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ORIGINAL ARTICLE

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Representing and analysing integrated engineering systems through combinatorial representations

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Abstract The current paper introduces a systematic method for representing and analysing coupled integrated engineering systems by means of general discrete mathematical models, called Combinatorial Representations, that can be conveniently implemented in computers. The combinatorial representation of this paper, which is based on graph theory, was previously shown to be useful in representing engineering systems from different engineering domains. Once all of the subsystems of an integrated multidisciplinary system are brought up to the common level of the combinatorial representation, they cease to be separated from one another and the analysis process is applied to all of the engineering elements disregarding the domain to which they belong.

During the development of the representation and study of its inherent properties, special attention was dedicated to developing an efficient analysis method. A vectorial extension of the mixed variable method known from electrical network theory was found to be the most suitable choice for this purpose.

In the paper, the approach is implemented by representing and analysing two systems: one that is a macro system comprised of truss, dynamic and electric elements, and another that is a comb-driven micro-resonator. The techniques presented in the paper are not limited to analysis only, but can be applied to many other aspects of engineering research. Among them is a systematic derivation of new ways of presenting engineering elements, one of which – the process of derivation of a new type of force representation entitled "face force" – is described in the paper.

Keywords Combinatorial representations · Integrated engineering systems · Graph theory · Mixed variable method

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Abbreviations

CR	Combinatorial Representations
FCFS	Flow Controlled Flow Source
FCPS	Flow Controlled Potential difference Source
FGR	Flow Graph Representation
MCA	Multidisciplinary Combinatorial Approach
PCFS	Potential difference Controlled Flow Source
PCPS	Potential difference Controlled Potential difference Source
PGR	Potential Graph Representation
RGR	Resistance Graph Representation

Other nomenclature

$\vec{B}$	Vector circuit matrix
b_i	Damping factor of the damper i
C^{Δ}	Chords of G^{Δ}
C'^{F}	Chords of G^{F} excluding independent sources
C_i	Capacity of capacitor i
C_{ij}	Initial capacity between combs of transducer ij
$circuit(i)$	Circuit defined by chord i of the graph
$cs\alpha$	$\cos\alpha \cdot \sin\alpha$
$cutset(i)$	Cutset defined by branch i of the spanning tree
d_{ij}	Gap between the fingers of opposite combs in transducer ij
$\vec{\Delta}(e)$	Potential difference of edge e
Δ^{Δ}	Set of independent potential difference sources
$\vec{\Delta}$	Vector of potential differences
E^{F}	Set of edges of graph G^{F}
E^{Δ}	Set of edges of graph G^{Δ}
$\vec{F}(e)$	Flow in edge e
$\vec{F}$	Vector of flows
G^{F}	Graph whose independent variables are flows
G_i	Stiffness coefficient of rod i
G_{R}	Resistance graph

G^{Δ}	Graph whose independent variables are potential differences
h_{ij}	The overlap between the fingers of opposite combs in transducer *ij*
H^{F}_{ij}	Ratio between the flows of edges *i* and *j*
H^{F}	Matrix of ratios between flows in the graph
H^{Δ}_{ij}	Ratio between the potential differences of edges *i* and *j*
$\boldsymbol{H}^{\Delta}$	Matrix of ratios between potential differences in the graph
$\mathbf{I}$	Unit matrix
k_i	Elasticity coefficient of the spring *i*
$K(e)$,$\boldsymbol{K}(e)$	Scalar and matrix presentations of the conductivity of edge e
$\boldsymbol{K}$	Conductivity matrix of the graph
L_i	Inductivity of coil *i*
m_i	Mass *i*
n_{ij}	Number of fingers in each comb of the transducer *ij*
P^{F}	Set of independent flow sources of the graph
$\vec{Q}$	Vector cutset matrix
$R(e)$, $\boldsymbol{R}(e)$	Scalar and matrix presentations of the resistance of edge *e*
$\boldsymbol{R}$	Matrix of resistances of the graph
s	Laplace transform operator
T^{F}	The spanning tree of graph G^{F}
T'^{Δ}	The spanning tree of graph G^{Δ} excluding the independent sources
$v_{0_{ij}}$	Initial voltage between the combs of the transducer *ij*

Subscripts

The first and the second subscripts beside the matrix name (mainly B and Q) indicate the set of edges the rows/columns of the matrix correspond to:

C	Chords
C'	Chords that are not sources
T	Branches of the spanning tree
T'	Branches of the spanning tree that are not sources

Superscripts

The first and the second superscripts beside the matrix name (mainly B and Q) indicate the graph the rows/columns of the matrix correspond to:

F	G^{F} (graph whose independent variables are flows)
Δ	G^{Δ} (graph whose independent variables are potential differences)

Line type attributes

dashed line	Chord
dotted line	Edge in E^{Δ}
double line	Branch of a spanning tree
bold line	Edge for which the value of the flow or potential difference is known
solid line	Edge for which the value of flow or potential difference is unknown

Introduction

This work is a part of a general approach, called Multidisciplinary Combinatorial Approach (MCA), in accordance with which engineering systems are represented and analysed through three systematic stages. First, discrete mathematical models, called Combinatorial Representations (CR), are developed on the basis of graph theory, matroid theory or discrete linear programming. Then, after the properties of the combinatorial representations are thoroughly investigated and the connections between them established, these representations are applied to represent and to analyse diverse systems from unrelated engineering fields. This approach has produced several important results, some of which appear in [1, 2, 3].

The paper employs only one of the features of this approach – representing tightly-coupled engineering systems by a single unified combinatorial representation, and deriving from it the combinatorial properties and the combinatorial analysis method for a particular system. The approach is demonstrated in two steps: first, it is applied to an integrated macro-system comprised of static, dynamic and electrical components coupled with one another; next it is applied to a known MEMS – a comb-driven micro resonator.

Performing analysis on the graph representation enables utilization of the discrete mathematical knowledge embedded in the graph. This knowledge is shown to lead to a "self-formulating method" that can easily be computerized to derive the analysis equations for integrated systems. Moreover, thorough investigation of the relations between the combinatorial representations opens up new avenues of research, by making knowledge from different engineering fields available one to another. Employing this principle, the paper introduces the derivation of a new type of representation for forces, based on the relations between the Flow Graph and Potential Graph combinatorial representations [1].

To facilitate reading of the paper and comprehension of the generality of the approach, the mathematical foundation is first provided, and with that as the starting point the engineering applications are developed. In the next section of the paper we briefly review the relevant theoretical material of the Multidisciplinary Combinatorial Approach (MCA). We start with a brief description of graph network theory terminology, and then introduce the main combinatorial representation of this paper – the Resistance Graph Representation (RGR). This combinatorial representation has already been applied to represent diverse engineering systems, including multidimensional trusses [4], dynamical systems [2], electrical systems [2].

In the section after that we introduce the vector mixed-variable method, which is the extension to multiple dimensions of the mixed variable method known in electrical network theory. Then, on the basis of the combinatorial properties embedded in the Resistance Graph Representation (RGR), the analysis procedure is developed.

The next section employs the fact that the Resistance Graph Representation (RGR) is a general combinatorial representation by which systems from various engineering fields can be represented. Therefore, integrated engineering systems consisting of elements from these fields can be represented by the Resistance Graph Representation. Analysis based on the vector mixed-variable method procedure is then performed in a unified way upon the Resistance Graph that represents an integrated system. In order to facilitate the explanation, the proposed method is first demonstrated on a macro-integrated engineering system. Afterwards, the method is applied to MEMS, by representing a comb-driven microresonator by the Resistance Graph Representation and deriving the characteristic equations in a systematic way.

