Extension of Graph Theory to the Duality Between Static Systems and Mechanisms

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This paper is a study of the duality between the statics of a variety of structures and the kinematics of mechanisms. To provide insight into this duality, two new graph representations are introduced; namely, the flow line graph representation and the potential line graph representation. The paper also discusses the duality that exists between these two representations. Then the duality between a static pillar system and a planar linkage is investigated by using the flow line graph representation for the pillar system and the potential line graph representation for the linkage. A compound planetary gear train is shown to be dual to the special case of a statically determinate beam and the duality between a serial robot and a platform-type robot, such as the Stewart platform, is explained. To show that the approach presented here can also be applied to more general robotic manipulators, the paper includes a two-platform robot and the dual spatial linkage. The dual transformation is then used to check the stability of a static system and the stationary, or locked, positions of a linkage. The paper shows that two novel platform systems, comprised of concentric spherical platforms inter-connected by rigid rods, are dual to a spherical six-bar linkage. The dual transformation, as presented in this paper, does not require the formulation and solution of the governing equations of the system under investigation. This is an original contribution to the literature and provides an alternative technique to the synthesis of structures and mechanisms. To simplify the design process, the synthesis problem can be transformed from the given system to the dual system in a straightforward manner. [DOI: 10.1115/1.2120827]

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1 Introduction

The principle of duality is an important concept that impacts engineering, in general [1], and mechanics, in particular Refs. [2,3]. The duality between statics and first-order instantaneous kinematics is documented in the literature [4–6]. The method of virtual work clearly indicates that there is an orthogonality between the static and kinematic variables underlying the behavior of a structure and the dual mechanism. The mathematical basis is linear algebra, and the dual relationships can be traced to the similarities in the corresponding matrices. The theory of screws has found widespread acceptance in the study of the duality between statics and kinematics [7–9]. For example, Duffy [10] provided an in-depth study of the duality between statics and first-order instantaneous kinematics and showed that the statics of a parallel manipulator is dual to the instantaneous kinematics of a serial manipulator. Waldron and Hunt [11] had previously showed the duality between the forward and inverse problems of the statics and instantaneous kinematics of parallel and serial manipulators. Then Davidson and Hunt [12] extended this work to the kinetostatics of spatial robots, and presented important relationships between kinematically equivalent serial and parallel manipulators.

Duality in graph theory is widely reported in the literature, and for a comprehensive list of publications the reader is referred to Swamy and Thulasiraman [13]. On the basis of graph theory, Shai [14] showed that there is a duality between determinate trusses and planar linkages. He also derived new techniques in structural mechanics from techniques that are commonly used in machine theory, presented a method for truss decomposition, and developed procedures to check the stability of trusses. Shai and Mohr [15] showed how the Henneberg method in statics can be transformed into kinematics, to yield an efficient method for the kinematic analysis of linkages and planetary gear trains. Based on the principle of duality, Shai [16] also presented a method to check the mobility of a mechanism and proposed systematic conceptual techniques for mechanism design. These techniques have resulted in the establishment of new engineering entities.

Two of the more important of these entities are the face force [17,18] and the equimomental line [19]. A face force is a force variable that is associated with a face of a static system, uniquely defines a force acting in the static system, and exhibits the property of an electrical potential. Due to this property, a face force can be used in a convenient manner to solve analytical problems and investigate synthesis problems in a static system [20]. An equimomental line is a static entity; i.e., it is a unique locus of points associated with two co-planar forces, and is dual to the concept of an instantaneous center of zero velocity in a planar mechanism [21]. Shai and Pennock [19] presented a theorem which states that the equimomental line for two co-planar forces, in a truss with two degrees of indeterminacy, must pass through a unique point. They also introduced the equimomental line theorem which states that the three equimomental lines defined by three co-planar forces must intersect at a unique point. These two entities; i.e., the face force and the equimomental line, can be used in combination to reveal important properties of both determinate trusses and mechanisms.

This paper extends our current knowledge of graph representations and duality transformations to relate the domains of statics and kinematics. A graph representation is basically a graph aug-

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mented by special mathematical properties and can be classified into several types, in accordance with the augmented properties. From a series of rules, each type of graph representation can be used to represent a number of engineering systems. The elements and properties of the system can be mapped, in an isomorphic manner, to the elements and properties of the graph representation. Duality transformations are mathematical techniques that relate the physical properties of the static system to the geometrical properties of the kinematic system. The goal of this paper is to establish relationships which will constitute a complete mapping between a system in one domain to a system in the other domain. The relationships will allow the engineer, or the designer, to transform a static system into the dual mechanism, or a mechanism into the dual static system.

The paper is arranged as follows. Section 2 defines a flow line graph representation (FLGR) and a potential line graph representation (PLGR) and discusses the duality between these two representations. Then Sec. 3 shows how to represent a two-dimensional pillar system by a FLGR and a planar linkage by a PLGR while Sec. 4 presents the duality between a planar linkage and a two-dimensional pillar system. This section also includes the special case of a statically determinate beam and a planetary gear train. Section 5 then extends the graph duality to spatial kinematics and illustrates the graph representations with an example of a six-link serial robot and a parallel (or platform-type) robot; namely, the Stewart platform. The parallel edges in a graph correspond to serial edges in the dual graph, therefore, the statics of a platform-type manipulator is dual to the kinematics of a serial robot. Section 6 uses the duality transformation to investigate the stability of a static system and the singular configurations of a mechanism. Then Sec. 7 employs the principle of duality for the conceptual design of mechanisms. Finally, Sec. 8 presents some important conclusions and suggestions for future research into the duality between static systems and mechanisms.

2 Graph Representations

2.1 Flow Line Graph and Potential Line Graph Representations. This section presents two original graph representations; namely, (i) the flow line graph representation (FLGR), and (ii) the potential line graph representation (PLGR). A line graph representation (LGR) has a strong correlation with the graph representation that is widely used in the kinematics of mechanisms [22]. The significant property of LGR, which distinguishes it from other graph representations, is the manner in which engineering systems are represented; i.e., vertices correspond to engineering elements and edges correspond to the interconnections between these elements. This property is embedded in both the FLGR and the PLGR. The FLGR will be used, in this paper, to represent three static systems; namely, pillar systems, beams, and platform-type structures. The PLGR will be used to represent three kinematic systems; namely, planar linkages, planetary gear trains, and serial robot manipulators.

The important properties of a FLGR and a PLGR are the flow law and the potential law, respectively, which can be defined as follows. The flow law states that the sum of flows of the edges forming a cutset in a graph G is equal to zero. A cutset is defined as the minimal set of edges, the removal of which from the graph G violates the connectivity property of the graph. The potential law states that the sum of potential differences of the edges forming a circuit (i.e., a closed path) in the graph G is equal to zero. A graph that satisfies the flow law is termed the flow graph representation (FGR) and a graph that satisfies the potential law is termed the potential graph representation (PGR). A brief background of these two graph representations is included here for the convenience of the reader, for a more detailed discussion and practical applications see Refs. [14,16].

A network graph is a flow graph, denoted here as G_F, if the flows in the edges are independent of the potential differences in these edges and satisfies the flow law. This law is a multidimensional generalization of Kirchhoff's current law [3] which is restricted to a one-dimensional system and, therefore, is appropriate for electrical circuits. The matrix form of the flow law can be written as

$$Q(G) \cdot \mathbf{F}(G) = 0 \tag{1}$$

where $Q(G)$ is the cutset matrix of the graph G [13] and $\mathbf{F}(G)$ is the vector of the flows through the edges of the graph (commonly referred to as the flow vector). An important property of the flow graph is that it must not contain cutsets consisting entirely of the flow sources; i.e., edges whose flows are known. If such cutsets of sources were to exist then there would be a linear dependence between the flows in these sources, which would violate the definition of the flow sources. Therefore, the spanning tree of the flow graph must be chosen in such a way that it does not include the flow sources. A spanning tree is defined as a connected graph without circuits and contains all the graph vertices. Edges belonging to the spanning tree are referred to as branches, while edges belonging to the graph are referred to as chords. The FGR can be used to represent a wide variety of two-dimensional and three-dimensional engineering systems, for example, determinate trusses, simple electrical circuits, and mass-cable systems in force equilibrium.

