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Multidimensional max-flow method and its application for plastic analysis

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Abstract

This paper expands the use of network flows to the multidimensional case, in which network flows are associated with vectors, instead of the conventionally used scalar values. A method for solving a multidimensional max-flow problem is systematically developed, on the basis of the primal-dual algorithm. It is demonstrated that upon reduction to a one-dimensional case, the method is transformed to the known Ford and Fulkerson algorithm.

This multidimensional flow network can be applied to a variety of engineering applications, as the variables underlying engineering systems frequently possess a vector form. In this paper, the multidimensional max-flow problem is shown to correspond to the problem of plastic analysis of trusses.

Keywords: Multidimensional network flows; Plastic analysis; Max-flow problem; Primal-dual algorithm; Trusses

1. Introduction

The paper introduces a multidimensional flow network problem, its engineering implementation and a method for its solution. The discussed network problem is shown to be the multidimensional expansion of the known max-flow problem. It is demonstrated that similarly to the one-dimensional case [1], the multidimensional max-flow problem is equivalent to the plastic problem in structural analysis. The method for solving the problem was developed through the primal-dual algorithm [2]. The algorithm is employed in the paper to replace the original problem by a series of transformed-simpler problems, which is performed in a systematic manner.

The primal-dual algorithm for solving linear programming problems was first formulated in [3]. If applied to the shortest path and max flow problems, primal-dual algorithm leads to the known algorithms of Dijkstra [4] and Ford and Fulkerson [5], respectively.

The paper first explains the essence of the multidimensional max-flow problem, both in the terminology of networks and the terminology of trusses. This way, the isomorphism between the networks and the engineering problems [6,24] is established. In order to develop the optimization algorithm for solving this problem, the paper adopts a less conventional formulation of the primal-dual algorithm [2], where the solution of the LP problem is obtained through iterative transformations of the original LP problem, to the two transformed ones (RP—Restricted primal, and DRP—Dual Restricted Primal). As is shown in the paper at each step of the method developed in the paper it is seen that in one-dimensional case the method reduces to the Ford and Fulkerson algorithm [5]. The correspondence between the max-flow problem and the problem of maximal loading upon trusses, also called a problem of load carrying capacity, [1] enables to apply the method to trusses, as is demonstrated in the paper. A short comparison between the derived method and the known methods of plastic analysis is provided. Two step-by-step examples of real two-dimensional truss maximal loading problems solved by means of the obtained method are demonstrated at the end of the paper.

Maximal load determination, or plastic analysis, is the problem of central interest for the mechanical engineering community [7].

The method that is systematically developed in the paper is different from other known plastic analysis methods, although certain similarities to some of them can easily be traced. The method is clearly related to the method of

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inequalities [8], based on treating the system of linear inequalities underlying the statical properties of the structure. The dual transformed problem, employed in the paper for detection of the halting point, uses a mechanism similar to the collapsed mechanism that is also used in the method of combining mechanisms [9].

The relation between linear and mathematical programming with the problem of plastic analysis was first described by Charnes and Greenberg [10], but can also attributed to Prager [1] and was further established in 1970 by Maier [11]. The duality of linear programming was introduced into the plastic analysis by Gavarini [12]. Algorithmic approach for solving the plastic problems with the help of linear programming formulation was started in 1979 by Kaneko [13] and continued by Franchi and Cohn [14]. Graph-theoretic approach to the problem was developed in 1971 by Fenves and Gonzales-Caro [15]. The primal-dual approach has been employed in the field of structural optimization [16], who mainly dealt with topology optimization and not with plastic analysis.

2. Formulation of the max-flow problem

Prager has shown that the one-dimensional truss maximal loading problem [1] is equivalent to the max-flow problem [5] in a network, whose edges are associated with the rods of the truss, and the vertices with the joints. The flows through the network edges are equal to the forces acting in the corresponding truss rods. Accordingly, the capacities of the edges are set equal to the yield limits of the rods.

In terminology of network flows the original max-flow problem can be formulated as follows:

Find the maximal flow in the source edge P of the directed network, so that the flows in the edges do not exceed the allowed capacities. The flow in the edge is considered positive if it flows in the direction of the edge, negative otherwise. Fig. 1(a) presents an example of flow network for which the max-flow problem will be formulated.