Leading on from this, we then have a section which highlights the fact that representing integrated systems by combinatorial representations is not limited only to modelling and analysing integrated engineering systems, but is also useful in other applications. In this section we introduce the derivation of a new engineering entity entitled the "face force", which enables one to obtain a new insight on the flow of the forces in structures, and so can reveal special properties inherent in an integrated system.

The mathematical foundation of the paper lies in graph theory, which is a well-known topic in discrete mathematics. This issue is widely-used in engineering, especially for analysis of electrical networks [5]. In 1955, Trent [6] was one of the first to establish a relationship between physical systems and graph theory, and since then it has been applied to many engineering fields. Andrews has developed a methodology for applying graphs to multidimensional dynamic systems [7, 8]. In structural mechanics two of the most significant works are by Fenves and Branin [9] and Kaveh [10]. In machine theory, the graph representation, in addition to analysis of mechanisms [11], was used as an abstract model of kinematic chains to aid in the creative stage of mechanism design [12]. In simulation, a technique at the level of the composition of system components has been developed [13]. Analogy, on the basis of graph theory, between different one-dimensional systems such as dynamics, electricity, and heat transfer appears in many books [14, 15].

Methods aimed at unified analysis of systems composed of elements from different engineering disciplines are mainly based on the idea of transforming all of the elements to equivalent elements from only one discipline. One of the earlier works concerning this issue was conducted by Kron [16], who approached it by transforming all of the engineering systems to equivalent electrical circuits. More recent work has been conducted by Senturia who suggested transforming elements in macromodels of MEMS to equivalent electrical elements upon which both analysis and design are then performed [17].

Network graphs

The combinatorial representation that is used in this paper is based on network and graph theories, the relevant details of which can be found in [2], or in books on graph theory, such as [18].

Before approaching the combinatorial representation itself, a brief review of the definitions specific to the Multidisciplinary Combinatorial Approach (MCA) is needed. The paper uses terms from network theory where graphs are usually characterized by matrices such as cutset, circuit and incidence matrices [19]. In the current approach, graphs are used to represent both the topology and the geometry of the engineering system, so the matrices are resolved into two corresponding types: vector matrices and scalar matrices [2]. The first type is actually the type of matrix that is used in network theory [19], where the term "vector" stands for the fact that these matrices provide information about the topological relations of the vectors of the network variables, without considering their geometry. The matrices of the second type – scalar – provide information about the geometry of the corresponding engineering elements. These matrices can be obtained from the vector matrices by multiplying each of their columns by a unit vector in the direction of the corresponding engineering element. The above definitions were found to be useful, since they help to reveal both topological and geometrical properties embedded in graphs.

There are several graph representations that are used for representing various engineering systems in the Multidisciplinary Combinatorial Approach (MCA) [2]. The combinatorial representation that is employed in this paper is the Resistance Graph Representation (RGR); therefore its underlying theory and properties are provided in the following subsection.

Resistance graph

The resistance graph, designated G_R, is a network graph, where in certain edges there is a dependence between flow and potential difference. Additionally, the flows and the potential differences in the resistance graph must satisfy two general laws: the *flow law* for flows and *potential law* for potential differences.

– *Flow Law*: the vector sum of the flows in every cutset of G_R is equal to zero.

The matrix form of the Flow Law is:

$$\vec{Q} \cdot \vec{F} = 0 \tag{1}$$

where $\vec{F}$ is the *flow vector* and $\vec{Q}$ is the *vector cutset matrix*.

– *Potential Law*: for every circuit in G_R, the sum of the potential differences of all the circuit edges is equal to zero.

In matrix representation, this law is written:

$$\vec{B} \cdot \vec{\Delta} = 0 \tag{2}$$

where $\vec{\Delta}$ is the *potential difference vector* and $\vec{B}$ is the *vector circuit matrix*.

One of the main properties of this representation is the *orthogonality principle*:

$$\vec{B} \cdot \vec{Q}^t = 0 \tag{3}$$

Several important relations between the matrices and the graph variables, namely the flows and the potential differences, can be derived from the orthogonality principle and the flow and potential laws, as follows:

$$\vec{B}_{\mathrm{T}} = -\vec{Q}^t_{\mathrm{C}} \tag{4}$$

$$\vec{\Delta} = \vec{Q}^t \cdot \vec{\Delta}_{\mathrm{T}} \tag{5}$$

$$\vec{F} = \vec{B}^t \cdot \vec{F}_{\mathrm{C}} \tag{6}$$

where the subscripts T and C indicate that the corresponding matrix (vector) includes only the columns (members) corresponding to the branches of the tree and the chords respectively.

The edges of the graph are classified in accordance with the relations between their flows and potential differences into four types of edges:

– resistance edges
– flow source edges
– potential difference source edges
– two-port edges

The definitions of these four types follow.

Resistance edges are edges where the dependence between the flow and potential difference is characterized by either a constant scalar (Eq. 7) or a constant matrix (Eq. 8), as follows:

$$\left|\vec{\Delta}(e)\right| = R(e) \cdot \left|\vec{F}(e)\right|; \quad \left|\vec{F}(e)\right| = K(e) \cdot \left|\vec{\Delta}(e)\right| \tag{7}$$

$$\vec{\Delta}(e) = R(e) \cdot \vec{F}(e); \quad \vec{F}(e) = K(e) \cdot \vec{\Delta}(e) \tag{8}$$

where $\vec{\Delta}(e)$ and $\vec{F}(e)$ are the potential difference and the flow of edge e.

Flow source edges are edges in which the flow is given and is independent of the potential difference in the edge.

Potential difference source edges are edges in which the potential difference is given and is independent of the flow in the edge.

Two-ports are the elements that contain two edges in the resistance graph.

The four variables possessed by the two edges are designated $\vec{\Delta}_1, \vec{\Delta}_2, \vec{F}_1, \vec{F}_2$. The two-port defines various mathematical relations between the pairs of these variables. In most cases, in accordance with these connections, the variables of one of the edges are determined by the variables of the other; so the former edge can be referred to as the ''source edge'' and the latter as the ''control edge''. The mathematical connections between the variables are expressed by terminal equations written in the form of matrix-vector multiplications.

The two-ports are classified in accordance with the sets of variable pairs that are interrelated. The most common types of two-ports [19] are given in Table 1, where in each interrelated variable pair, the first argument corresponds to the control edge and the second to the source edge.