A network graph G is a potential graph, denoted here as G_Δ, if the potential differences in the edges are independent of the flows in these edges and satisfies the potential law. This law is a multidimensional generalization of Kirchhoff's voltage law which is restricted to a one-dimensional system. The matrix form of the potential law can be written as

$$B(G) \cdot \mathbf{\Delta}(G) = 0 \tag{2}$$

where $B(G)$ is the circuit matrix of the graph G [13] and $\mathbf{\Delta}(G)$ is the vector of potential differences of the edges of the graph (commonly referred to as the potential difference vector). In the summation of potential differences of all the circuit edges, the potential of each vertex will appear twice but with opposite signs. The reason is either: (i) It is a head vertex on one occasion and a tail vertex on the other occasion, or (ii) it is in the direction of the circuit on one occasion and in the opposite direction on the other occasion. Therefore, the summation of the potential differences is equal to zero. An important property of a PGR is that there can be no circuits consisting of only potential difference sources; i.e., the potential differences in the edges are known. If such circuits were to exist then there would be a linear dependence between the potential differences in these sources which would violate the definition of the potential difference sources. Therefore, the spanning tree of a PGR must be chosen in such a way that it includes all the potential difference sources. The PGR can be used to represent a wide variety of multidimensional engineering systems, such as, linkages, simple electrical circuits, gear trains, and mass-cable systems in force equilibrium.

A FLGR is a combination of a FGR and a LGR and possesses the following properties: (i) A FLGR is a directed graph; (ii) each edge of a FLGR is associated with a multidimensional vector called the flow; (iii) flows of the graph edges satisfy the flow law; and (iv) the relation between the edges of the FLGR and the different components of their flow can be written as

$${}^{A}\mathbf{F}_i = \mathbf{r}_i \times {}^{L}\mathbf{F}_i \tag{3}$$

where ${}^{A}\mathbf{F}_i$ and ${}^{L}\mathbf{F}_i$ are the angular and linear components, respectively, of the flow in edge i due to the moment exerted by element i. The analysis of a FLGR (i.e., determining the flows in all the edges) can be performed in a manner similar to the analysis of a FGR [12]. The procedure is to select a spanning tree and then solve the set of flow law equations for the cutsets defined by each of the tree branches.

A PLGR is a combination of a LGR and a PGR and possesses the following properties: (i) A PLGR is a directed graph; i.e., a graph where the end vertices associated with each edge constitute

Table 1 Correspondence between the properties of FLGR and the dual PLGR

Properties of FLGR	Properties of the dual PLGR
The Flow in edge i	The Potential Difference of edge i^*
Angular component of flow in edge i	Linear component of potential difference of edge i^*
Linear component of flow in edge i	Angular component of potential difference of edge i^*
Set of edges that constitute a cutset	Set of edges that constitute a circuit
The flow in each of the cutsets satisfies the flow law	The potential difference in each circuit satisfies the potential law
$\mathbf{r}_i$—vector relating the angular and linear components of the flow in edge i	$\mathbf{r}_{i^*}$—vector relating the linear and angular components of the potential difference in edge i^*
Set of edges incident to the vertex	Set of edges limiting a face (closed area in the graph)

an ordered pair, referred to as a head vertex and a tail vertex of an edge; (ii) each vertex of a PLGR (corresponding to an element in the engineering system) is associated with a multidimensional vector called the potential; (iii) each edge of a PLGR (corresponding to a relation between elements) is associated with a multidimensional vector called the potential difference. The potential difference is equal to the vector difference between the potential of the tail vertex and the potential of the head vertex; (iv) potential differences of the graph edges satisfy the potential law; and (v) the relation between the edges of PLGR and different components of their potential difference can be written as

$$^{L}\mathbf{\Delta}_i = \mathbf{r}_i \times {}^{A}\mathbf{\Delta}_i \tag{4}$$

where $^{L}\mathbf{\Delta}_i$ and $^{A}\mathbf{\Delta}_i$ are the linear and angular components, respectively, of the potential difference of edge i, and $\mathbf{r}_i$ is the constant vector associated with edge i. The analysis of a PLGR (i.e., determining the potential differences in all the edges) can be performed in a manner similar to the analysis of a PGR. The procedure is to: (i) Select a spanning tree upon which the set of basic circuits is defined; (ii) write a vector equation of the circuit law for each basic circuit, and (iii) solve the resulting set of linear equations.

The following subsection will show, based on the graph theory duality, the existence of a duality relation between the FLGR and the PLGR. This relation will constitute the mathematical basis for the duality between the static systems and the mechanisms that are presented in Sec. 3.

2.2 The Duality Between the FLGR and the PLGR. Given a flow line graph (denoted here as G^{F}), the dual graph (denoted here as $G^{*\mathrm{F}}$) can be constructed and the potential differences in the edges of $G^{*\mathrm{F}}$ can be equated to the flows in the corresponding edges of G^{F}. From the properties of a dual graph it follows that these potential differences must satisfy the potential law in $G^{*\mathrm{F}}$. Therefore, the dual graph $G^{*\mathrm{F}}$ can be regarded as a valid potential line graph (denoted here as G^{Δ}). The conclusion is that for each flow line graph there exists a dual potential line graph, and vice versa. According to the rules for constructing the dual graph, each element and variable in the original graph correspond to an element and a variable in the dual graph. The specific correspondence between the two representations can be described as follows. For each edge i in a FLGR there is a corresponding edge i^* in the dual PLGR. This, and additional correspondences between the graph properties, as they follow from graph theory and the concept of duality, are presented in Table 1.

Table 2 presents practical examples of: (i) The one-dimensional case; (ii) the two-dimensional case, and (iii) the three-dimensional case. The examples are: (i) A determinate beam and a planetary gear train; (ii) a pillar system and a planar linkage; and (iii) a platform-type robot (namely, the Stewart platform) and a serial six-link open-chain robot. The first and fourth columns of this table show figures of these dual engineering systems, and the two middle columns show the FLGR of the static system and the PLGR of the corresponding mechanism.

For purposes of illustration, the following section will show how a static pillar system can be represented by a FLGR and how a planar linkage can be represented by a PLGR.

3 A Pillar System and a Planar Linkage

3.1 Representing a Static Pillar System by a FLGR. Consider a system of solid plates (or platforms), interconnected by rods (or pillars) which are subjected to a number of external forces and internal reaction forces. In order for this system to be in static equilibrium, the forces acting on each plate must satisfy the force and moment equilibrium equations; i.e.,

$$\sum_{i=1}^{n} {}^{L}\mathbf{F}_i = 0 \tag{5a}$$

and

$$\sum_{i-1}^{n} \mathbf{r}_i \times {}^{L}\mathbf{F}_i = 0 \tag{5b}$$

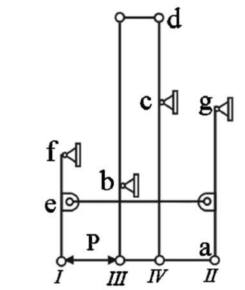
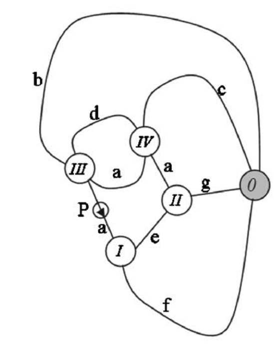
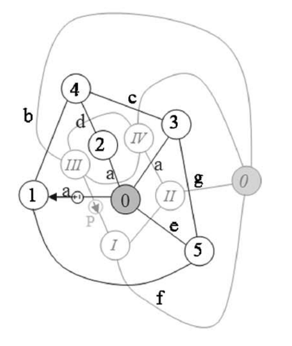
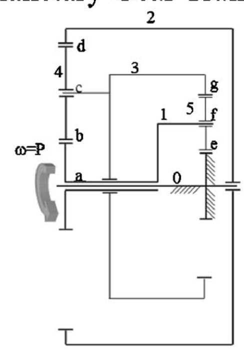
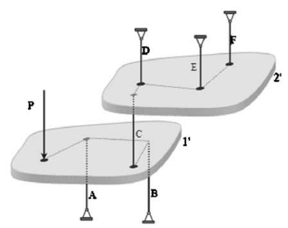
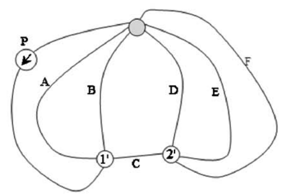
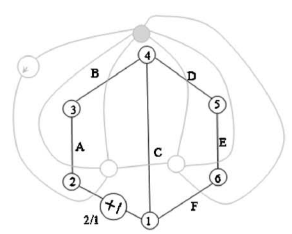
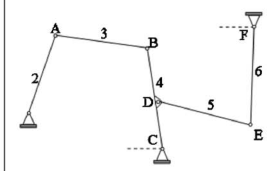
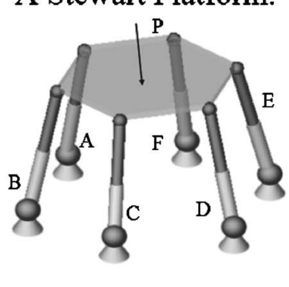
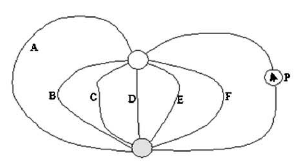
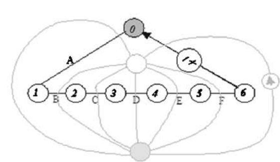
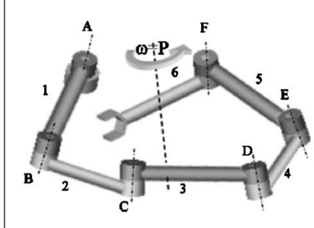
where $^{L}\mathbf{F}_i$ is the ith force acting on the plate (for example, an external applied force or a reaction force from a rod), and $\mathbf{r}_i$ is the vector from a reference point to a point on the line along which the force $^{L}\mathbf{F}_i$ is acting.