The one-dimensional engineering problem that corresponds to the above formulated max-flow problem is formulated as follows:

Find the maximal loading P that can be applied upon the one-dimensional truss, so that the yield limit is not exceeded in any of the rods. The force in the rod is considered positive if the rod is in compressed state, negative otherwise. Fig. 1(b) shows a one-dimensional truss, whose maximal load problem is equivalent to the maximal flow problem for the network of Fig. 1(a).

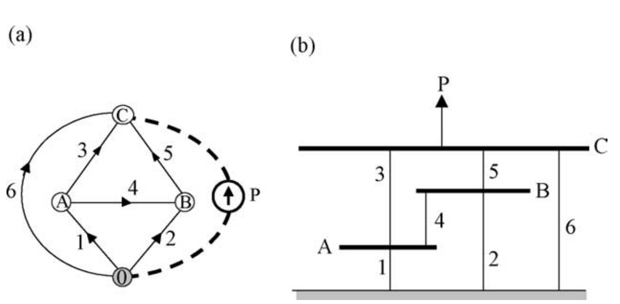


Fig. 1. The correspondence between the max-flow problem and the problem of maximal loading (a) the flow network problem and (b) corresponding maximal loading problem upon a one-dimensional truss.

Representing a problem by a network enables formulating the problem in linear programming through the variables associated with the network:

$$\begin{aligned} &\text{Max } P \\ &\forall v \in \mathbf{V} \quad \sum_{e \in \mathbf{E}'} I(e,v)\cdot F(e) + I(p,v)\cdot P = 0 \\ &\forall e \in \mathbf{E}' \quad F(e) \le b^{+}(e) \\ &\forall e \in \mathbf{E}' \quad -F(e) \le b^{-}(e) \\ &\forall e \in \mathbf{E}' \quad F(e) \le > 0 \\ &\qquad\quad P > 0 \end{aligned} \tag{1}$$

where: $\mathbf{V}$ is the set of vertices of the network, $\mathbf{E}'$ is the set of edges that are not sources. $I(e,v)$ is a function of adjacency between edge e and vertex v. It equals 1 if v is the head vertex of e, -1 if it is the tail vertex, 0 otherwise. $F(e)$—is the flow in edge e, while $b^{+}(e)$ and $b^{-}(e)$ are the upper and the lower bounds of the flow through edge e.

It is well known that linear programming provides one with additional insight upon the optimization problem from the aspect of the 'dual problem'—another problem, the optimal solution of which possesses the same value of the target function [17]. The dual problem is of a great use in linear programming, and in particular in the primal-dual algorithm adopted in this paper. Eq. (2) shows the LP formulation of the dual problem, as it is derived from the original formulation of Eq. (1) by applying the LP duality transformation rules.

$$\begin{aligned} &\text{Min} \sum_{\forall \text{ edge } e=(t,h) \text{ in the network}} \gamma^{+}(t,h) b^{+}(t,h) \\ &\quad + \sum_{\forall \text{ edge } e=(t,h) \text{ in the network}} \gamma^{-}(t,h) b^{-}(t,h) \\ &\pi(h) - \pi(t) + \gamma^{+}(t,h) - \gamma^{-}(t,h) \ge 0 \\ &\quad (\text{for each edge } e=(t,h)) \\ &-\pi(t_p) + \pi(h_p) \ge 1 \ (\text{for the source edge } p=(t_p,h_p)) \\ &\pi <> 0 \quad \gamma^{+}(t,h) \ge 0 \quad \gamma^{-}(t,h) \ge 0 \end{aligned} \tag{2}$$

where: h and t are the head and the tail vertices of an edge e; $\pi(v)$—is a variable, associated with every vertex of the network and thus can be seen as a potential variable of the network; $\gamma^{-}(t,h)$ and $\gamma^{+}(t,h)$ are associated with every edge of the network. Since in the constraint they are related to the subtraction between the corresponding potentials, they can be considered positive and negative potential differences of the edge.