Table 1 Different types of two-ports

Two-port	Interrelated pairs (control, source)	Terminal equations
Potential difference-Controlled Flow Source (PCFS)	(Δ_1, F)	$\begin{pmatrix}\vec{F}_1\\ \vec{F}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ K_{12} & 0\end{pmatrix} \cdot \begin{pmatrix}\vec{\Delta}_1\\ \vec{\Delta}_2\end{pmatrix}$
Flow-Controlled Potential difference-Source (FCPS)	(F_1, Δ_2)	$\begin{pmatrix}\vec{\Delta}_1\\ \vec{\Delta}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ R_{12} & 0\end{pmatrix} \cdot \begin{pmatrix}\vec{F}_1\\ \vec{F}_2\end{pmatrix}$
Potential difference-controlled flow mutual source (gyrator)	(Δ_1, F_2), (F_1, Δ_2)	$\begin{pmatrix}\vec{F}_1\\ \vec{F}_2\end{pmatrix} = \begin{pmatrix}0 & K_{21}\\ K_{12} & 0\end{pmatrix} \cdot \begin{pmatrix}\vec{\Delta}_1\\ \vec{\Delta}_2\end{pmatrix}$
Potential difference-Controlled Potential difference Source (PCPS)	(Δ_1, Δ_2)	$\begin{pmatrix}\vec{\Delta}_1\\ \vec{F}_1\end{pmatrix} = \begin{pmatrix}H^{\Delta}_{21} & 0\\ 0 & 0\end{pmatrix} \cdot \begin{pmatrix}\vec{\Delta}_2\\ \vec{F}_2\end{pmatrix}$
Flow-Controlled Flow Source (FCFS)	(F_1, F_2)	$\begin{pmatrix}\vec{\Delta}_1\\ \vec{F}_1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & H^{\mathrm{F}}_{21}\end{pmatrix} \cdot \begin{pmatrix}\vec{\Delta}_2\\ \vec{F}_2\end{pmatrix}$
Potential difference and flow controlled potential difference and flow source (ideal transformer)	(Δ_1, Δ_2), (F_1, F_2)	$\begin{pmatrix}\vec{\Delta}_1\\ \vec{F}_1\end{pmatrix} = \begin{pmatrix}H^{\Delta}_{21} & 0\\ 0 & H^{\mathrm{F}}_{21}\end{pmatrix} \cdot \begin{pmatrix}\vec{\Delta}_2\\ \vec{F}_2\end{pmatrix}$
Fully coupled two-port	$(\Delta_2$ and F_2, $\mathrm{F}_1)$, $(\Delta_2$ and F_2, $\Delta_1)$,	$\begin{pmatrix}\vec{\Delta}_1\\ \vec{F}_1\end{pmatrix} = \begin{pmatrix}R_{21} & H^{\Delta}_{21}\\ H^{\mathrm{F}}_{21} & K_{21}\end{pmatrix} \cdot \begin{pmatrix}\vec{F}_2\\ \vec{\Delta}_2\end{pmatrix}$

Self-formulating method for obtaining the analysis equations for the Resistance Graph Representation

The mixed variable method is a well-known method widely used in electrical network theory [19]. In this method, the independent variables (main unknowns) are chosen to be partially potential differences and partially flows. After finding these independent unknowns, all of the flows and all of the potential differences of the graph can be easily determined. This section introduces a vectorial extension of this method, where the variables are vectors, in contrast to the scalars in electrical theory. In order to distinguish this extension from the original method, it will be referred as the *vectorial mixed variable method*. The mathematical derivation of the method follows.

First, all of the edges of the resistance graph are divided into two groups – E^F and E^Δ, in accordance with Table 2, where the symbols √ or - designate whether a specific type of edge should or should not be included in the corresponding set of edges. Having chosen the two groups of edges, two graphs – G^F and G^Δ – are built. Graph G^F is obtained by contracting the edges of E^Δ from the original graph, namely $G^F = G_R \circ E^\Delta$, while G^Δ is obtained by deleting the edges of E^F, namely $G^\Delta = G_R - E^F$.

It has been proved in [1] that the contraction (deletion) operation preserves the validity of the flow (potential) law; so the flow (potential) law is preserved valid as well for G^F (G^Δ).

Once the two graphs are built, the spanning trees (or the spanning forests in the case where one or both graphs are disconnected) are to be chosen within them. The union of these trees (forests) yields a spanning tree (forest) for the original graph. The choice of the spanning tree (forest) should comply with the last two columns of Table 2. The information given in this table is dictated by the derivation process that follows and the justification for these specific choices is provided below.

The edges of the graphs now belong to the following four groups:

- T^Δ, C^Δ: the branches and the chords corresponding to the spanning forest in G^Δ
- T^F, C^F: the branches and the chords corresponding to the spanning forest in G^F

Furthermore, by separating the flow and potential difference source edges from the rest, the following six groups are obtained:

- Δ^Δ and P^F: the potential difference and flow sources
- T'^Δ, C^Δ, T^F, C'^F: the above-mentioned four groups that include neither flow nor potential difference sources

The potential law in the matrix form is now rewritten, with the circuit matrix and the potential difference vector divided in accordance with edge classification into these six groups:

$$\vec{B}\cdot\vec{\Delta} = 0$$

$$\Rightarrow \begin{array}{cc} & \begin{array}{cccccc} \Delta^\Delta & T'^\Delta & T^F & C^\Delta & C'^F & P^F \end{array} \\ \begin{array}{c} C^\Delta \\ C'^F \\ P^F \end{array} & \begin{pmatrix} \vec{B}^{\Delta\Delta}_{C\Delta} & \vec{B}^{\Delta\Delta}_{CT'} & 0 & I & 0 & 0 \\ \vec{B}^{F\Delta}_{C'\Delta} & \vec{B}^{F\Delta}_{C'T'} & \vec{B}^{FF}_{C'T} & 0 & I & 0 \\ \vec{B}^{F\Delta}_{P\Delta} & \vec{B}^{F\Delta}_{PT'} & \vec{B}^{FF}_{PT} & 0 & 0 & I \end{pmatrix} \end{array} \cdot \begin{pmatrix} \vec{\Delta}^\Delta \\ \vec{\Delta}^\Delta_{T'} \\ \vec{\Delta}^F_T \\ \vec{\Delta}^\Delta_C \\ \vec{\Delta}^F_{C'} \\ \vec{\Delta}_P \end{pmatrix}$$

$$= 0 \tag{9}$$

The superscripts and the subscripts beside the submatrices and subvectors indicate, respectively, the subgraphs and groups of edges to which the rows and the columns of the submatrix correspond. For instance, $\vec{B}^{F\Delta}_{C'T'}$ is that part of the vector circuit matrix whose rows correspond to the chords of G^F and columns to the branches of G^Δ.

The first row of Eq. 9 gives:

$$\vec{\Delta}^\Delta_C = -\vec{B}^{\Delta\Delta}_{C\Delta}\cdot\vec{\Delta}^\Delta - \vec{B}^{\Delta\Delta}_{CT'}\cdot\vec{\Delta}^\Delta_{T'} \tag{10}$$

Table 2 Classification of the graph edges

Type of edge		E^F	E^Δ	Tree (forest)	Chord
Resistance edge		√	√	√	√
Flow source edge		√	–	–	√
Potential difference source edge		–	√	√	–
Flow Controlled Flow Source (FCFS)	control	√	–	–	√
	source	–	√	√	–
Potential difference Controlled Potential difference Source (PCPS)	control	–	√	√	–
	source	√	–	–	√
Flow Controlled Potential difference Source (FCPS)	control	√	–	–	√
	source	√	–	–	√
Potential difference Controlled Flow Source (PCFS)	control	–	√	√	–
	source	–	√	√	–
Transformer	edge of port 1	√	–	–	√
	edge of port 2	–	√	√	–
Gyrator	both edges	√	√	√	√
Fully coupled two-port	edge of port 1	√	–	–	√
	edge of port 2	–	√	√	–

The second row of Eq. 9 gives:

$$\vec{B}^{\mathrm{FF}}_{\mathrm{C'T}} \cdot \vec{\Delta}^{\mathrm{F}}_{\mathrm{T}} + \vec{\Delta}^{\mathrm{F}}_{\mathrm{C'}} = -\vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'}\Delta} \cdot \vec{\Delta}^{\Delta} - \vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'T'}} \cdot \vec{\Delta}^{\Delta}_{\mathrm{T'}} \tag{11}$$