The procedure for constructing the FLGR that represents a static pillar system is:

(i) Represent each plate by a vertex in the graph. In addition, create a vertex corresponding to the fixed plane, referred to as the global reference vertex.
(ii) Represent each rod, external force, internal reaction force, and support in the system by an edge in the graph. The end vertices of the edge correspond to the plates between which the represented force is acting. In other words, a vertex is added to the graph for each plate, and an edge is added to the graph for each rod, reaction force, and internal reaction force.
(iii) The constant vector $\mathbf{r}_i$ of the edge is set equal to the vector from the reference point to an arbitrary point on the rod.
(iv) The linear component of the flow of edge i is equal to the force i acting on the plates. The angular component of the flow of edge i, according to Eq. (4), is equal to the moment caused by the force i on the plate, expressed in a local reference frame. Note that Eq. (3) and the flow law, applied to the constructed graph, yields Eqs. (5). This indicates that the FLGR is a proper isomorphic representation of the static pillar system.

For purposes of illustration, consider the planar static pillar system comprised of three horizontal plates (assumed to be massless and denoted as 1, 2, and 3), see Fig. 1. The plates are connected to one another and to the ground by eight vertical rods. The

Table 2 Three dual systems and the corresponding graph representations

The Static System.	The FLGR.	The PLGR.	The Dual Mechanism.
A Determinate Beam. 			A Planetary Gear Train. 
A Pillar System. 			A Planar Linkage. 
A Stewart Platform. 			A Serial Robot. 

rods are connected to the plates by revolute joints and are responsible for the force transfer between the plates. Two external forces, denoted as P_D and P_G, are applied vertically downwards at points D and G in plates 1 and 2, respectively.

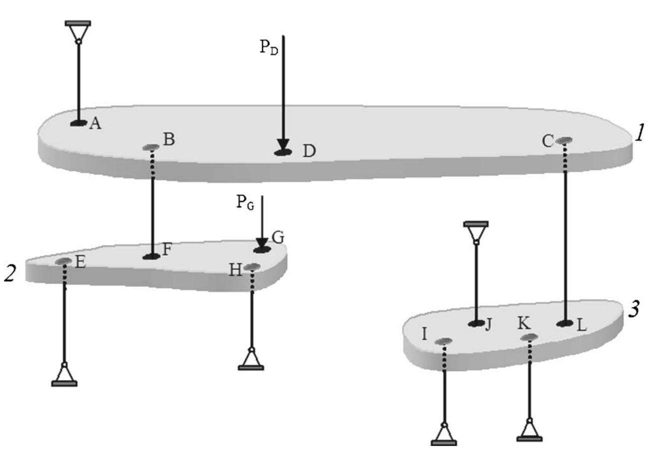


Fig. 1 The static pillar system

The FLGR corresponding to this static pillar system is shown in Fig. 2. The global reference vertex is denoted as O in the figure, see the gray vertex.

3.2 Representing a Linkage by a PLGR. Consider a linkage with i links ($i=1,2,\ldots,n$ where 1 denotes the fixed link) connected by revolute joints. The summation of the relative angular velocities between neighboring links i and $i+1$ must satisfy the two conditions:

$$\sum_{i=1}^{n} \boldsymbol{\omega}_{i+1,i} = 0 \tag{6a}$$

and

$$\sum_{i=1}^{n} (\mathbf{r}_{i+1,i} \times \boldsymbol{\omega}_{i+1,i} + \mathbf{v}_{i+1,i}) = 0 \tag{6b}$$

where $\mathbf{v}_{i+1,i}$ is the velocity of an arbitrary point fixed in link $i+1$ relative to the velocity of the same point fixed in link i, and $\mathbf{r}_{i+1,i}$ is the vector from a fixed reference point to the kinematic pair connecting links i and $i+1$ (i.e., the axis of rotation of links i and $i+1$).

On the basis of the properties of a PLGR, the ability of the

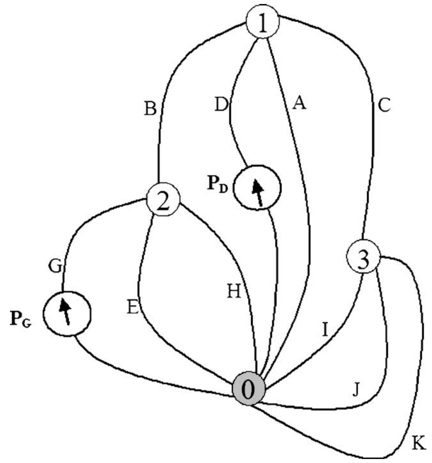


Fig. 2 The FLGR corresponding to the static pillar system

graph representation to map the physical behavior of a specified linkage can be postulated. The procedure for constructing the PLGR corresponding to the linkage is:

(i) Associate a vertex in the graph with each link;
(ii) associate an edge in the graph with each kinematic pair. The end vertices correspond to the two links connected by a kinematic pair;
(iii) the constant vector $\mathbf{r}_i$ of the edge is set equal to the vector connecting a common reference point to some point on the rotation axis of the kinematic pair;
(iv) the angular component of the potential difference of the edge is set equal to the relative angular velocity between the corresponding links. Equation (4) and the potential law of the PLGR must satisfy Eqs. (6). This indicates that the PLGR is a proper isomorphic representation of the linkage.

For purpose of illustration, consider the six-bar linkage commonly referred to as the Stephenson-III six-bar linkage, see Fig. 3(*a*). The PLGR of this six-bar linkage is shown in Fig. 3(*b*).

This section showed that a FLGR and a PLGR are dual representations which correspond to a static pillar system and a linkage, respectively. Therefore, static pillar systems and linkages can be regarded as dual engineering systems; i.e., for a given static pillar system there is a corresponding dual linkage, and vice versa. The following section will derive the properties of this duality from the properties of the corresponding graph representations.

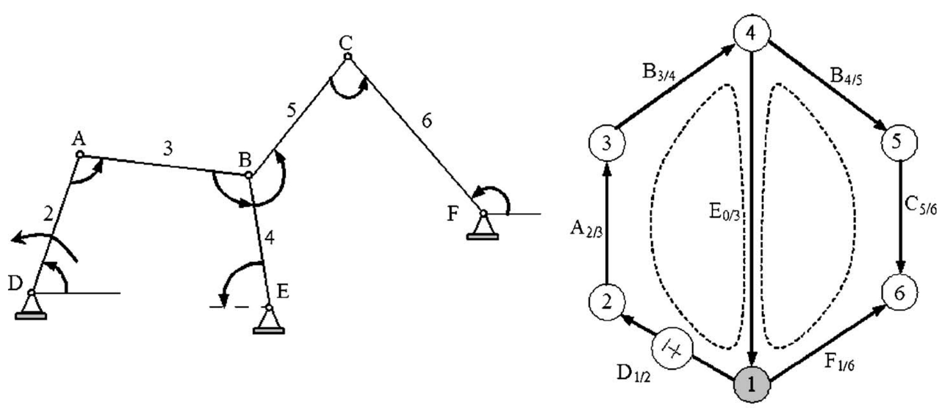
4 The Duality Between a Linkage and a Static Pillar System

The edges in a FLGR correspond to the forces acting in a static pillar system and the edges in the dual PLGR correspond to the kinematic pairs in the dual linkage. From the duality between the two systems, there is a force i in the static pillar system for each kinematic pair i^* in the dual linkage, and vice versa. The duality relation can be used to transform the properties of a FLGR and the dual PLGR, listed in Table 1, to the properties of a static pillar system and the dual linkage, see Table 3.

The procedure to construct a planar linkage that is dual to a two-dimensional static pillar system is:

(i) Construct the FLGR corresponding to the given pillar system;
(ii) construct the dual PLGR by following a standard procedure in graph theory where each face of the original graph is associated with a vertex in the dual graph, and vertices corresponding to adjacent faces are connected by edges; and
(iii) build the linkage from the dual PLGR.

