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Paper 7

Duality between Statical and Kinematical Engineering Systems

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Abstract

The work presented in the paper is one of the outcomes of a general research called Multidisciplinary Combinatorial Approach (MCA) aimed to assist in obtaining a general perspective over engineering systems. The approach is focused on developing general discrete mathematical models, called Combinatorial Representations and associating them with different engineering systems. Once the engineering system is associated with a specific combinatorial representation, analysis and other forms of engineering reasoning can be computerized and conducted solely upon the combinatorial representation.

The work introduced in this paper enables to transfer knowledge from various domains of engineering to structural mechanics and vice versa. The paper is concentrated on transferring knowledge by means of a new duality relation between statical and kinematical systems that has been established in 2001.

Keywords: graph theory, combinatorial representations, duality, beams, trusses, gear systems, mechanisms.

1 Introduction

The duality relations between statical and kinematical systems are established in the paper through systematic mathematical processes based on relations between generalized discrete mathematical models, called Combinatorial Representations. The first duality presented in the paper is between determinate trusses and mechanisms [1]. It is shown that a combinatorial representation of a mechanism is a potential graph representation (PGR), whereas the representation of a truss is the flow graph representation (FGR). It is then proved that the potential and flow graph representations are mutually dual, thus making the trusses and mechanisms dual as well, as is depicted in the example of Figure 1a. Another duality relation that has been recently established is the duality between statical column systems and mechanisms, and its special case - the duality between statical beams and gear systems (Figure. 1b).

Establishing the duality between statics and kinematics provides a channel for knowledge and information transfer, yielding immediate practical and theoretical implications, some of which are described in this paper.

One of the significant theoretical implications is proving the correspondence between theorems and methods in the two fields. In the paper this issue is demonstrated by showing that Maxwell-Cremona diagram for forces in trusses and velocity polygon of mechanisms are actually dual methods.

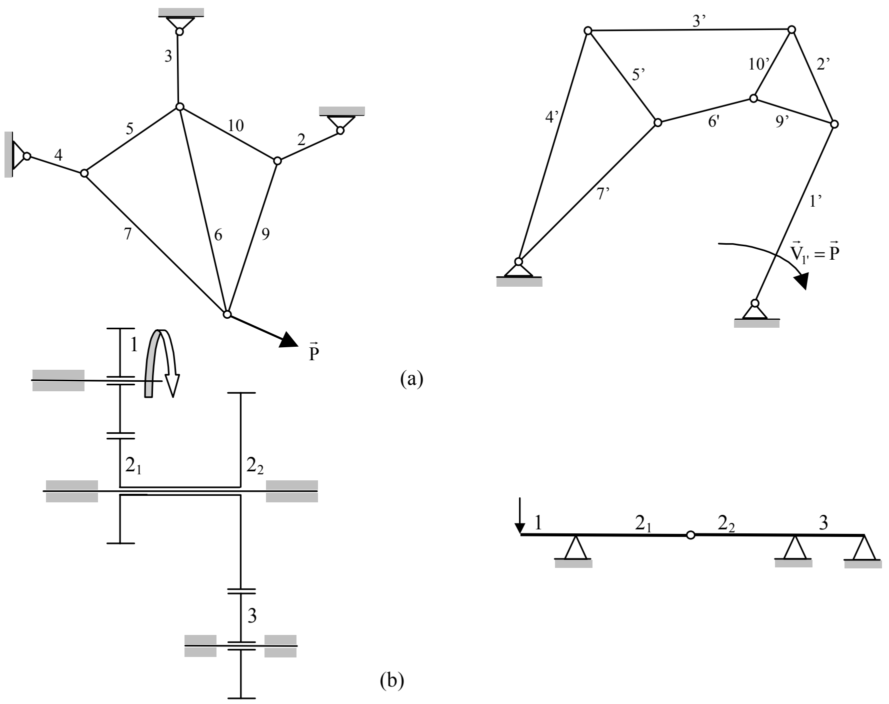

Figure 1: Example of dual systems from static and kinematical domains. (a) Truss and its dual mechanism, (b) Beam and its dual planetary gear system.