Applying Eq. 4 to the flow law gives:

$$\vec{Q} \cdot \vec{F} = \left(I| - \vec{B}^{t}_{\mathrm{T}}\right) \cdot \vec{F}$$

$$= \begin{array}{c} \\ \Delta \\ T^{\prime\Delta} \\ T^{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \Delta & \mathrm{T}^{\Delta} & \mathrm{T}^{\mathrm{F}} & \mathrm{C}^{\Delta} & \mathrm{C}^{\prime\mathrm{F}} & \mathrm{P} \end{array} \\ \begin{pmatrix} I & 0 & 0 & -(\vec{B}^{\Delta\Delta}_{\mathrm{C}\Delta})^{t} & -(\vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'}\Delta})^{t} & -(\vec{B}^{\mathrm{F}\Delta}_{\mathrm{P}\Delta})^{t} \\ 0 & I & 0 & -(\vec{B}^{\Delta\Delta}_{\mathrm{CT'}})^{t} & -(\vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'T'}})^{t} & -(\vec{B}^{\mathrm{F}\Delta}_{\mathrm{PT'}})^{t} \\ 0 & 0 & I & 0 & -(\vec{B}^{\mathrm{FF}}_{\mathrm{C'T}})^{t} & -(\vec{B}^{\mathrm{FF}}_{\mathrm{PT}})^{t} \end{pmatrix} \end{array}$$

$$\times \begin{pmatrix} \vec{F}^{\Delta}_{\Delta} \\ \vec{F}^{\Delta}_{\mathrm{T'}} \\ \vec{F}^{\mathrm{F}}_{\mathrm{T}} \\ \vec{F}^{\Delta}_{\mathrm{C}} \\ \vec{F}^{\mathrm{F}}_{\mathrm{C'}} \\ \vec{F}_{\mathrm{P}} \end{pmatrix} = 0 \tag{12}$$

The second row of Eq. 12 gives:

$$\vec{F}^{\Delta}_{\mathrm{T'}} - (\vec{B}^{\Delta\Delta}_{\mathrm{CT'}})^{t} \cdot \vec{F}^{\Delta}_{\mathrm{C}} = (\vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'T'}})^{t} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{C'}} + (\vec{B}^{\mathrm{F}\Delta}_{\mathrm{PT'}})^{t} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{P}} \tag{13}$$

The last row of Eq. 12 gives:

$$\vec{F}^{\mathrm{F}}_{\mathrm{T}} = (\vec{B}^{\mathrm{FF}}_{\mathrm{C'T}})^{t} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{C'}} + (\vec{B}^{\mathrm{FF}}_{\mathrm{PT}})^{t} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{P}} \tag{14}$$

The following four equations are derived from Eqs. 7, 8 and the equations of Table 1. They describe the flow/potential relationships in the resistance and the two-port edges.

$$\vec{\Delta}^{\mathrm{F}}_{\mathrm{T}} = R^{\mathrm{F}}_{\mathrm{T}} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{T}} \tag{15}$$

$$\vec{F}^{\Delta}_{\mathrm{C}} = K^{\Delta}_{\mathrm{C}} \cdot \vec{\Delta}^{\Delta}_{\mathrm{C}} \tag{16}$$

$$\vec{\Delta}^{\mathrm{F}}_{\mathrm{C'}} = R^{\mathrm{F}}_{\mathrm{C'}} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{C'}} + H^{\Delta} \cdot \vec{\Delta}^{\Delta}_{\mathrm{T'}} \tag{17}$$

$$\vec{F}^{\Delta}_{\mathrm{T'}} = K^{\Delta}_{\mathrm{T'}} \cdot \vec{\Delta}^{\Delta}_{\mathrm{T'}} + H^{\mathrm{F}} \cdot \vec{F}^{\mathrm{F}}_{\mathrm{C'}} \tag{18}$$

It can be seen from Eqs. 15, 16, 17 and 18 that the most comprehensive relations between the system variables can be established if $\vec{\Delta}^{\Delta}_{\mathrm{T'}}$ and $\vec{F}^{\mathrm{F}}_{\mathrm{C'}}$ are chosen to be the independent variables of the analysis equations. Eqs. 17 and 18, expressing the relations of these variables, are rewritten in a more convenient form in Eq. 19:

$$\begin{pmatrix} \vec{\Delta}^{\mathrm{F}}_{\mathrm{C'}} \\ \vec{F}^{\Delta}_{\mathrm{T'}} \end{pmatrix} = \begin{pmatrix} R^{\mathrm{F}}_{\mathrm{C'}} & H^{\Delta} \\ H^{\mathrm{F}} & K^{\Delta}_{\mathrm{T'}} \end{pmatrix} \cdot \begin{pmatrix} \vec{F}^{\mathrm{F}}_{\mathrm{C'}} \\ \vec{\Delta}^{\Delta}_{\mathrm{T'}} \end{pmatrix} \tag{19}$$

Eq. 19 possesses a general form of the two-port terminal equations that appear in Table 1, and indeed, they all are included in this equation. Therefore, the edges representing the ports can be classified into the edge groups in such a manner that their terminal equations fit into Eq. 19. This was the main motivation behind the choices made in Table 2.

Performing algebraic substitutions and rearrangement of terms in Eqs. 10 and 11, and applying Eq. 4 yields the following final equation of the vector mixed-variable method:

$$\begin{pmatrix} R^{\mathrm{F}}_{\mathrm{C'}} + \vec{B}^{\mathrm{FF}}_{\mathrm{C'T}} \cdot R^{\mathrm{F}}_{\mathrm{T}} \cdot \left(\vec{B}^{\mathrm{FF}}_{\mathrm{C'T}}\right)^{t} & \vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'T'}} + H^{\Delta} \\ \vec{Q}^{\Delta\mathrm{F}}_{\mathrm{T'C'}} + H^{\mathrm{F}} & K^{\Delta}_{\mathrm{T'}} + \vec{Q}^{\Delta\Delta}_{\mathrm{T'C}} \cdot K^{\Delta}_{\mathrm{C}} \cdot \left(\vec{Q}^{\Delta\Delta}_{\mathrm{T'C}}\right)^{t} \end{pmatrix} \cdot \begin{pmatrix} \vec{F}^{\mathrm{F}}_{\mathrm{C'}} \\ \vec{\Delta}^{\Delta}_{\mathrm{T'}} \end{pmatrix} = \begin{pmatrix} -\vec{B}^{\mathrm{F}\Delta}_{\mathrm{C'}\Delta} & -\vec{B}^{\mathrm{FF}}_{\mathrm{C'T}} \cdot R^{\mathrm{F}}_{\mathrm{T}} \cdot \left(\vec{B}^{\mathrm{FF}}_{\mathrm{PT}}\right)^{t} \\ -\vec{Q}^{\Delta\Delta}_{\mathrm{T'C}} \cdot K^{\Delta}_{\mathrm{C}} \cdot \left(\vec{Q}^{\Delta\Delta}_{\Delta\mathrm{C}}\right)^{t} & -\vec{Q}^{\Delta\mathrm{F}}_{\mathrm{T'P}} \end{pmatrix} \cdot \begin{pmatrix} \vec{\Delta}^{\Delta} \\ \vec{F}^{\mathrm{P}} \end{pmatrix} \tag{20}$$

A self-formulating method for building the analysis equations is obtained by employing the topological meaning of the matrices that appear in Eq. 20. The topological interpretation of a general element located in row i and column j in the analysis matrix appear in Fig. 1.

Representing and analysing macro-integrated and micro-integrated engineering systems

The current section applies the Resistance Graph Representation (RGR) to represent, in a unified way, engineering systems which consist of interrelated elements from a variety of domains, such as electricity, dynamics and multidimensional indeterminate structures.

Representation of some of the above elements by a graph is well-known in the literature [20]. Table 3 provides a review in the terminology of this paper in order to supply the reader with a quick reference for dealing with different engineering domains. In this table, for each engineering element the physical interpretation of its corresponding variables (flow and potential difference) is given, and for each edge, the terminal equation, namely the equation describing the relation between the flow and potential difference of that edge, is given.