To illustrate this procedure, consider the problem of obtaining a planar linkage that is dual to the two-dimensional static pillar system shown in Fig. 4(*a*). Following the steps that are outlined in Sec. 2.1 will give the corresponding FLGR, see Fig. 4(*b*). The



(a) A Two-Dyad Six-Bar Linkage. (b) The PLGR.

Fig. 3 The PLGR of the six-bar linkage

Table 3 Correspondence between the static pillar system and the dual linkage

Properties of a Static Pillar System	Properties of the Dual Linkage
Force i	Relative angular velocity of the kinematic pair i^*
$\mathbf{r}_i$—vector to the line of action of force i from the common reference point	$\mathbf{r}_{i^*}$—vector to the axis of relative rotation of the kinematic pair i^* from the common reference point
Plate/set of forces acting on the plate	Face/set of links limiting the face

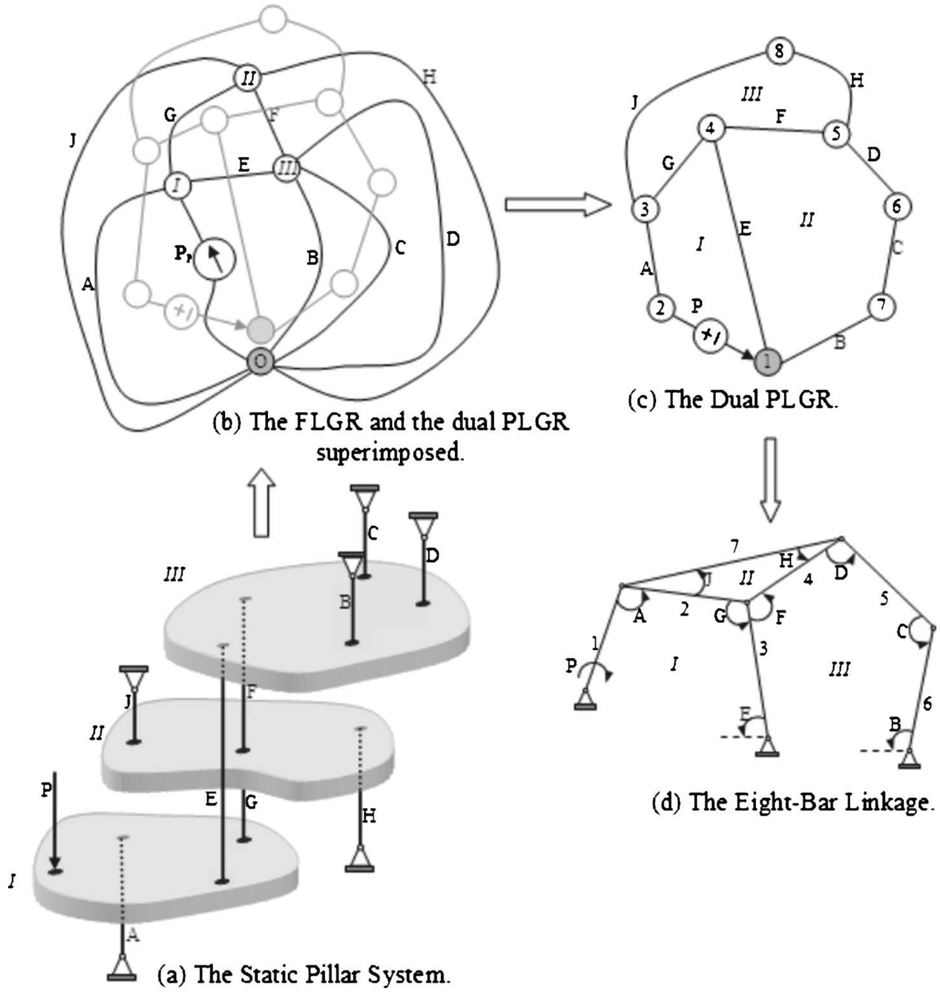


Fig. 4 The two-dimensional static pillar system and the dual linkage

dual PLGR can then be constructed, as explained above, and is shown in Fig. 4(*c*). The corresponding dual linkage, which is an eight-bar linkage, is shown in Fig. 4(*d*).

The procedure to obtain a static pillar system that is dual to a planar linkage is:

(i) Construct the PGR and the dual FGR;
(ii) construct the PLGR and the dual FLGR;
(iii) build the determinate truss corresponding to the dual FLGR; and
(iv) build the static pillar system corresponding to the dual FLGR.

To illustrate this procedure consider the single-degree-of-freedom double flier eight-bar linkage shown in Fig. 5(*a*). The PGR, with the dual FGR superimposed, is shown in Fig. 5(*b*). Then the PLGR, with the dual FLGR superimposed, is shown in Fig. 5(*c*). Finally, the corresponding determinate truss and the static pillar system are shown in Figs. 5(*d*) and 5(*e*), respectively.

4.1 A Special Case. A special case of the duality relation, which is of practical importance, is where the location vectors are parallel (for example, a beam subjected to several point loads). Since the links of the mechanism must be parallel, and the mechanism must also satisfy Eq. (3), then the mechanism is a planetary gear train. To illustrate this duality, consider the determinate beam shown in Fig. 6(*a*). The FLGR is shown in Fig. 6(*b*) and the dual PLGR is shown in Fig. 6(*c*). Finally, the corresponding planetary gear train is shown in Fig. 6(*d*).

Since graph representations are valid for representing multidimensional systems then the techniques presented in this section can be extended to spatial systems. In a one-plate pillar system, for example, inclining the pillars at arbitrary angles to the plate will produce the well-known Stewart platform [23]. The static behavior of the extensible legs of the platform can be represented by pillars. The following section will discuss the duality between serial robots and platform-type robots. Then the duality between the Stewart platform and a serial robot is shown to be a special case of the multi-dimensional expansion of the proposed techniques.

5 The Duality Between a Platform-Type Robot and a Serial Robot

Consider the six-degree-of-freedom Stewart platform shown in Fig. 7(*a*). The FLGR corresponding to this platform is shown in Fig. 7(*b*). Due to existence of only one plate, all the vectors associated with the graph edges lie in a common plane defined by the plate. The dual PLGR, with the FLGR superimposed, is shown in Fig. 7(*c*). The graph is built of six serially connected edges, therefore, it corresponds to a mechanism with six serially connected links. Since the relative angular velocities between the connected links correspond to the forces in the extensible legs of the Stewart platform, then the links must be interconnected by spherical joints, and the rotation axes must be defined by the angles of the