Further application of the duality relation is demonstrated by developing rules for validity checking. It is shown that through the duality relation, methods for checking the mobility of mechanisms become applicable to checking the stability of trusses and vice versa.

One of the most promising practical applications of the approach is opening a new avenue of research for facilitating the process of design of engineering systems. Employing the dualism, civil engineer faces an opportunity to search for a solution to his problem also among known mechanisms. Once a mechanism performing the desired task is found, the designer is just left to transform it to the corresponding dual static system.

2 Combinatorial Representations and their Duality

The results reported in this paper are based on four Combinatorial Representations - Potential Graph Representations (PGR), Potential Line Graph Representations (PLGR), Flow Graph Representations (FGR) and Flow Line Graph Representations (PLGR). Tables 1 and 2 summarize the properties of these representations, accompanied with their relations to relevant engineering fields. Further information appears in detail in ([1], [2] and [3]).

Tables 1 and 2 employ some basic terms of graph theory, thus following are few necessary definitions: Network graph – is a directed graph G = <**V**, **E**>, where **V** is the vertex set and **E** is the edge set. The vertex upon which the arrow is directed is called the "head vertex" and the other is called the "tail vertex". Each edge e is assigned a vector called 'flow' $\vec{F}(e)$ and each vertex v is assigned a vector called 'potential' $\vec{\pi}(v)$. Subtraction of the potential of the tail vertex from the potential of the head vertex for a specific edge is called the 'potential difference' - $\vec{\Delta}(e)$ of that edge.

Combinatorial **R**epresentations (**CR**)		**F**low **G**raph **R**epresentation - G_F (**FGR**)	**P**otential **G**raph **R**epresentation- G_Δ (**PGR**)
Main property		Flow law - The vector sum of the flows in every cutset of G_F is equal to zero. $\vec{\mathbf{Q}}(G_F)\cdot\vec{\mathbf{F}}(G_F)=\mathbf{0}$	Potential law – The vector sum of the potential differences in every circuit of G_Δ is equal to zero. $\vec{\mathbf{B}}(G_\Delta)\cdot\vec{\Delta}(G_\Delta)=\mathbf{0}$
Represented engineering system		Determinate truss.	Mechanism.
Engineering interpretation of the graph elements	Edge	Truss element: rod, reaction, external force in the truss. The flow in the edge is interpreted as the force in the corresponding truss element.	Link in the mechanism. The potential difference is interpreted as the relative velocity between the end joints of the link.
	Vertex	Pinned joint of the truss	Joint in the mechanism. The potential is interpreted as the linear velocity of the joint.

Table 1. Flow and Potential Graph Representations and their usage.

Combinatorial Representations (CR)		Flow Line Graph Representation – G_{LF} (FLGR)	Potential Line Graph Representation- $G_{L\Delta}$ (PLGR)
Main property		Terminal equation - Flows in all edges of G_{LF} have two orthogonal components: rotational and linear, related as follows: $\vec{F}_{Ri} = \vec{r}_i \times \vec{F}_{Li}$, where $\vec{r}_i$ is some constant vector associated with the edge. Flow law - The vector sum of the flows in every cutset of G_F is equal to zero. $\vec{\mathbf{Q}}(G_{LF}) \cdot \vec{\mathbf{F}}(G_{LF}) = \mathbf{0}$	Terminal equation – Potential Differences in all edges of $G_{L\Delta}$ have two orthogonal components: rotational and linear, related as follows: $\vec{\Delta}_{Li} = \vec{r}_i \times \vec{\Delta}_{Ri}$, where $\vec{r}_i$ is some constant vector associated with the edge. Potential law – The vector sum of the potential differences in every circuit of G_Δ is equal to zero. $\vec{\mathbf{B}}(G_{L\Delta}) \cdot \vec{\Delta}(G_{L\Delta}) = \mathbf{0}$
Represented engineering system		Pillar structure	Mechanism
Engineering interpretation of the graph elements	Edge	A pillar, reaction or external force applied to the horizontal plate. The linear flow in the edge is interpreted as the force in the corresponding element. The rotational flow is the moment exerted by the element upon the plates to which it is connected. The constant vector $\vec{r}_i$ thus defines the location of pillar in relation to the plate.	Kinematical couple of mechanism links. The rotational potential difference is interpreted as the relative angular velocity between the end joints of the link. The linear potential difference is the torsor of the kinematical pair. The constant vector $\vec{r}_i$ is thus equal to the radius vector from the corresponding junction to some common reference point.
	Vertex	A plate supported by the pillars	Link in the mechanism. The potential is interpreted as the angular velocity of the link.