Using the information provided in Table 3, the integrated system is transformed into the graph representation. Then, applying a unified analysis method while disregarding the type of physical element to which the specific edges correspond, yields a unified treatment of the system. This ability opens up an avenue of practical research, since the integrated systems can now be analysed, designed or stored in a database as one whole, in a systematic and convenient manner.

Example of analysis of an integrated system using the Resistance Graph Representation

Figure 2 shows an example of an integrated engineering system to be analysed by means of the unified procedure

Chords $i, j \in C'^{F}$. [i,j] = proper signed sum[1] of the resistances in all the edges which belong to both circuit (i) and circuit (j) in G^F if i=j it is equal to the positive sum of all the edge resistances in circuit (i). If there exists an FCPS for which j is the control edge and i is the source edge then R_{ij} is added.	Chord $i \in C'^{F}$ and branch $j \in T'^{\Delta}$. [i,j]=1 with a properly chosen sign[2], if branch j is included in circuit (i) of G^R, 0 otherwise. If there exists a PCPS, for which j is the control edge and i the source edge *then* H^{Δ}_{ij} is added.	·	$\vec{F}^{F}_{C'}$ the vector of flow magnitudes in the chords of G^F that are not flow sources.	=	Chord $i \in C'^{F}$ and potential difference source $j \in \Delta^{\Delta}$. [i,j] = 1 with minus of a properly chosen sign[2] *if* potential difference source j is included in circuit (i)	Chord $i \in C'^{F}$ and flow source $j \in P^{F}$. [i,j] = minus of a properly signed sum[1] of the resistances of edges that belong to both circuit (i) and circuit (j).	·	Δ^{Δ} the potential differen-ce sources in G^{Δ}.
Branch $i \in T'^{\Delta}$ and chord $j \in C'^{F}$. [i,j]=1 with a properly chosen sign[2] if chord j is included in cutset (i) of G^R, 0 otherwise. *If* there exists an FCFS where i is the source edge and j is the controlled edge *then* H^{F}_{ij} is added.	Branches $i, j \in T'^{\Delta}$. [i,j]= the properly signed sum[1] of the conductivities in all the edges which belong to both cutset (i) and cutset (j) of G^{Δ}. *If i=j then* it is equal to the positive sum of all the conductivities of edges of cutset (i). If there exists a PCPS for which j is the control edge and i is the source edge then K_{ij} is added.		$\vec{\Delta}^{\Delta}_{T'}$ the components of potential difference vectors in the branches of G^{Δ} that are not potential sources.		Branch $i \in T'^{\Delta}$ and potential source $j \in \Delta^{\Delta}$ [i,j]= minus of the proper signed sum[1] of the conductivities of all the edges belonging to both cutset (i) and cutset (j).	Branch $i \in T'^{\Delta}$ and flow source $j \in P^{F}$. [i,j] = 1 with an opposite of the properly chosen sign[2] if the flow source edge j is in cutset (i) of G_R.		F^{P} the flow sources in G^F.

[1]The proper sign for an edge common to two cutsets (circuits) is plus, if it is oriented similarly in relation to both cutsets (circuits), minus otherwise.

[2]The proper sign is plus if the branch (chord) i is in the direction of circuit (j) (cutset (j)), minus otherwise.

Fig. 1 The topological interpretation of the mixed variable method equations

described above. The system consists of coupled elements of a multidimensional indeterminate truss, a dynamic system and an electrical control circuit. The truss is acted upon by two external loads – the given force P_1 and the force applied to the truss by the dynamic system through spring 7. The deformation of rod 1 resulting from these loads is coupled with the voltage source 12 of the control circuit. The current produced in the ampmeter 15 controls the external force P_2 acting upon the dynamic system, where mass 8 has a given initial velocity.

The graph representing the system, and the division of the edges to E^{Δ} and E^{F}, is shown in Fig. 3, where the edges belonging to E^{Δ} are designated by dotted lines. From Fig. 3 one can then obtain the graphs G^{Δ} and G^{F}. These graphs are shown in Fig. 4 and Fig. 5 with spanning forests highlighted.

The graph of Fig. 4 consists of the following sets of edges:

Table 3 Edges and vertices in RGR representing an engineering system

Domain	Engineering system interpretation		Type of element	Flow-Potential Difference relation
	Edge and flow	Vertex and potential		
Electrical circuit	Edge corresponds to electrical element: resistor, capacitor, coil, current or voltage source. Flow corresponds to electrical current through the element.	Vertex corresponds to the junction in the circuit. Potential corresponds to the electric potential of the junction.	Resistor	$\Delta_i = F_i R_i$
			Capacitor	$F_i = C_i \frac{d\Delta_i}{dt} = C_i s \Delta_i$
			Coil	$\Delta_i = L_i \frac{dF_i}{dt} = sL_i F_i$
Dynamic system	Edge corresponds to dynamic element: mass, damper, spring external force, initial tension or velocity. Flow corresponds to the internal force in the element.	Vertex corresponds to a junction having independent velocity. Potential corresponds to the velocity of the junction.	Mass	$\Delta_i = \frac{F_i}{m_i s}$
			Spring	$\Delta_i = \frac{sF_i}{k_i}$
			Damper	$\Delta_i = b_i F_i$
Static system	Edge corresponds to a system element with an internal force and the flow to the force in that element.	Vertex is a joint connecting system elements. The potential corresponds to the displacement of the joint.	Rod	$\begin{pmatrix} F_x(e) \\ F_y(e) \end{pmatrix} = G(e) \cdot \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} \cdot \begin{pmatrix} \Delta_x(e) \\ \Delta_y(e) \end{pmatrix}$
			Reaction	No relation

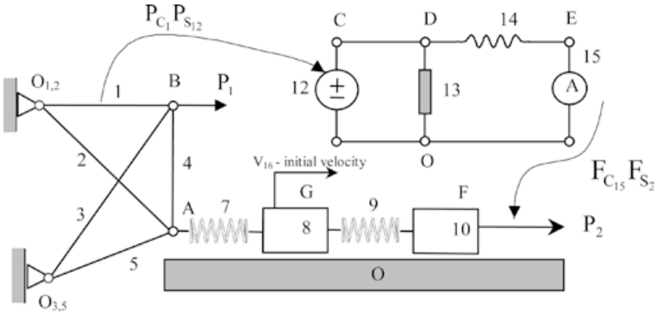

Fig. 2 Integrated engineering system, containing elements of a truss, a dynamic system and an electrical circuit

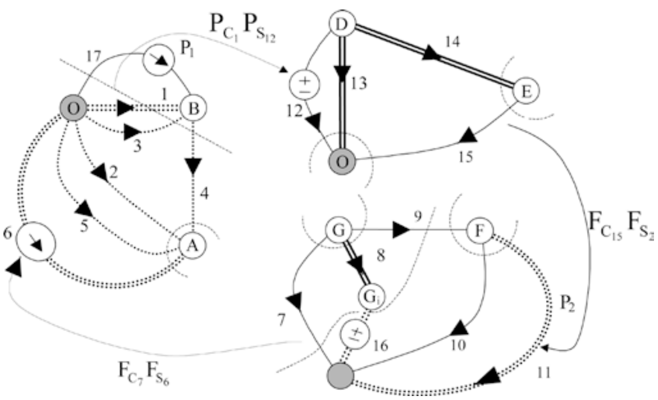

Fig. 3 Graph representing the system of Fig. 2

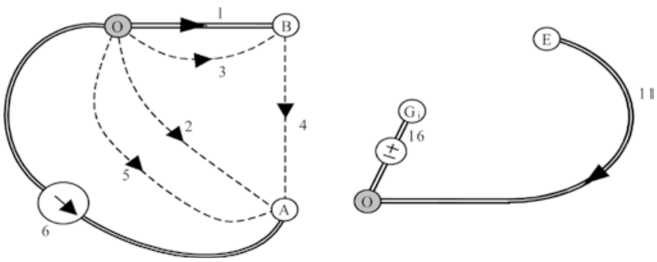

Fig. 4 The graph G^{Δ}

$$T'^{\Delta} = \{1, 6, 11\}$$
$$\Delta^{\Delta} = \{16\}$$
$$C^{\Delta} = \{2, 3, 4, 5\}$$

The graph in Fig. 5 consists of the following sets of edges:

$$T^{F} = \{8, 13, 14\}$$
$$P = \{17\}$$
$$C'^{F} = \{7, 9, 10, 12, 15\}$$

Now, by means of the topological rules provided in Fig. 1, one can obtain the members of the analysis matrices directly from the graph, as shown in Fig. 6. Alternatively, these matrices could be obtained from Eq. 20.