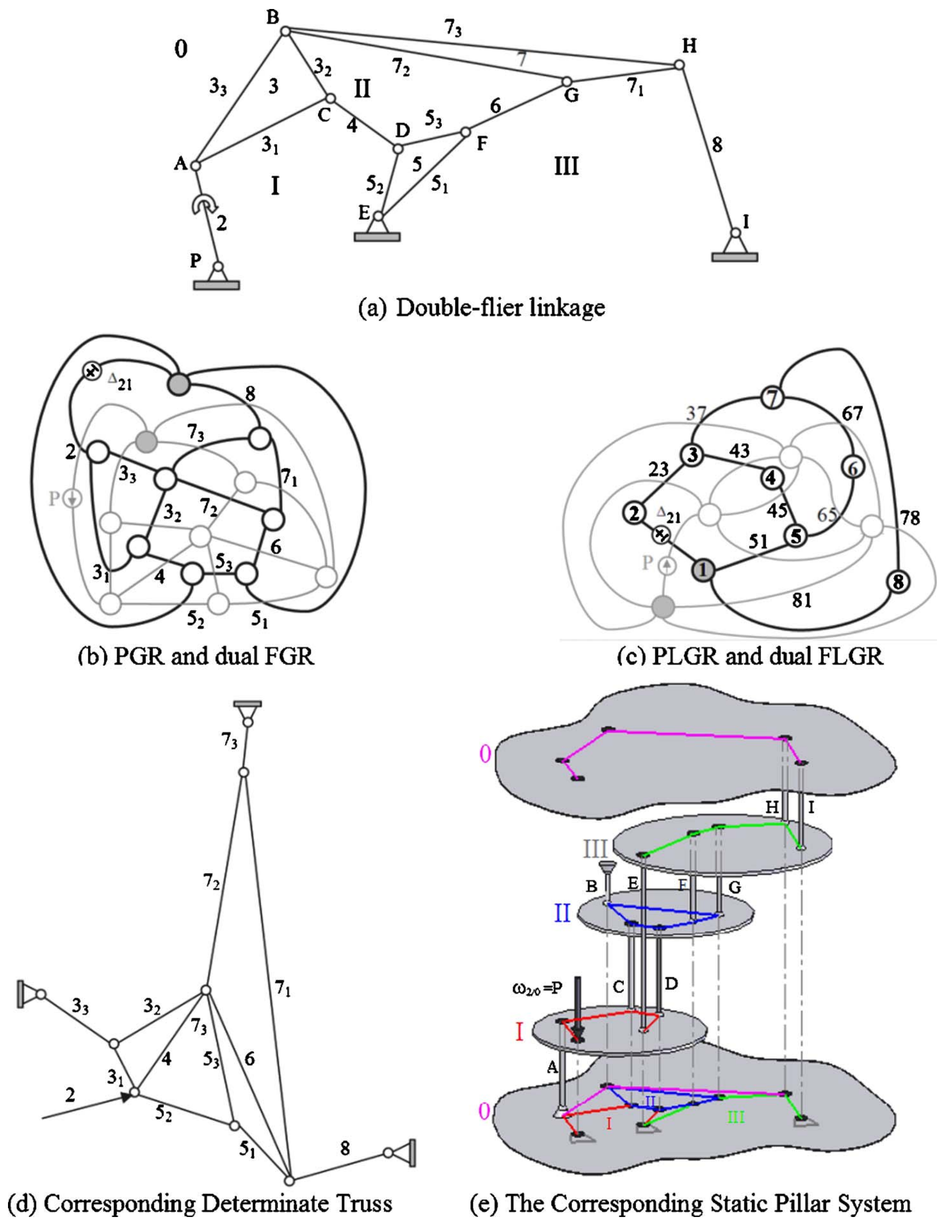


Fig. 5 The double flier eight-bar linkage and the corresponding static pillar System

corresponding pillars. The dual mechanism is a six-link serial robot where the links are connected by prismatic joints, see Fig. 7(*d*).

The technique presented here can be applied to more general problems than the Stewart platform and the dual serial robot. As an example, consider the two-platform robot shown in Fig. 8(*a*). The FLGR corresponding to this robot is shown in Fig. 8(*b*) and the PLGR dual to the FLGR is shown in Fig. 8(*c*). The graph is built of fourteen edges and corresponds to a two-degree-of-freedom open-chain linkage (or serial robot) with twelve links (which is the same as the number of extensible legs in the original two-platform robot). This spatial linkage has two input, or driving, links (which is the same as the number of external forces) as shown in Fig. 8(*d*).

The principle of duality not only allows a system to be transformed from one domain to another domain but also addresses special properties of the system. The following section will investigate special configurations of a six-bar linkage and a six-link serial robot. The procedure is to study the dual system where the dual property is more apparent.

6 A Study of Special Configurations Using the Duality Transformation

If a static pillar system is not stable (or not rigid) then the dual linkage is not mobile (i.e., the linkage is instantaneously locked), and vice versa. This feature can be employed to check the stability of a static system or to check the mobility of a linkage. Consider, for example, the planar pillar system in the configuration shown in Fig. 9(*a*). Checking the stability of this pillar system can be a difficult task. However, the linkage that is dual to this static system is the Stephenson-III six-bar linkage, see Fig. 3(*a*), which is in the configuration shown in Fig. 9(*b*). With link 2 as the input link, it is known that the linkage is not movable (i.e., stationary) in this configuration. In kinematics, this special configuration is commonly referred to as a dead-center position [24]. The only constraint for the six-bar linkage to be in a dead-center position is that the instant center I_{15} be located on link 3 (or link 3 extended). Since links 3, 4, and 6 (or the extension of these three links) intersect at a single point then the relative instant center of link 2 and coupler link 5 (denoted as I_{25}) is coincident with the absolute

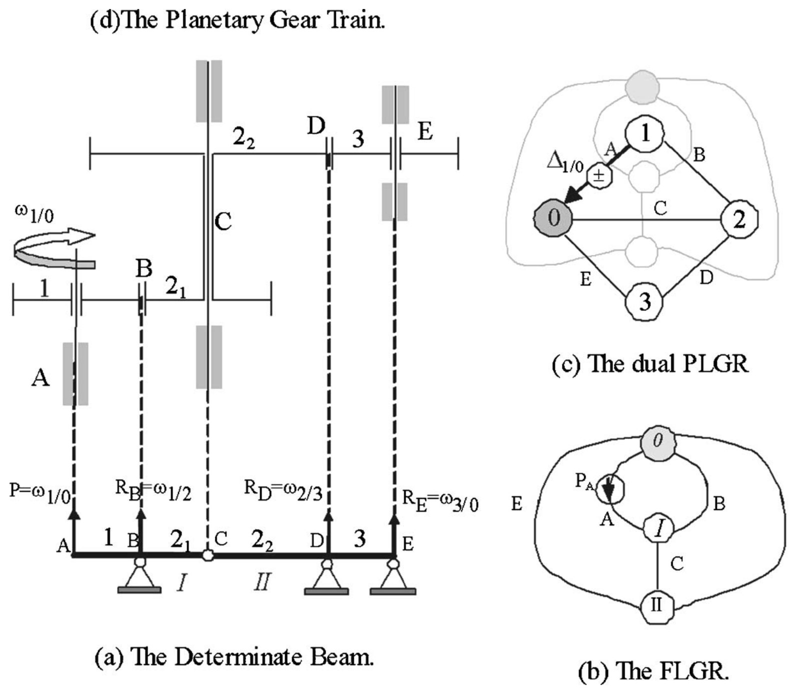


Fig. 6 A determinate beam and the dual planetary gear train

instant center for link 5 (denoted as I_{15}). Therefore, the dual linkage is in a dead-center position which implies that the static pillar system in the given configuration is not stable.

The same approach can also be applied to spatial mechanisms and structures. Consider, for example, the six-link serial robot in the configuration shown in Fig. 10(*a*). The robot consists of two sets of three links that are inter-connected by two universal joints. It is difficult to visualize if the given configuration of the robot is a singular configuration. However, consider the corresponding configuration of the dual system, i.e., the Stewart platform, as shown in Fig. 10(*b*). Since legs 5′ and 6′ are in the same plane as the platform then the Stewart platform is known to be in a singular