The optimal solutions to the original and the dual problems are related through the following complementary slackness relations [18]:

$$\text{a. } \gamma^{+}(e)(b^{+}(e)-F(e))=0 \quad \text{b. } \gamma^{-}(e)(b^{-}(e)-F(e))=0 \tag{3}$$

The dual problem is defined upon the variables that are associated with the network, both through the problem formulation and the complementary slackness conditions. Thus, the problem can be interpreted as a problem that is also associated with the network, and accordingly with the represented truss.

In the terminology of network flows, the dual problem can be interpreted as follows:

Find a minimal cutset in the network—so that the sum of the capacities of its edges is minimal [19]. According to the complementary slackness conditions, an edge belongs to a minimal cutset if and only if in the maximum flow solution, the flows in these edges are equal to their capacity (the edges are saturated). In the terminology of Eq. (2), $\pi(v)$ has the value of 1 if vertex v is from one side of the minimal cutest (same side as the tail vertex of the source edge), 0 otherwise. $\gamma^{+}(t,h)/\gamma^{-}(t,h)=$equals to 1 if edge (t,h) belongs to the minimal cutset and is oriented in forward/backward direction relatively to the cutset.

The corresponding one-dimensional engineering problem would accordingly be:

In the plastic analysis, the rods reaching the yield limit are in the plastic mode—they can assume any kinematicaly required deformation. The problem is to find the 'minimal work plastic mechanism', defined by a set of rods to be put in the plastic mode in order to turn the truss into a mechanism [1]. The energy function is defined by the sum of the yield limits of the plastic rods—that is equal to the work done upon these rods if the joint of the external force moves a unit of length in the direction of the external force.

2.1. Two-dimensional expansion of the problems

The two-dimensional expansion of the network flow defines the flows associated with the edges to be two-dimensional vectors. The continuity condition for the flows in the network would now possess a vector form, meaning that the vector sum of the flows at each network vertex is zero. By allowing the flows to be vectors, the network can now be used to represent multidimensional systems, such as plane and spatial trusses, where each rod and thus its internal force possess different inclination. Representing trusses through network graphs is widely reported in the literature [6,20,21]. Since the angle of action of each force in the truss can be considered constant, the flows of the networks considered in this paper will be directed at a predetermined angle.

Accordingly the two-dimensional formulation of the network problem is as follows:

Given a two-dimensional network, where the flows are vectors with predetermined angles, find the maximal magnitude of the flow in the source edge P, so that the magnitudes of the flows in the edges do not exceed the allowed capacities.

The corresponding two-dimensional engineering problem is then:

Find the maximal loading P that can be applied upon the two-dimensional truss, so that the yield limit is not exceeded in any of the rods [22]. Fig. 2(a) and (b) present an example of a two-dimensional network and the corresponding two-dimensional truss for which the optimization problems can be formulated.

Eq. (4) presents the LP formulation of two-dimensional optimization problems defined in the above:

$$\begin{aligned} &\text{Max } P \\ &\forall v\in V \quad \sum_{e\in E'} I(e,v)\cdot F(e)\cdot\cos(\alpha(e)) + I(p,v)\cdot P\cdot\cos(\alpha(p))=0 \\ &\forall v\in \mathbf{V} \quad \sum_{e\in E'} I(e,v)\cdot F(e)\cdot\sin(\alpha(e)) + I(p,v)\cdot P\cdot\sin(\alpha(p))=0 \\ &\forall e\in E' \quad F(e)\le b^{+}(e) \\ &\forall e\in E' \quad -F(e)\le b^{-}(e) \\ &\forall e\in E' \quad F(e)\le>0 \\ &\qquad P>0 \end{aligned} \tag{4}$$

where $F(e)$ is the magnitude of the flow in edge e and $\alpha(e)$ is the angle outlining its direction.

The first two constraints in Eq. (4) formulated for each of the truss vertices are the flow continuity conditions. The first constraint is responsible for the continuity of the flows in the x axis, namely, that the amount of flow in the x axis entering the vertex is equal to the amount of flow in the x axis leaving it. The flow magnitudes are multiplied by the cosines of the angles of the corresponding edges and by a sign function, $I(e,v)$, indicating whether vertex v is the tail or the head vertex of edge e. Similarly, the second constraint is responsible for the flow continuity in the y axis.