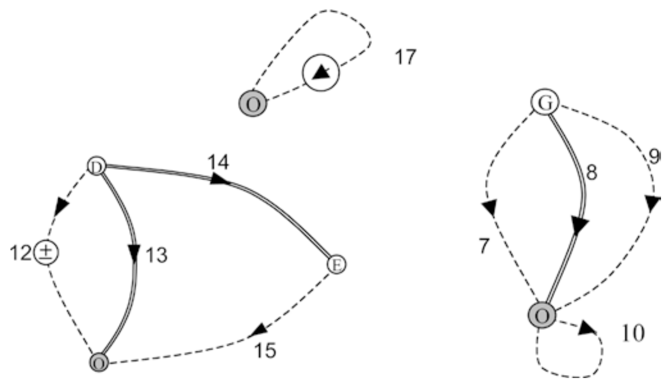

Fig. 5 The graph G^{F}

Example of analysis of an integrated microelectromechanical system using the Resistance Graph Representation

The same approach can be applied to some microelectromechanical systems, as we demonstrate in this section. Consider the comb-driven microelectromechanical resonator [21] shown in Fig. 7.

The principle underlying this engineering system is as follows: the electrical signal, that can be a signal consisting of periodic signals with different frequencies and amplitudes, is supplied to the system by the voltage source V_{in}. The signal is then converted by means of a comb transducer 2–3 into a mechanical force acting at point C upon the resonating beam. In this case, the beam acts as a classical resonator, by responding with a non-negligible displacement only to the force applied at its resonant frequency. In turn, the displacement of the beam is converted into an electrical signal V_{out} by means of the second comb transducer 6–7.

The graph representing the system is shown in Fig. 8. Edges 1 and 2 of the graph represent the electrical circuit of the input terminal. Edges 3, 4, 5 and 6 represent the mechanical part of the system, whereas 4, 5 and 6 form a subgraph corresponding to the beam. The resistance of each of the latter three edges is of a distinct type – edge 4 is an elastic lumped member, edge 5 is a damping lumped member and 6 is a lumped mass. The two comb transducers are represented in the graph of Fig. 8 by two fully coupled two-ports: (2,3) and (6,7). Employing the relation developed in [22] for electrostatic comb transducers, the terminal equation of the two port (2,3) is as follows:

$$\begin{pmatrix} \Delta_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{C_{23}} & \frac{\Gamma_{23}}{C_{23}} \\ \frac{\Gamma_{23}}{C_{23}} & \frac{\Gamma_{23}^2}{C_{23}} \end{pmatrix} \cdot \begin{pmatrix} F_2 \\ \Delta_3 \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} F_2 \\ \Delta_3 \end{pmatrix}$$

Here

$$\Gamma_{23} = \frac{\varepsilon_0 n_{23} h_{23} v_{0_{23}}}{d_{23}}$$

and:

– C_{23} is the initial capacity between the combs of the transducer 23

$$\mathbf{R}^{F}_{C'}+\vec{\mathbf{B}}^{FF}_{C'T}\cdot\mathbf{R}^{F}_{T}\cdot\left(\vec{\mathbf{B}}^{FF}_{C'T}\right)^{t}\qquad \vec{\mathbf{B}}^{F\Delta}_{C'T}+\mathbf{H}^{\Delta}\qquad -\vec{\mathbf{B}}^{F\Delta}_{C'\Delta}\qquad -\vec{\mathbf{B}}^{FF}_{C'T}\cdot\mathbf{R}^{F}_{T}\cdot\left(\vec{\mathbf{B}}^{FF}_{PT}\right)^{t}$$

$$\begin{array}{cc|ccccc|ccccc}
 & & \multicolumn{5}{c|}{C'^{F}} & \multicolumn{5}{c}{T'^{\Delta}} \\
 & & 7 & 9 & 10 & 12 & 15 & 1x & 1y & 6x & 6y & 11 \\
\hline
 & 7 & R_7+R_8 & R_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 9 & R_8 & R_8+R_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
C'^{F} & 10 & 0 & 0 & R_{10} & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & 12 & 0 & 0 & 0 & R_{13} & R_{13} & h_{1,12}\cdot\cos\alpha_1 & h_{1,12}\cdot\sin\alpha_1 & 0 & 0 & 0 \\
 & 15 & 0 & 0 & 0 & R_{13} & R_{13}+R_{14} & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 1x & 0 & 0 & 0 & 0 & 0 & K_1\cdot\cos^2\alpha_1+K_3\cdot\cos^2\alpha_3+K_4\cdot\cos^2\alpha_4 & K_1\cdot cs\alpha_1+K_3\cdot cs\alpha_3+K_4\cdot cs\alpha_4 & -K_4\cdot\cos^2\alpha_4 & -K_4\cdot cs\alpha_4 & 0 \\
 & 1y & 0 & 0 & 0 & 0 & 0 & K_1\cdot cs\alpha_1+K_3\cdot cs\alpha_3+K_4\cdot cs\alpha_4 & K_1\cdot\sin^2\alpha_1+K_3\cdot\sin^2\alpha_3+K_4\cdot\sin^2\alpha_4 & -K_4\cdot cs\alpha_4 & -K_4\cdot\sin^2\alpha_4 & 0 \\
T'^{\Delta} & 6x & \cos\alpha_6 & 0 & 0 & 0 & 0 & -K_4\cdot\cos^2\alpha_4 & -K_4\cdot cs\alpha_4 & K_5\cdot\cos^2\alpha_5+K_2\cdot\cos^2\alpha_2+K_4\cdot\cos^2\alpha_4 & K_5\cdot cs\alpha_5+K_2\cdot cs\alpha_2+K_4\cdot cs\alpha_4 & 0 \\
 & 6y & \sin\alpha_6 & 0 & 0 & 0 & 0 & -K_4\cdot cs\alpha_4 & -K_4\cdot\sin^2\alpha_4 & K_5\cdot cs\alpha_5+K_2\cdot cs\alpha_2+K_4\cdot cs\alpha_4 & K_5\cdot\sin^2\alpha_5+K_2\cdot\sin^2\alpha_2+K_4\cdot\sin^2\alpha_4 & 0 \\
 & 11 & 0 & -1 & 1 & 0 & h_{11,15}\cdot & 0 & 0 & 0 & 0 & 0
\end{array}
\cdot
\begin{pmatrix} F_7\\ F_9\\ F_{10}\\ F_{12}\\ F_{15}\\ \Delta_{1x}\\ \Delta_{1y}\\ \Delta_{2x}\\ \Delta_{2y}\\ \Delta_{11}\end{pmatrix}
=
\begin{array}{c|c}
\Delta & P\\
16 & F_{17}\\
\hline
1 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\
0 & -\cos(\alpha_P)\\ 0 & -\sin(\alpha_P)\\ 0 & 0\\ 0 & 0\\ 0 & 0
\end{array}
\cdot
\begin{pmatrix}\Delta_{16}\\ F_{17}\end{pmatrix}$$

$$\vec{\mathbf{Q}}^{\Delta F}_{T'C'}+\mathbf{H}^{F}\qquad \mathbf{K}^{\Delta}_{T'}+\vec{\mathbf{Q}}^{\Delta\Delta}_{T'C}\cdot\mathbf{K}^{\Delta}_{C}\cdot\left(\vec{\mathbf{Q}}^{\Delta\Delta}_{T'C}\right)^{t}\qquad -\vec{\mathbf{Q}}^{\Delta\Delta}_{T'C}\cdot\mathbf{K}^{\Delta}_{C}\cdot\left(\vec{\mathbf{Q}}^{\Delta\Delta}_{\Delta C}\right)^{t}\qquad \vec{\mathbf{Q}}^{\Delta F}_{T'P}$$