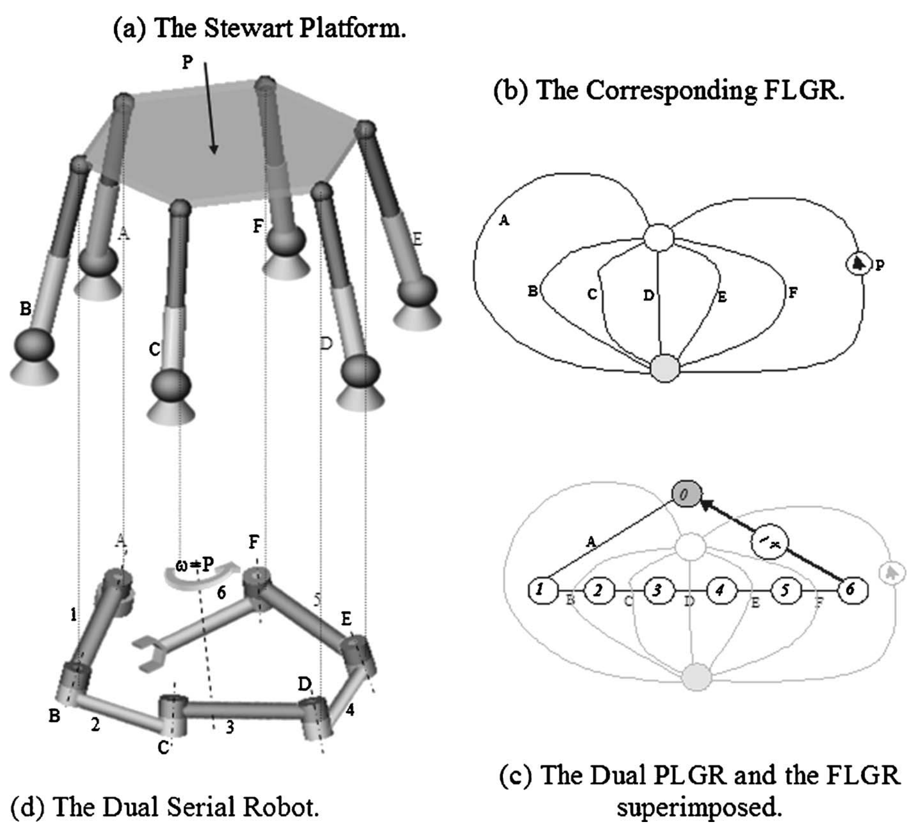


Fig. 7 A multi-dimensional pillar system and the dual serial robot

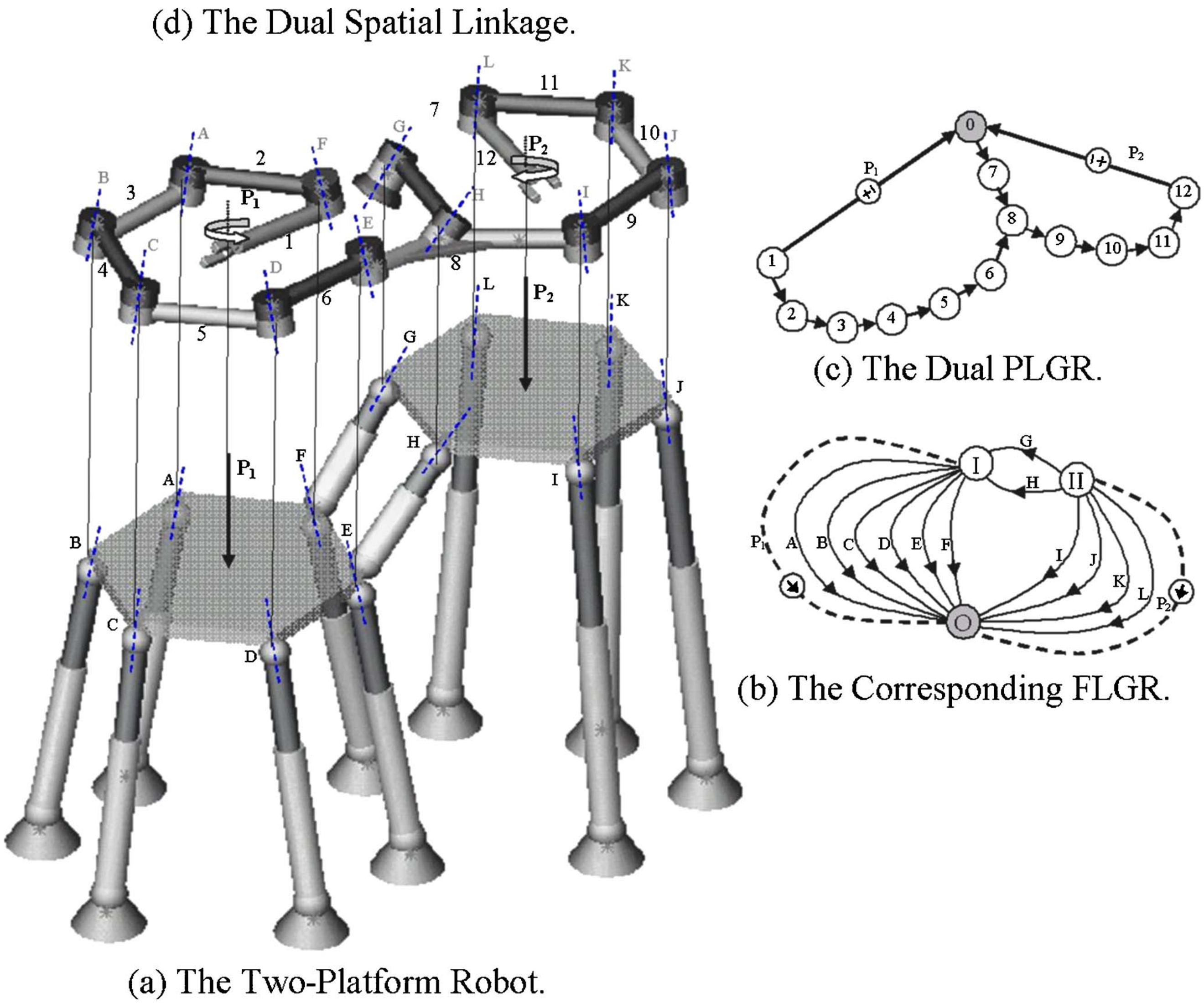


Fig. 8 A two-platform robot and the dual spatial linkage

configuration [25]. The conclusion is that the serial robot must be in a singular configuration and the end-effector (or terminal link) cannot be rotated or translated along the terminal joint axis.

Two examples of novel platform systems are presented in Figs. 11(*a*) and 11(*b*). These topologically identical systems are comprised of two concentric spherical platforms that are interconnected by rigid rods. Note that all the rods are oriented radially relative to the platforms.

To determine if the two platform systems are statically stable, the procedure is to construct and investigate the dual kinematic system. The first step is to construct the FLGR representation of the two platform systems, which is common to both due to their topological isomorphism. The FLGR of the two systems and the dual PLGR are shown in Figs. 12(*a*) and 12(*b*), respectively.

The PLGR can be interpreted as a representation of two spherical six-bar linkages which are topologically identical but differ in their geometry. The two linkages are represented by the PLGR that is shown in Fig. 12(*b*). Figure 13(*a*) shows the spherical linkage which is dual to the platform system in Fig. 11(*a*) and 13(*b*) shows the spherical linkage which is dual to the platform system in Fig. 11(*b*).

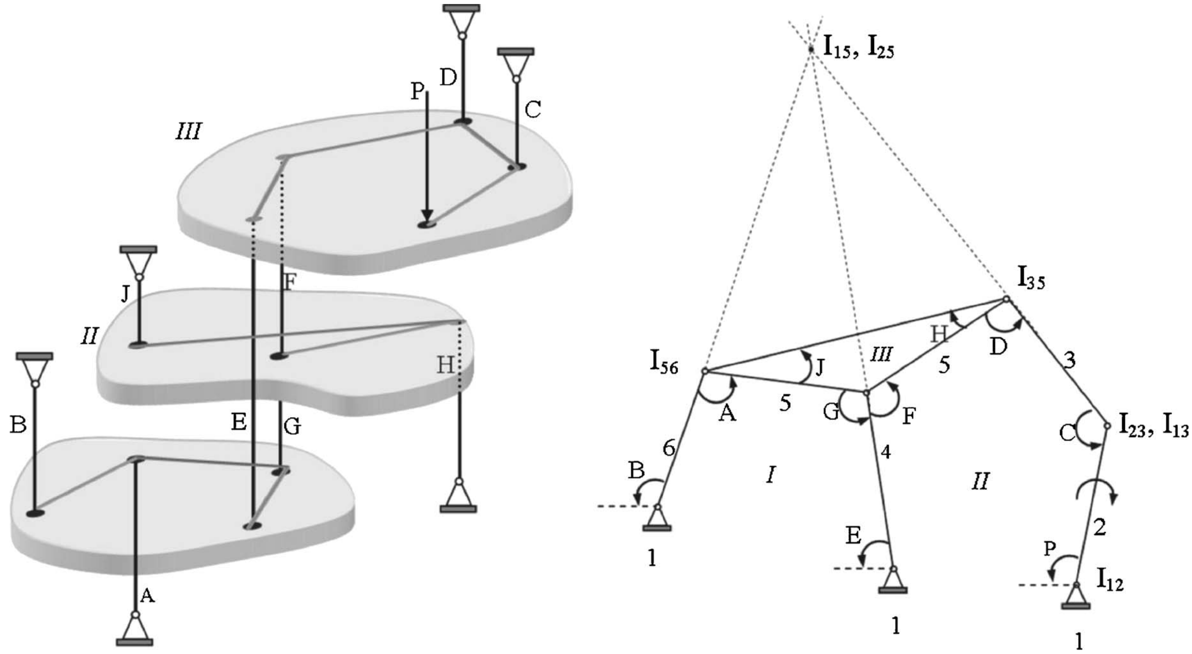
Note that the two linkages are actually two configurations of the same spherical six-bar linkage. Also, note that Fig. 13(*b*) represents a special configuration of the linkage; i.e., the arc of link 4, the arc of link 5, and the arc connecting kinematic pairs 1-6 and 2-3 intersect at a unique point, denoted as point K on the figure.

In this configuration, the absolute instantaneous screw axis of link 6 (denoted as S_{16}) is coincident with the relative instantaneous screw axis between the input links 2 and 6 (denoted as S_{26}). Therefore, the linkage is in a stationary (or a locked) configuration. This implies that the second platform system, see Fig. 11(*b*), is in a singular configuration, i.e., not statically stable. Finally, note that linkage shown in Fig. 13(*a*) is in an arbitrary configuration, therefore, the first platform system in Fig. 11(*a*) is in a stable configuration.

The following section will explain how the ability to transform knowledge between engineering fields can be employed for conceptual design. The concepts from the dual domain are transformed to solve the original engineering design problem. The section will include two practical examples. The first example is the problem of amplifying the input force on a static beam system. The second example is to devise a system for measuring the three-components of the angular velocity of a rigid body and the three components of the linear velocity of a point fixed in the body.