Table 2. Flow and Potential Line Graph Representations and their usage.

Separation of the CR to two tables is not arbitrary. It can be seen from these Tables that there is a certain correspondence between the properties described in their left and right sides. Indeed it can be proved that the corresponding CR are actually dual:

Duality between flow and potential graphs [1]. Given a flow graph G_F execute the following steps: build its dual graph G_F^*, equate the potential differences in the edges of G_F^* to the flows in the corresponding edges of G_F. It then follows from the properties of dual graphs [4] that these potential differences satisfy the potential law in G_F^*. Thus, G_F^* can be considered as a valid potential graph G_Δ. Finally, it can be postulated that for each flow graph G_F there exists a dual potential graph G_Δ and vice versa.

From the duality between the flow and potential graph representations one can deduce the duality relation between trusses and mechanisms, as is outlined in the diagram in Figure 2.

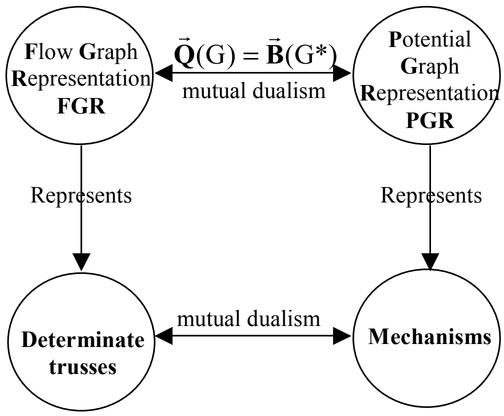

Figure 2: Diagram explaining the mutual dualism between trusses and mechanisms.

Similar procedure can be performed to prove the duality between Potential and Flow Line Graph Representations [3].

3 Duality between statical and kinematical engineering systems.

From the duality between PGR and FGR, one can conclude that the engineering systems they represent, namely mechanisms and trusses are also dual. This duality relation is defined as follows:

- there is a link in the mechanism for each rod, mobile support reaction or external force in the truss.

- the vector of the relative velocity of each mechanism link is equal to the vector of the force acting in the corresponding element in the truss.

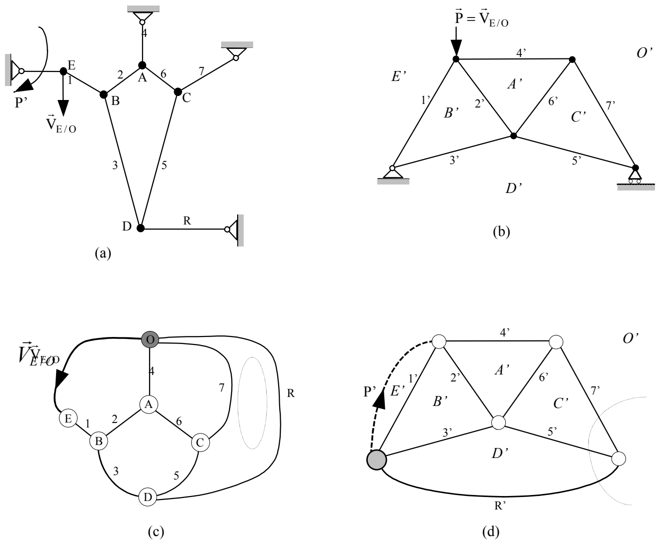

Figure 3: Duality between mechanisms and trusses - (a) Mechanism, (b) Dual Truss, (c) Potential Graph Representation of the Mechanism, (d) Flow Graph Representation of the truss.

Figure 3 shows an example of a mechanism and its corresponding dual truss, obtained through building the corresponding graph representations.