Third and fourth constrains in Eq. (4) set the minimal and maximal boundaries for the flow—in the truss they correspond to the maximal tension and the maximal compression that can be applied upon the corresponding rod.

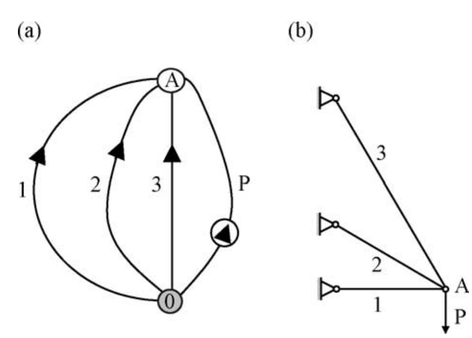


Fig. 2. Multidimensional expansion of the max-flow problem formulation. (a) The two-dimensional network and (b) the corresponding two-dimensional truss.

The LP formulation of the dual problem is provided in Eq. (5).

$$
\begin{aligned}
& \text{Min} \sum_{e\in \mathbf{E}'} b^{+}(e)\cdot\gamma^{+}(e) + \sum_{e\in \mathbf{E}'} b^{-}(e)\cdot\gamma^{-}(e) \\
\forall (e=\langle t,h\rangle)\in \mathbf{E}' \quad & (\pi_x(t)-\pi_x(h))\cdot\cos(\alpha(e)) + (\pi_y(t)-\pi_y(h))\cdot\sin(\alpha(e)) + \gamma^{+}(e)-\gamma^{-}(e)=0 \\
\text{for } (p=\langle 0,v_p\rangle) \quad & \pi_x(v_p)\cdot\cos(\alpha(p))+\pi_y(v_p)\cdot\sin(\alpha(p))\geq 1 \\
\forall v\in \mathbf{V} \quad & \pi_x(v),\pi_y(v)\leq\geq 0 \\
\forall e\in \mathbf{E}' \quad & \gamma^{+}(e),\gamma^{-}(e)\geq 0
\end{aligned}
\tag{5}
$$

In the dual problem, there are four types of variables: π_x, π_y and γ^{+}, γ^{-}. The first two are associated with each of the network vertices, while the latter two are associated with edges. The first constraint in Eq. (5) is associated with each network edges defines π_x and π_y as the potentials in the two dimensions of the vertices of the network, while γ^{+} and γ^{-} are the magnitudes of the potential difference of the edge. In the truss problem, the potentials can be seen as the displacements of the corresponding joints in x and y directions, γ^{+} as an elongation of the rod and γ^{-} as a shortening of the rod. Accordingly, the second constraint in Eq. (5) can be interpreted as a requirement that the displacement of the joint upon which the external force is applied is higher or equal than the unit of length. Fig. 3 demonstrates such an interpretation of the problem variables upon a deformed truss rod.

Therefore in the terminology of the two-dimensional networks, the dual problem can be formulated as follows:

Find a minimal two-dimensional cutset in the network—a set of edges, removing of which will block the flow through the network.

It should be noted, that the two-dimensional cutset differs in its properties from the cut-sets accustomed in regular networks, as not every connected network can be used to transfer two-dimensional flow.

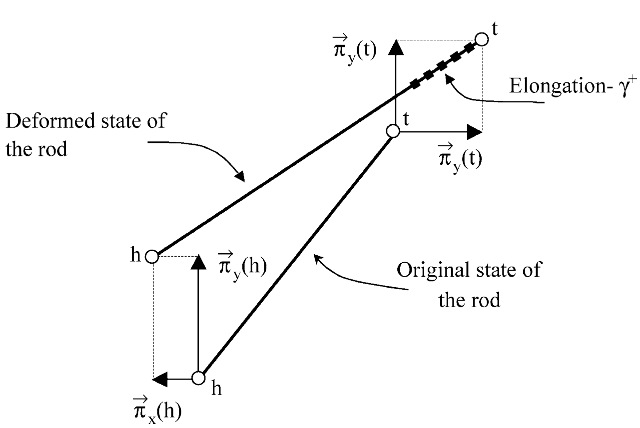


Fig. 3. Interpretation of the network variables as truss rod deformation.