Fig. 6 The set of linear equations corresponding to the engineering system presented in Fig. 2

– $v_{0_{23}}$ is the initial voltage between the combs of the transducer 23
– n_{23} is the number of fingers in each comb of the transducer 23

Additional constants involved are outlined in Fig. 7 Similarly, the terminal equation of the two port (6,7) is:

$$\begin{pmatrix}\Delta_6\\ F_7\end{pmatrix}=\begin{pmatrix}\frac{1}{C_{67}} & \frac{\Gamma_{67}}{C_{67}}\\ \frac{\Gamma_{67}}{C_{67}} & \frac{\Gamma_{67}^2}{C_{67}}\end{pmatrix}\cdot\begin{pmatrix}F_6\\ \Delta_7\end{pmatrix}\equiv\begin{pmatrix}e & f\\ g & h\end{pmatrix}\cdot\begin{pmatrix}F_6\\ \Delta_7\end{pmatrix}$$

The constants corresponding to the resistance edges 4, 5 and 6 are:

$$R_4=\frac{s}{k_4}$$

$$R_5=b_5$$

$$R_6=\frac{1}{m_6 s}$$

Applying the partition rules of the graph edges given in Table 2, the graph edges are classified into the following two sets:

$E^F=\{2,6\};\ E^\Delta=\{1,3,7,8,4,5\}$,

which yields the two graphs shown in Fig. 9.

The spanning tree chosen within the two graphs is highlighted in Fig. 9, and is expressed in the following six sets of edges:

$T^F=\phi$
$P^F=\phi$
$C'^F=\{2,6\}$

$T'^\Delta=\{3,7\}$
$\Delta^\Delta=\{1\}$
$C^\Delta=\{4,5,8\}$

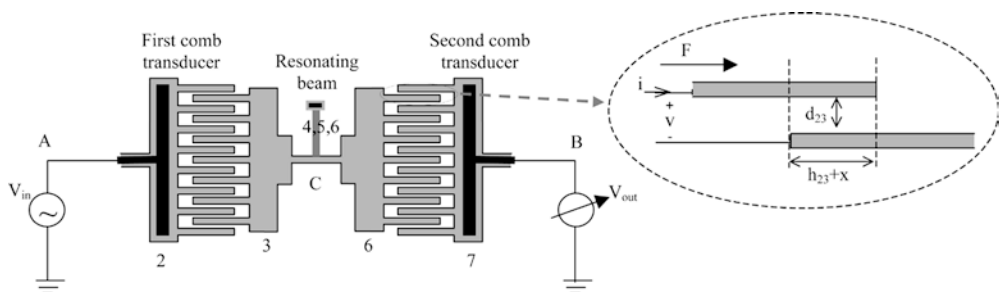

Fig. 7 A schematic diagram of a comb-driven microelectromechanical resonator

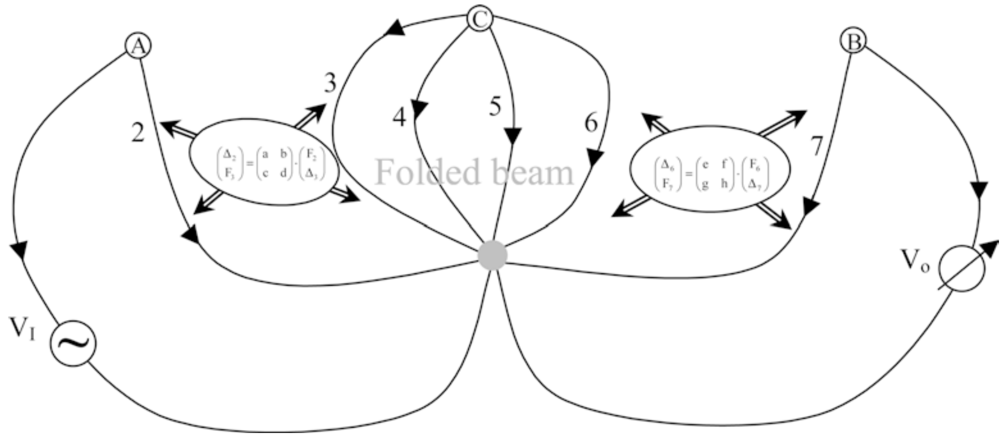

Fig. 8 The Resistance Graph Representation of the microelectromechanical system presented in Fig. 7

The matrices corresponding to the graph are:

$$H^{\Delta}=\begin{pmatrix} b & 0 \\ 0 & f \end{pmatrix} \quad B^{F\Delta}_{C'\Delta}=\begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad R^{F}_{C'}=\begin{pmatrix} a & 0 \\ 0 & e+R_6 \end{pmatrix} \quad B^{F\Delta}_{C'T'}=\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$Q^{\Delta F}_{T'C'}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H^{F}=\begin{pmatrix} c & 0 \\ 0 & g \end{pmatrix} \quad K^{\Delta}_{T'}=\begin{pmatrix} d & 0 \\ 0 & h \end{pmatrix} \quad Q^{\Delta\Delta}_{T'C}=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{FF}_{C'T}\cdot R^{F}_{T}\cdot \left(B^{FF}_{C'T}\right)^{t}=[0] \quad B^{\Delta\Delta}_{C'T}=\begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad B^{\Delta\Delta}_{C\Delta}=-\left(Q^{\Delta\Delta}_{\Delta C'}\right)^{t}=\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad K^{\Delta}_{C}=\begin{pmatrix} 1/R_4 & 0 & 0 \\ 0 & 1/R_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Substituting these matrices into Eq. 20 gives the final set of analysis equations:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & e+R_6 & -1 & f \\ c & 1 & d+1/R_4+1/R_5 & 0 \\ 0 & g & 0 & h \end{pmatrix}\cdot\begin{pmatrix} F_2 \\ F_6 \\ \Delta_3 \\ \Delta_7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\Delta_1$$

Obtaining new engineering entities through combinatorial representations

Previous sections have introduced a generalized approach for dealing with integrated systems by means of combinatorial representations. This result was achieved by bringing different engineering domains up to a common combinatorial level. Raising the problem up to the common level opens up new avenues of research, the analysis process being only one of them. Specifically, this higher level of representation enables us to gain new insights into the theory underlying engineering systems. An example is demonstrated in this section, by introducing the derivation of a new type of force in structures called the "face force".

Earlier publications [1, 3] showed that trusses and mechanisms can be represented by the Flow Graph Representation (FGR) and the Potential Graph Representation (PGR) respectively. At the common combinatorial level it was proved that these representations are dual, so in turn trusses and mechanisms are dual at the engineering level. The summary of the duality relation between mechanisms and trusses is outlined in Table 4.