In the same way as it was done for mechanisms and trusses, one may conclude the duality between mechanisms and pillar structures from the duality between PLGR and FLGR.
This duality implies that for each mechanism there is a corresponding dual pillar structure and vice versa. Table 3, summarized the relations between the two engineering systems.

Properties of a pillar structure	**Properties of the dual mechanism**
Plate / set of forces acting on the plate.	Face / set of links limiting the face.
Force i.	Relative angular velocity of the kinematical pair i^*.
Plane parallel to the plates of the system.	Plane of the mechanism links.
$\vec{r}_{i,j}$ = radius vector between two adjacent joints upon which internal/external forces are acting.	Li,j – vector parallel to a link in the mechanism whose end kinematic joints are i and j.
$\vec{r}_i$ - radius vector to the point of action of force i.	$\vec{r}_{i^*}$ - radius vector of the kinematical pair i^*.

Table 3. Correspondence between the properties of pillar structure and its dual mechanism.

Figure 4 shows an example of a mechanism and its dual pillar structure.

Interesting result emerges, when the duality relation is applied to a one-dimensional case of engineering systems. The one-dimensional plate acted upon by a number of vertical forces is actually a beam acted upon external forces and reactions. The one dimensional mechanism on the other hand has all links parallel to one another, combining it to the fact that it also satisfies torsor circuit rule, leads us to conclude that it is actually a gear system. Consequently, one-dimensional determinate beams are dual to the gear mechanisms, as is shown in the Figure 1 above.

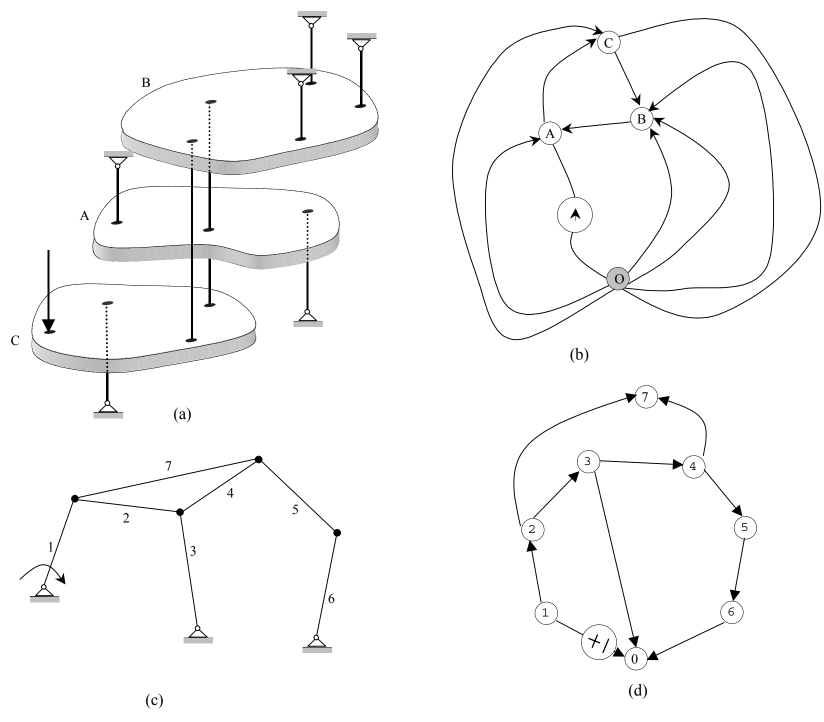

Figure 4: (a) Pillar structure, (b) Flow Graph Representation of the Pillar structure, (d) Potential Graph Representation of the Mechanism, (c) Dual Mechanism.

4 Practical aspects of the duality relations.

4.1 The mutual dualism between Maxwell-Cremona and image velocity diagrams

Image velocity diagram is a known graphical method for velocity analysis of mechanisms [5]. Once the image velocity diagram of a mechanism is constructed, the relative velocities of every mechanism link can be measured directly from it.
Knowing that the relative velocities in the mechanism links are equal to the internal forces in the rods of the dual truss, it can be concluded that the image velocity can be used to perform the static analysis of a truss. Furthermore, one can build the diagram directly from the truss, without even considering its dual mechanism as is explained in the current subsection. This process becomes clear when summarizing all the properties of the image velocity diagram while simultaneously rewriting them in the terminology of structural analysis, as is done in Table 4.