In the case of a two-dimensional truss the dual problem is formulated similarly to the one-dimensional case: find the 'minimal work plastic mechanism', defined by a set of rods to be put in the plastic mode in order to turn the truss into a mechanism. For each rod there are two possible plastic modes—compression mode, where the rod can be shortened to any extent, and another is tension mode where the rod can be lengthened to any extent. In the engineering terminology, these modes can be considered as replacement of the rods with cables and struts depending on the specific plastic mode chosen. The energy function is defined as the sum of the products of the yield limits of the chosen rods by their elongation in the mechanism when the joint of external force is moved by a unit of length in the direction of the force.

Such an interpretation is reinforced by the complementary slackness conditions, from which it follows that the variable $\gamma^{+/-}(e)$ can differ from zero only if the corresponding edge in the dual problem is saturated (i.e. reaches the maximal or minimal boundary, respectively). Accordingly, rod e can be deformed only if it is saturated. The rod can be shortened (elongated) only if it is saturated by the compressive (tensile) force and accordingly can be considered as a cable (strut).

This mechanism is actually similar to the so-called collapsed mechanism used in the method of combining mechanisms for solving the maximal loading problem [26].

3. Derivation of the two-dimensional max-flow method from the primal-dual algorithm

3.1. Description of the primal-dual algorithm in the terminology of this paper

Following are the steps of the primal-dual algorithm developed by Papadimitriou and Steiglitz [2]. The mathematical linear programming formulations underlying each of the following steps can be found in the latter reference.

The main idea of this algorithm is that for each given original optimization problem, it constructs, in a systematic manner, a transformed optimization problem that in many cases is easier to be solved. Once solved, the solution of the transformed problem is augmented to the solution of the original problem after multiplication by a calculated coefficient.

(1) The original, or primal—in terminology of Papadimitriou and Steiglitz [2] problem. The original problem is stated in the terminology of linear programming, including the constraint inequalities for the problem variables and the objective function, the value of which is to be optimized.
(2) Some initial solution is given to the problem variables, so that the problem constraints are satisfied. In many cases,

a feasible initial solution is when all the variables are set to be equal to zero.

(3) Set of admissible constraints—J: Upon the substitution of the current values of the variables into the constraints of the original problem, all the constraints that are satisfied in equality form a set, designated by J.

(4) Transformed problem, called in terminology of Papadimitriou and Steiglitz [2]—Dual Restricted Primal problem. In accordance with the current state of the variables of the original problem—another LP problem is built by means of the following rules:
   a. For each variable in the original problem there is a corresponding variable in the transformed problem.
   b. All the variables in the transformed problems are restricted to be no greater than one.
   c. For each constraint of the original problem, which belongs to J, there is a constraint in the transformed problem having the same left-hand side, while its right hand-side is zero.

(5) The dual transformed problem (Restricted Primal in the terminology of Papadimitriou and Steiglitz [2]). If the objective function of the dual transformed problem and thus of the transformed problem is equal to zero, it indicates that the optimal solution for the original problem was found.

(6) The optimal solution to the transformed problem is found. The transformed problem possesses a restricted form in comparison to the original problem. Only a part of the constraints appearing in the original problem appears in the transformed problem, while their right hand side containing the numeric information about the problem is eliminated. Thus, finding the optimal solution to the transformed problem should be a simpler task than finding the solution to the original problem.

(7) Constant parameter, θ, is evaluated from the solutions of the original and the transformed problems, by means of Eq. (6).

$$\theta = \underset{\forall j \notin J}{\text{Min}} \left[\frac{(\text{right hand side of constraint } j \text{ of the original problem}) - (\text{left hand side of constraint } j \text{ of the original problem})}{(\text{left hand side of constraint } j \text{ of the transformed problem})} \right] \tag{6}$$

The augmentation coefficient evaluated in Eq. (6) assures that the objective function of the main problem is augmented by the maximum possible positive value.

(8) The solution of the original problem is augmented through Eq. (7) and a new solution is obtained.

$$\begin{aligned}&(\text{New value of the variable of the original problem})\\ &\quad = (\text{Old value of the variable of the original problem})\\ &\qquad + \theta^* (\text{The corresponding variable}\\ &\qquad\quad \text{in the transformed problem})\end{aligned} \tag{7}$$

(9) Return to stage 3 and repeat the process.