7 Employing the Duality for Conceptual Design

The principle of duality can also be applied to conceptual design by allowing the engineer to use known concepts or even



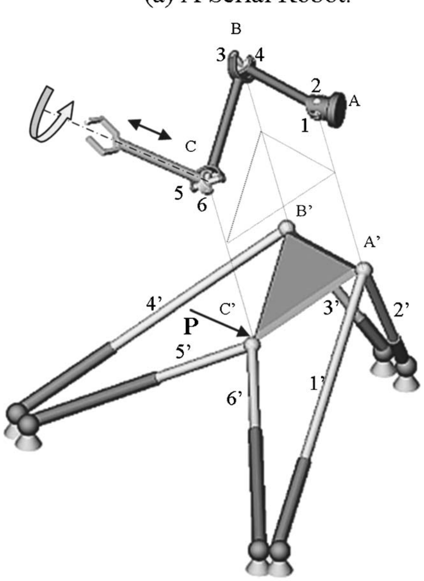
(a) Unstable Configuration. (b) Dead-Center Position.

Fig. 9 Static pillar system and dual linkage

patents in other engineering domains. The concepts can be obtained from existing devices in other domains, through the dual transformations. To illustrate this idea consider the problem of designing a static beam system to amplify the amplitude of an input force, see Fig. 14. In this case, the designer can take advantage of existing concepts in machine theory, instead of searching for a solution in the static arena. Since lever systems are dual to planetary gear trains, as shown in the previous sections, then the procedure is to investigate these mechanisms.

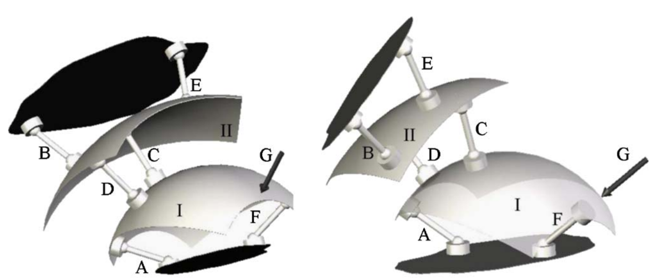
Transforming this problem from the realm of statics to the realm of kinematics would require a search for an existing design of a mechanism that can amplify the input angular velocity. There are many solutions to this synthesis problem, for example, con-

(a) A Serial Robot.



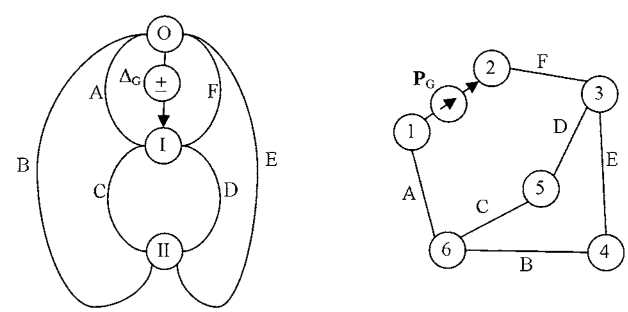
(b) The Dual Stewart Platform.

Fig. 10 The mobility of the serial robot



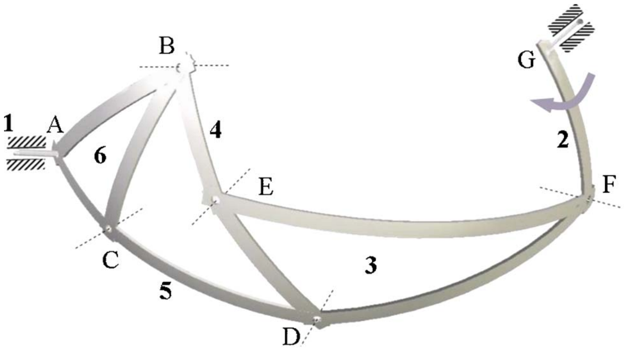
(a) The First Platform System. (b) The Second Platform System.

Fig. 11 Two topologically identical concentric platform systems

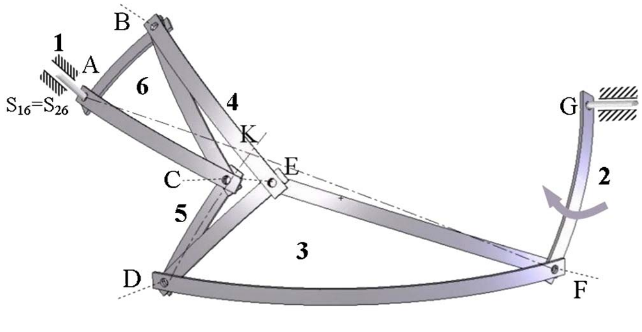


(a) The FLGR of the Platform Systems. (b) The Dual PLGR.

Fig. 12 The FLGR and the dual PLGR of the platform systems



(a) The Spherical Linkage Dual to the First Platform System.



(b) The Linkage Dual to the Second Platform System.

Fig. 13 The linkages dual to the two platform systems

sider an electrical screwdriver, see Fig. 15(a). A schematic drawing of the planetary gear train within this screwdriver is shown in Fig. 15(b).

The process of transforming this concept through the corresponding graphs to arrive at a satisfactory static system (satisfying the given requirements) is as follows. First, transform the planetary gear train into the corresponding PLGR, see Fig. 16(a). The transformation is performed by following the steps explained in Sec. 5 and the resulting graph representation is shown in Fig. 16(b). The graph possesses both the topological and the geometrical information describing the planetary gear train, namely the locations of the joints. This is accomplished with the constant vector variable, $\mathbf{r}_i$, associated with each edge of the graph as shown in Fig. 16(b).

The next transformation is the duality transformation from the PLGR into the dual FLGR. For the graph representation shown in Fig. 16(b) construct the dual representation of type FLGR. As explained in Sec. 2.1, the dual representation is constructed by associating each face of the original PLGR representation with a vertex in the dual graph, and each edge in the former with an edge in the latter. Furthermore, the constant vector $\mathbf{r}_i$ associated with the edge in the new representation can be regarded as the vector associated with the corresponding edge in the original graph representation, see Fig. 17(a).

The final stage is performing the inverse transformation on the graph representation obtained in Fig. 17(b), which presents the construction process in which a specific static system is constructed from the graph. From Sec. 2.3, the beam elements of the new system are associated with the vertices of the graph representation, while the connections between beam elements, enabling one element to exert force upon the other, are associated with the graph edges. The geometrical locations of these connections are determined by the constant vectors, $\mathbf{r}_i$, associated with the corresponding edges. The static system that is derived by applying this transformation is presented in Fig. 18(b). For the convenience of the reader the original gear train, see Fig. 15(b), is repeated here as Fig. 18(a).

The forces (i.e., external forces, or internal reaction forces) acting in the joints of the dual static system correspond to the relative angular velocities in the corresponding kinematic pairs of the planetary gear train. Accordingly, as the main property of the original system was the ratio between the input and the output angular velocities, in the new transformed static system this property becomes the ratio between the input and the output forces. Therefore, the new system constitutes a special force amplification system that was systematically obtained by transforming knowledge between the domain of kinematics and the domain of statics.

The following problem will provide further insight into the design technique and illustrate the application of the concepts to three-dimensional systems. The problem is to design a mechanism that can measure the three-components of the angular velocity of a rigid body and the three components of the linear velocity of a point fixed in this body. Instead of attempting to solve this problem directly in the domain of kinematics, the problem will be transformed into the domain of statics. In this domain, the problem is to devise a system for measuring the three components of a force and the three components of a moment acting on the platform. The problem of a six-component force sensor is widely

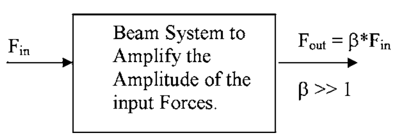
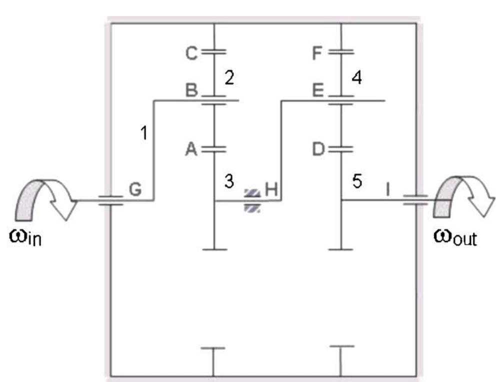


Fig. 14 A static beam system to amplify the input force

(a) Electrical Screwdriver.



(b) Schematic Description of the Underlying Mechanism.