From Table 4, it follows that the image velocity method completely coincides with the known Maxwell-Cremona diagram algorithm for static analysis of determinate structures [6]. Consequently, Maxwell-Cremona and Image velocity methods are mutually dual methods.

Image velocity properties	
In mechanism terminology	**In terminology of the dual truss**
Each point corresponds to a joint in a mechanism.	Each point corresponds to a non-bisected area in the truss closed by truss elements.
The driving link is represented by a line in its velocity direction and length proportional to its velocity value. The line connects the points corresponding to the end joints of the link.	The external force is represented by a line in its direction with length proportional to its magnitude. The line connects the points corresponding to the areas separated by the external force.
The relative velocity of a link is represented by a line parallel to the link relative velocity, which connects the points corresponding to the end joints of the link.	The axial force in a rod is represented by a line parallel to the rod, connecting the points corresponding to the areas separated by it.

Table 4. Properties of the image velocity diagram in the terminology of both mechanisms and the dual trusses.

Figure 5 presents a four bar chain, its dual truss and the image velocity diagram. One can verify that this diagram also presents the static analysis diagram of the dual truss, namely its Maxwell-Cremona diagram.

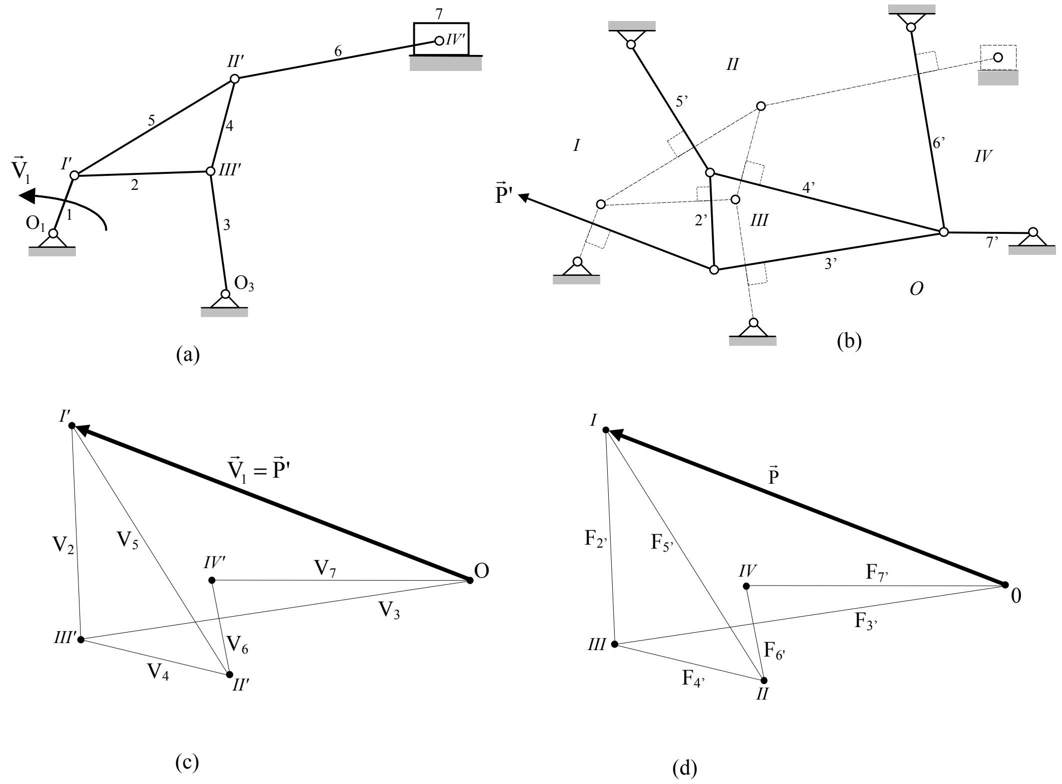

Figure 5: The correspondence between image velocity and Maxwell-Cremona diagrams. (a) The four bar chain. (b) Its dual truss. (c) The image velocity (d) Maxwell-Cremona diagram.