The complete correspondence between trusses and mechanisms implies that for each variable in one system there exists a variable in the other possessing the same value. Nevertheless, a closer look into this relation

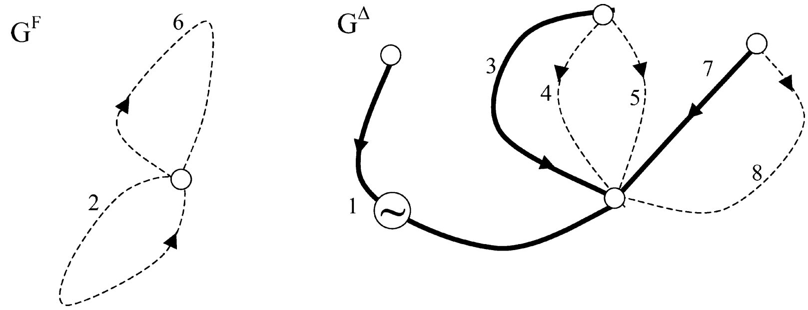

Fig. 9 G^F and G^{Δ} corresponding to the graph of Fig. 8

Table 4 Summary of the duality relations between mechanisms and trusses

Mechanism	Dual truss
Vertex	Face
Link	Rod
Relative linear velocity of the link	Internal force in the rod

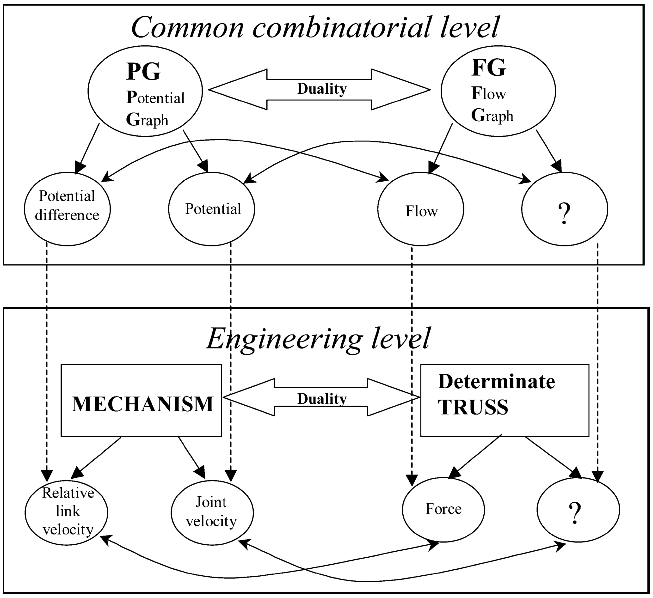

Fig. 10 Duality between trusses and mechanisms

reveals that there is no known type of variable in the truss to correspond to the linear velocity of a joint in the mechanism, as depicted in Fig. 10.

This gap reveals a new type of variable that can be used for analysis of trusses and gives a new insight into the analysis process. Furthermore, the derivation of this new variable is straightforward, since it can be done simply by employing the dualism relations (Table 4) and translating the properties of the linear velocity of a joint to the terminology of trusses, as is done in Table 5.

It can be concluded from Table 5 that the new variable established in trusses can be thought of as a multidimensional generalization of the "mesh current" in electrical circuits [19]. Here it will be called the "face force" in the truss. The action of the face forces in the truss is outlined in the example of Fig. 11.

Establishing the "face force" variable is not only important from a theoretical point of view, but has practical analysis applications as well [3].

It is interesting to note that Gabriel Kron once wrote: "It is easy to say 'mesh current' and let it go at that. But to say 'force acting in a closed mesh' requires a philosophical dissertation that would leave the mechanical engineer unresponsive." [16]. It seems that the concept foreseen by Kron is actually the one derived in this section.

Conclusions

The paper has introduced a graph representation that enables one to represent, as a single mathematical system, physical systems consisting of tightly-coupled components from different engineering disciplines. Doing so enables us to obtain a generalized perspective for a system, and then apply a unified analysis process to it, without needing to consider which engineering field the system elements correspond to. In order to reduce the computation complexity of the system analysis, a vector mixed variable method was developed.

The idea was demonstrated first on an integrated system consisting of elements from statics, dynamics and electricity. Then, the same unified approach was applied to represent and analyse a MEMS comb-driven micro-resonator.

Furthermore, it was shown that this approach can also contribute to the theoretical foundation of engineering. In the previous section it was shown that a new force entity can be derived from the relations between different combinatorial representations.

All of the results introduced in the current paper were derived by applying only one combinatorial representation, namely the Resistance Graph Representation (RGR). Since this work is part of a general approach that includes several interrelated combinatorial representations, it is expected that in the near future knowledge and methods from other fields of engineering will be applied to integrated engineering systems on the basis of the connections between these representations.

Table 5 Establishing the properties of the new variable in trusses

Properties of the joint velocity in mechanism	Translation to the terminology of the truss
The variable is a two-dimensional vector corresponding to each joint in the mechanism	The new variable is a two-dimensional vector corresponding to each face (area bounded by the truss rods) of the truss
Relative velocity of a mechanism link is equal to the difference between the linear velocities of the end joints of this link	The force in the corresponding truss rod is equal to the difference between the new variables corresponding to the faces separated by this rod
The linear velocity of a fixed joint is equal to zero	The new variable corresponding to the external face of the truss is equal to zero

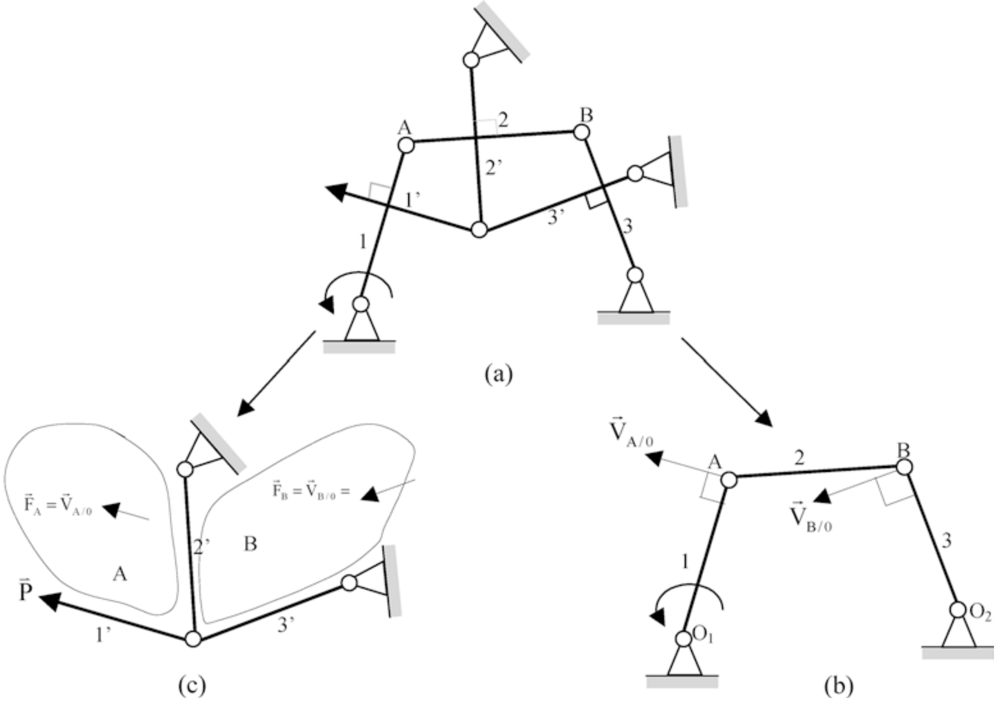

Fig. 11 The face forces in the dual truss: **a** the mechanism and its dual truss superimposed; **b** the mechanism; **c** the dual truss

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