The above algorithm is summarized in the diagram of Fig. 4:

Each iteration of the above algorithm causes monotonic increase of the objective function (step 8), which as is proved in [2] reaches the optimum value at a finite amount of time.

3.2. Deriving the two-dimensional network flow through the primal dual algorithm and its application to plastic analysis

The formulation of the two-dimensional max-flow problem, upon substitution to the paradigm of the primal-dual algorithm yields relevant formulations for underlying problems of the algorithm. This way the primal-dual method for solving the two-dimensional max-flow problem is obtained. Following is the description of the steps of the algorithm. Each step is described through a two columns table—giving the formulation of the step in the terminology of the two-dimensional networks and the terminology of the two-dimensional trusses. The one-dimensional interpolation is also provided in case it differs from the two-dimensional formulations to show that the derived algorithm is actually the extension of the one-dimensional algorithm [5] to multiple dimensions.

Step 1. Setting the initial feasible solution.

Two-dimensional network terminology	Truss terminology
Set all the flows in the network to be zero vectors	Set all the forces in the truss to be zero vectors

Step 2. Constructing the set of admissible constraints—J (Table 1).

Step 3. Building the transformed problem.

Applying the construction rules appearing in [2] upon the original problem, yields the following LP formulation of the transformed problem:

$$\text{Max } P^T$$

$$\begin{aligned}
&\forall v \in \mathbf{V} \quad \sum_{e \in E'} I(e,v) \cdot F^T(e) \cdot \cos(\alpha(e)) + I(p,v) \cdot P^T \cdot \cos(\alpha(p)) = 0\\
&\forall v \in \mathbf{V} \quad \sum_{e \in E'} I(e,v) \cdot F^T(e) \cdot \sin(\alpha(e)) + I(p,v) \cdot P^T \cdot \sin(\alpha(p)) = 0\\
&\forall e \in \mathbf{J}^+ \quad 0 \geq F^T(e) \geq -1\\
&\forall e \in \mathbf{J}^- \quad 1 \geq F^T(e) \geq 0\\
&\forall e \in \mathbf{E}' \quad F^T(e) \leq > 0\\
&\qquad\quad 1 \geq P^T > 0
\end{aligned} \tag{8}$$