4.2 Checking stability through dual mechanisms

Sometimes the problem of checking stability of statical system requires its thorough numerical investigation. Employing the duality relations may in some cases assist in avoiding such a procedure.

Consider, for example, two determinate trusses of Figure 6a and 6b. Only the first one of these two is stable. Reaching this conclusion without performing calculations, is not easy even for experts in mechanical engineering. On the other hand, considering the mechanisms dual to these trusses makes the task easier. Figure 6c shows the mechanism dual to truss of Figure 6a, whereas Figure 6d shows the mechanism dual to truss of Figure 6b. In the mechanism of Figure 6d, links 1 and 9 are co-linear, in contrast to the mechanism of Figure 6c. Therefore, it is easy to derive that the dual mechanism of the first truss on Figure 6d is locked while the dual mechanism of the second truss is not. This makes it possible to postulate that the truss of Figure 6a is not stable, whereas the truss of Figure 6b is stable. This example strengthens the claim that there are properties which are hard to detect in the primal representation, whereas they are transparent in the dual.

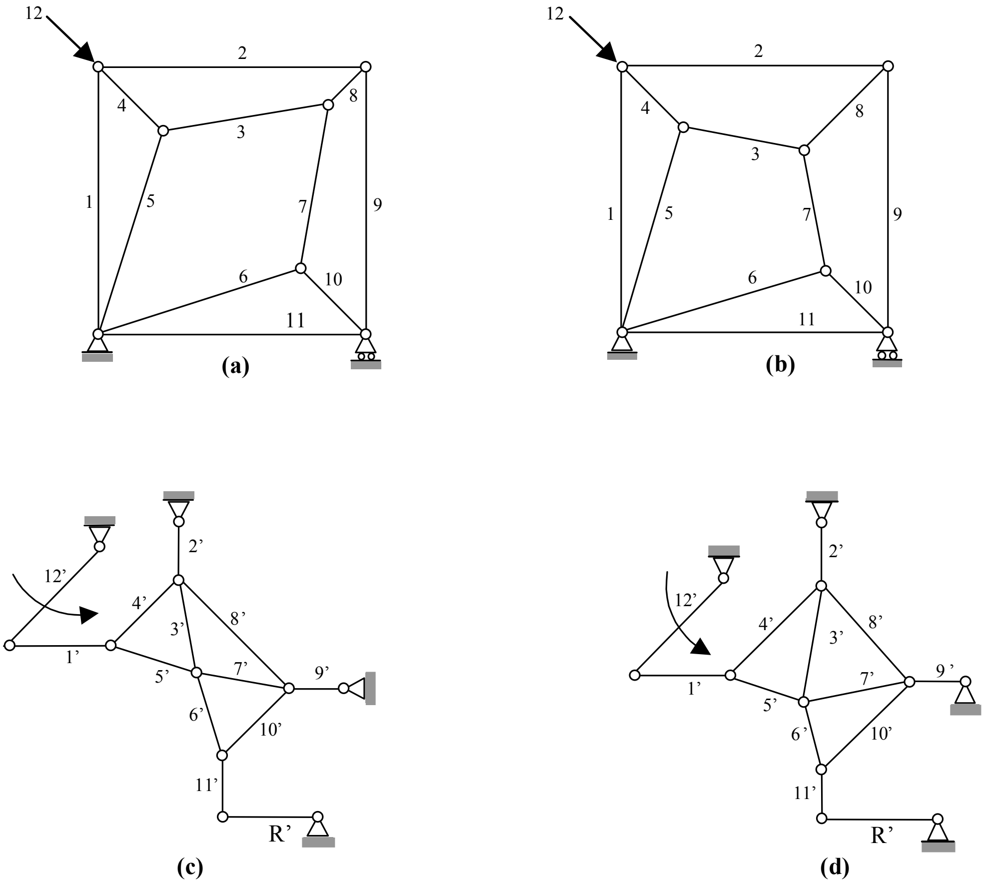

Figure 6: Example of stable and non-stable trusses and their dual mechanisms.
(a) A stable truss (b) A non-stable truss (c, d) Corresponding dual mechanisms.

