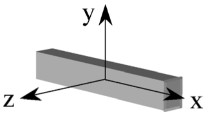
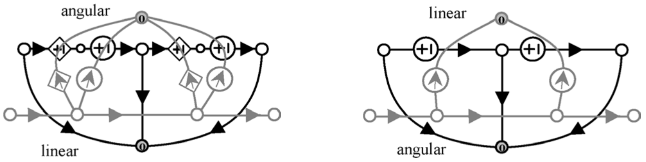
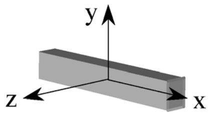
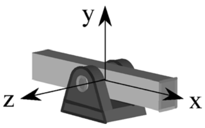
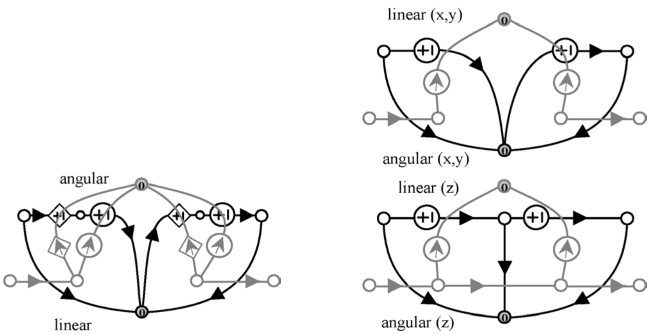
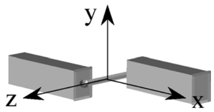
Fig. 14. Flows in the dual frame as potentials in the original frame. (a) The internal moments in the dual frame are equal to (b) the linear deflections of the original beam. (c) The internal linear forces in the dual frame are equal to (d) the slopes of the original frame.

Relation 4.

(1) The linear internal force vector in a section of the dual frame is equal in value to the vectorial angular displacement of the corresponding section of the original frame.
(2) The angular internal force vector in a section of the dual frame is equal in value to the vectorial linear displacement of the corresponding section in the original frame.

This result coincides with the conjugate frame method (Abdul-Shafi, 1985), an extension of the conjugate beam method. Again, not only the duality resulted through a determinate procedure of graph theory, but it also voids the complication involved with the conjugate frame method, that is revolving the structure. The k-dimension force/moment in the dual frame is equal to the

Table 2
Duality between structural elements through the PGR-to-FGR duality

Original structural element	Original PGR (black) and the dual FGR (gray) superimposed	Dual structural element
 Free end joint		 Constrained joint
 Constrained joint		 Free end joint
 Common joint		 Common joint
 Mixed joint		 Mixed joint

k-dimension slope/deflection in the original frame. Neither the structure nor the axis system needs to be revolved, unlike the conjugate frame method. Fig. 14, demonstrates the correspondence between different types of variables of the original and the dual frames.

4.1. *Expanding the approach to a general frame/beam structure*

The approach, which was demonstrated so far via a specific L-shaped frame, can be generalized to any skeletal structure. Table 2 provides a set of conversion rules that can be used to transform between a general skeletal structure and its conjugate (dual) counterpart.

Generally, as illustrated in the first three rows of the table, each of the six linear and angular degrees of freedom related to an end joint of a segment has 3 alternative conditions: either it is free, constrained, or common to another segment. In the linear or angular PGR, these conditions are represented, respectively, as a free potential vertex, a zero potential vertex, or a common equipotential vertex. Through the dual angular or linear FGR it is deduced that the dual structural element is, respectively, a constrained joint, a free joint, or a common joint. Note that the dual of an original linear or angular condition is an angular or linear dual condition, respectively. On the basis of the three basic conditions, the dual of all other types of structural elements can be deduced, as demonstrated by the mixed joint example in the bottom of the table.

5. Conclusions

The paper has shown a new perspective on conjugate theorems from the aspect of graph theory duality principle. Giving this new perspective to the conjugate theorem was found to contribute not only to the engineering theory but also, to the understanding of the mathematical relations underlying the duality between graph representations. Till now, this approach has led to establishing of the duality relation between determinate trusses and linkages, duality between gear systems and beams, pillar systems and kinematical systems and other known (Waldron and Hunt, 1991) and new (Shai, 2001) relations between engineering systems. It should be noted that although known methods in structural mechanics (Shames and Dym, 1995) enable to perform analysis of beams and frames more efficiently than graph representations, the latter enable alternative insight upon the behavior of the systems. Moreover the graphs provide another view on the mathematical foundation underlying theorems in structural mechanics, as was demonstrated throughout the paper.

Understanding the discrete mathematical foundation underlying the conjugate theorems brought up a new transformation channel, in which one represents the potentials of the original system (in the paper – displacements) and from it derives the dual engineering system, from which the information about its flows (forces) is obtained. Existing approaches correlating with the presented result rely on correspondences between the vector spaces spanned by the variables underlying the behavior of two systems, or the same system. Nevertheless, employing graph-theoretical tools is what makes possible to establish systematically the topologies of the new systems.

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