Fig. 15 The planetary gear train in an electrical screwdriver

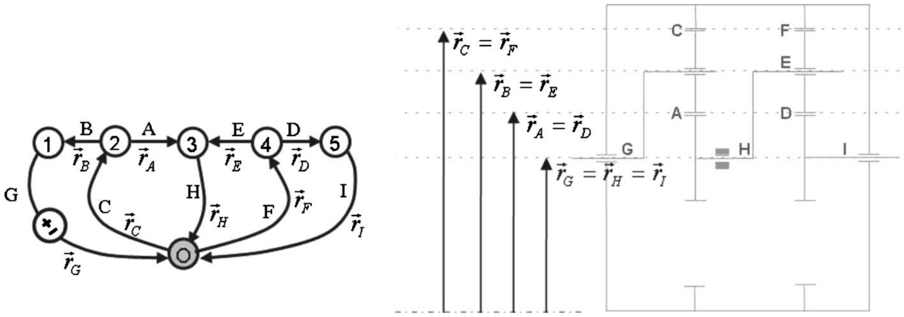


Fig. 16 The graph representing the planetary gear train

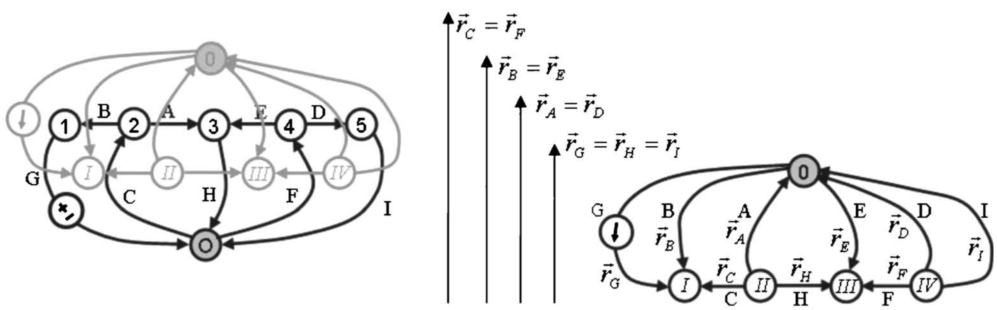
known in engineering and possesses a number of practical solutions, including the so-called 3-2-1 platform [26]. A sketch of the 3-2-1 platform is shown in Fig. 19(*a*).

Consider point *B* on the platform which is subjected to an unknown external force **F** and an applied moment **M**. The linear components (i.e., the forces)—F_x, F_y, and F_z, and the angular components (i.e., the moments)—M_x, M_y, and M_z, can be measured in an efficient manner using this platform system. The procedure is by direct measurement of the internal stresses in the six rods supporting the platform. Then to obtain the solution to the original design problem in the domain of kinematics, a mechanism can be constructed that is dual to the 3-2-1 platform. Since the 3-2-1 platform is topologically identical to the Stewart platform then the FLGR and the dual PLGR of this platform are as shown in Figs. 7(*b*) and 7(*c*), respectively. From the dual PLGR in Fig. 7(*c*), the mechanism that satisfies the design requirements of the design problem is shown in Fig. 19(*b*) and is a six-link serial robot manipulator.

This serial robot manipulator can measure the three components of the angular velocity of the terminal link (link 4) and the three components of the linear velocity of point B fixed in the terminal link. This is accomplished through the direct measurement of the angular velocities in each of the six kinematic pairs in the robot. Therefore, this design will satisfy the requirements that are specified in the original synthesis problem.

8 Conclusions

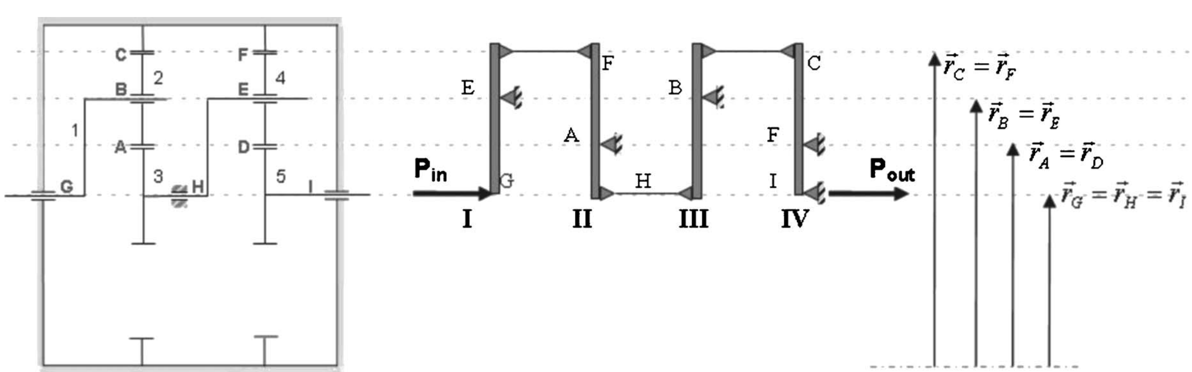
This paper shows that many engineering systems can be transformed from the domain of kinematics to statics, and vice versa, using dual graph representations. These representations will transform all of the properties, topology, and geometry of the particular system under investigation. For purpose of illustration, three types of systems are presented; namely: (i) The one-dimensional case where the duality is between beams and planetary gear trains; (ii) the two-dimensional case where the duality is between vertical



(a) The Original and Dual Representations Superimposed.

(b) The Dual Representation Containing the Corresponding Constant Vectors.

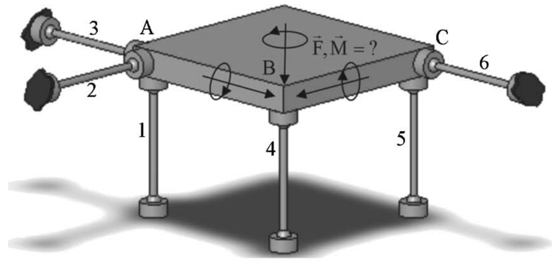
Fig. 17 The dual graph representation of the planetary gear train



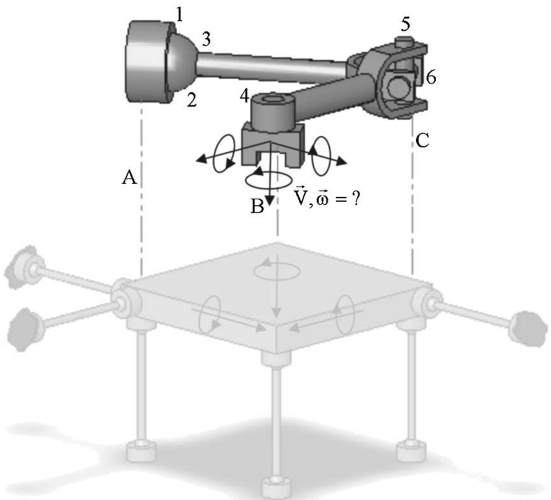
(a) The Original Gear Train System.

(b) The Static System Resulting from the Duality Transformation.

Fig. 18 The original gear train and the resulting static beam system



(a)



(b)

Fig. 19 (*a*). The 3-2-1 platform. (*b*). A six-link serial robot manipulator.

pillar systems and planar linkages; and (iii) the three-dimensional case where the duality is between serial robot manipulators and platform-type robots, such as the Stewart platform. A two-platform system is also included in the paper to illustrate that the approach can be applied to more general systems than serial and parallel robots.

An important contribution of this paper is that the dual transformation, as presented here, allows one system to be transformed to another system without the need to write the governing equations of either system. This property is shown to be valuable for conceptual design, where the concept is transformed from the known dual system to the original design problem. The property is also useful in revealing unstable configurations of a static system and singular configurations of a mechanism. The authors believe that the work presented in this paper offers the potential for creating many novel spatial mechanisms with interesting and practical properties. The general idea of inventing a new device from an existing design by using the principle of duality is appealing. Also, the designer will have the option of transforming a synthesis problem from one domain to another, based on personal preference.

The techniques presented in this paper are different, in several aspects, with previous research in this area. The important difference is that dual relations are derived from a graph-theoretic approach. This approach gives a more general perspective of the duality between the statics and kinematics of engineering systems. In previous research, the major focus was to derive the equations underlying the physical behavior of the system and then the duality could be revealed from the correlation between these equations. However, in this paper, the engineering system is associated with the dual system through the corresponding graph representaions. One need only write the equations of the two dual systems to confirm the resulting correlation. A significant advantage of the approach presented here is that the designer can transform from one system to the other without the need to perform a complete static or kinematic analysis of the given system. This approach can be applied to many practical engineering systems. For illustration, the principle of duality is applied to platform-type robots, such as the Stewart platform, and to serial robots and can also be applied to multiple platform systems which will yield multiple serial robots. The dual transformation can also reveal special properties in a system. For purposes of illustration, the stability (or rigidity) of a static system is used to provide insight into the singular configurations (or mobility) of a linkage, and vice versa.

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