4.3 Employing duality for designing engineering systems

The duality connection between mechanisms and structures can be applied for synthesis of new engineering systems. The main idea behind this approach lies in the fact that if a mechanism possesses some special engineering properties, then its dual truss or pillar structure possess the exact same properties. In the following example the idea is employed to solve a static design problem.

Suppose one needs to design a static system, such that when a small force is applied to one of its joints, a much greater force is produced in one of its rods. Such a static system can be obtained immediately by using the duality between trusses and mechanisms. This is done by first finding a known mechanism having similar velocity characteristics, namely, a mechanism that for a small relative velocity in its driving link produces in its other link a much greater relative velocity. One of many known mechanisms satisfying this requirement is presented on Figure 7a. The velocity of link 1 of this mechanism is considerably larger than that of the link 5. The truss dual to this mechanism is presented in Figure 7d. According to the duality property, the truss possesses the same force characteristics as the velocity characteristics of the mechanism, i.e. a small external force P causes a much greater force in rod 1.

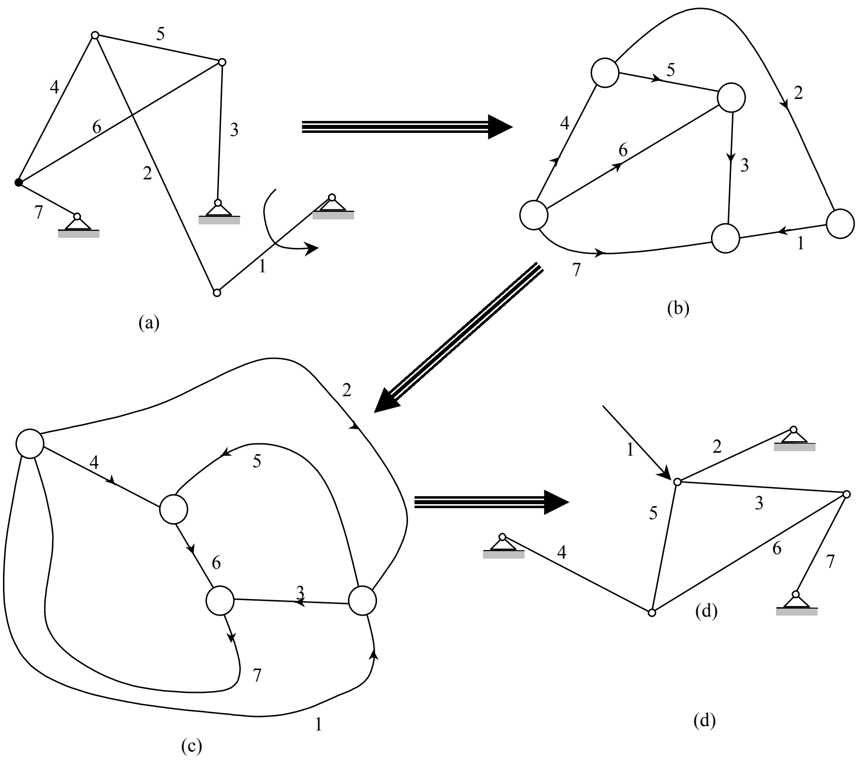

Figure 7: Truss design. (a) Original mechanism, (b) Corresponding potential graph representation. (c) Dual flow graph representation and (d) the resulting truss.

5 Further research.

The general vector formulation of the graph representations involved should not limit us to two-dimensional systems. The variables of FLGR and PLGR both may acquire arbitrary spatial orientations without contradicting the embedded rules and properties of these representations. Consider for example a spatial structure presented in Figure 8:

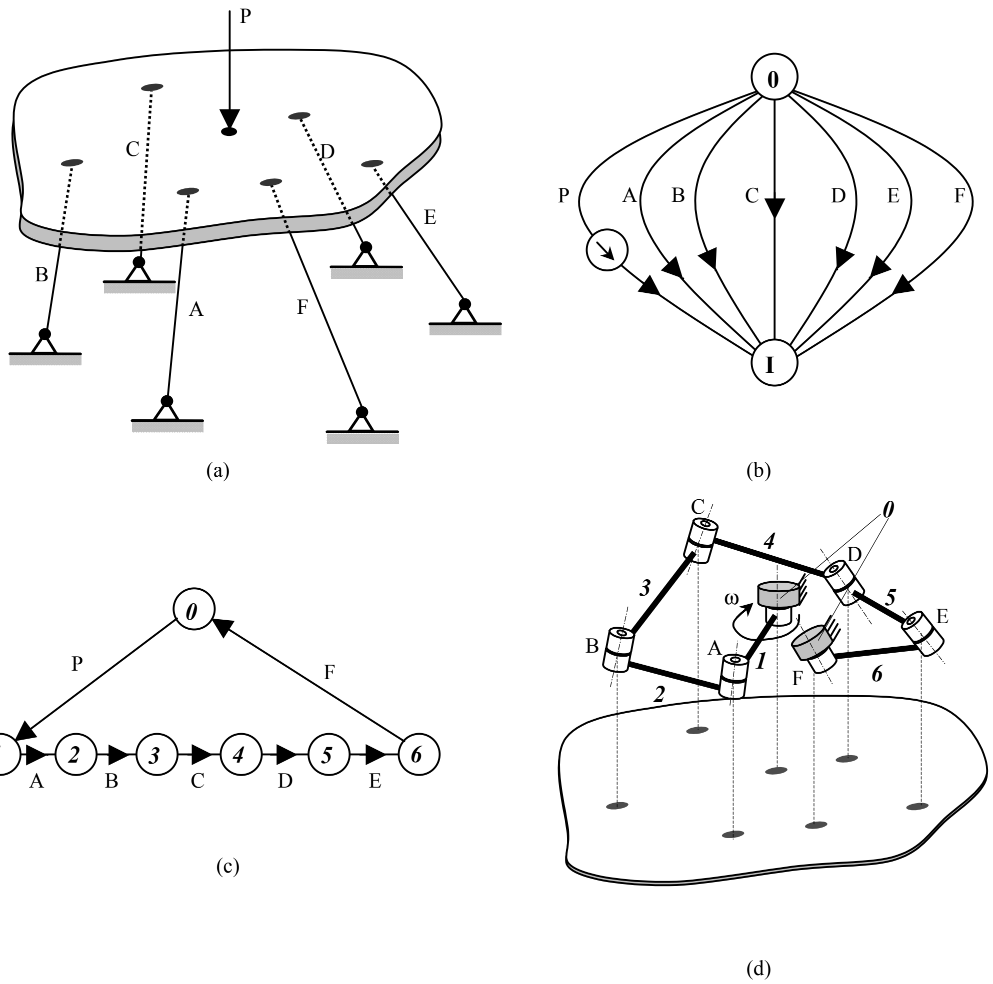

Figure 8. Stewart platform and its dual. (a) Stewart platform, (b) Corresponding FLGR, (c) Dual PLGR and (d) the corresponding dual spatial mechanism.

The structure shown in Figure. 8 presents a statical aspect of the known Stewart platform. The PLGR dual to the graph appearing in Figure 8b appears in Figure 8c. The graph is built of 6 serially connected edges and thus corresponds to a mechanism with 6 serially connected links, also known as a serial manipulator mechanism. By means of this relation one can now convert all the reasoning processes upon the Stewart platform to processes upon the dual mechanism, thus gaining new abilities that were not previously available. This result resonates with a research done in robotics for establishing duality between serial and parallel manipulators [7,8] that was mainly based on the screw theory.

6 Conclusions

The paper has introduced some of the theoretical and practical contributions of two duality relations: between mechanisms and determinate trusses, and between mechanisms and pillar structures. It should be noted that although the highlight of the paper was the contribution to the field of civil engineering, the general approach introduced is applicable to many other engineering fields as well. Up until today the approach has been employed in statics, kinematics, electronics and hydraulics. In additional to theoretical contribution of the approach it may have implications in design, analysis, optimization, checking validity and other aspects of engineering research.

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