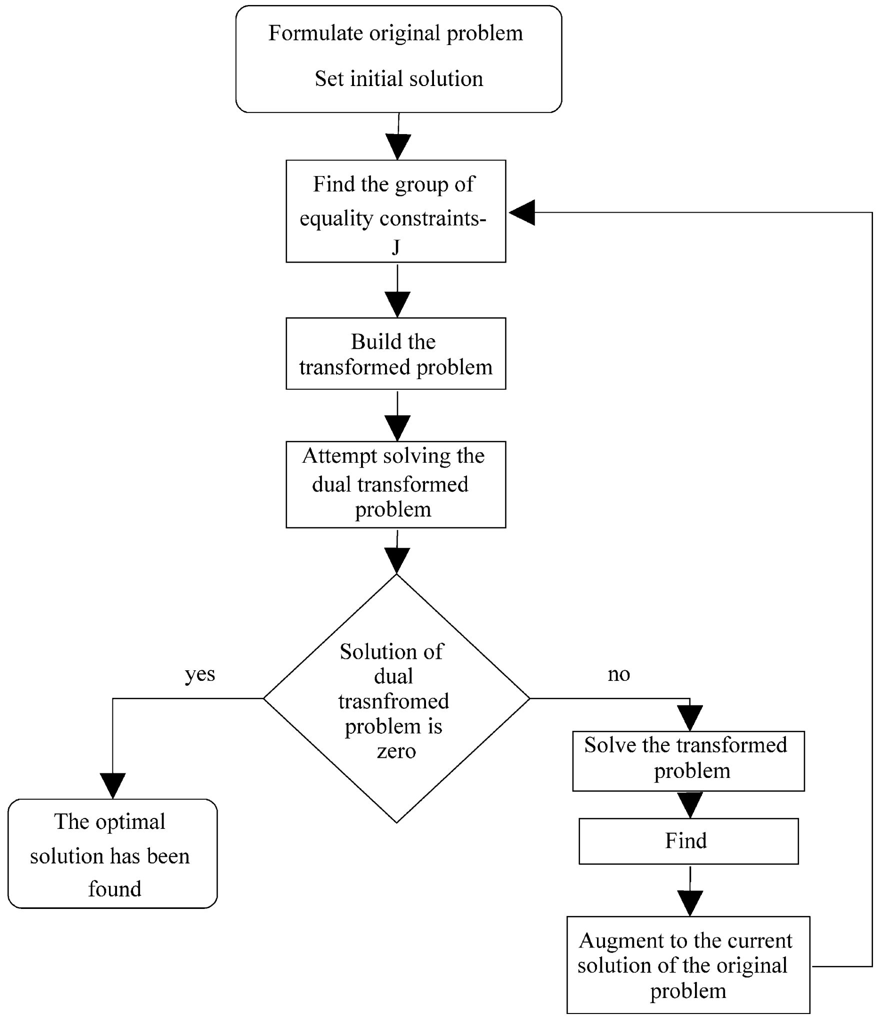


Fig. 4. Outline of the primal-dual algorithm in the terminology adopted in this paper.

where the superscript 'T' beside the variables indicates the belonging to the transformed problem.

The problem formulated in Eq. (8) possesses the same types of variables as the original problem (Eq. (1)), subjected to somewhat different constraints: while the flow continuity constrains remained intact, the flow magnitude limitations are different. Table 2 summarizes the transformed problem in the terminologies of the two-dimensional network, and the engineering domain—trusses.

Step 4. Checking the halting condition.

The check of whether the current solution is the optimal solution of the original problem can be performed by verifying that the optimal solution of either the transformed or the dual transformed problems is zero. The two possibilities are described in the two rows of Table 3.

Step 5. Solving the transformed problem (Table 4).

Step 6. Augmenting the solution (Table 5).

Step 7. Proceed to step 2 for the next iteration.

Step 8. End

As the mathematical foundation of the algorithm presented above is the primal-dual method, as is explained in the previous section, it will arrive at the value of the maximal loading upon a truss in a finite amount of time.

Table 1
The set of admissible constraints-J

Two-dimensional network terminology	Truss terminology
Set J is a set of edges for which the constraints limiting their capacity are found to be in equality at the current stage of the computation. Since there are two types of such constraints, for positive and negative flows, the set J can be divided into two: set J^+ and set J^-, respectively.	Set J is a set of rods where at the current stage of the computation the internal force has reached the maximal yield limit. Since there are two types of such constraints, for compression and tension forces, the set J can be divided into two: set J^+ and set J^-, respectively

Table 2
The formulations of the transformed problems

Two-dimensional network terminology	Truss terminology
Build a transformed network as follows: **If** the flow in the original network edge is saturated at the current stage, then substitute it by a unidirectional edge allowing only flow opposite to the saturation flow, **otherwise** the edge remains as it is. The magnitudes of the flows in the saturated edges are limited to be no greater than 1	**Build** a transformed truss as follows: **If** the force in the rod in the current state of the original truss is saturated with the compression (tension) force, then substitute it by a cable (strut), i.e., it can sustain only tension (compression) force, **otherwise** the rod remains as is. All the forces in the saturated rods are limited to be no greater than 1, either if the force is tension or compression

Table 3
The halting condition in terminologies of networks and the trusses

Two-dimensional network terminology	Truss terminology
Dual transformed problem: **If** the transformed network is not valid, i.e. there is no admissible flow that can be initiated from the source **then** the optimal solution, or the maximal flow has been reached—**stop**. For the one-dimensional case—the condition becomes the failure to find a directed path through the network	*Dual transformed problem:* **If** the resultant transformed truss is not valid, i.e. no positive external force can be applied on it, **then** the optimal solution, or the maximal loading has been reached—**stop**
Transformed problem: If the saturated edges form a vector cutset (a set of edges blocking the admissible flow) then the optimal solution has been found (in one-dimensional case the blocking is produced by those edges whose removal affects the connectivity of the graph)	*Transformed problem:* If the transformed truss allows the external forces to move a unit displacement without producing forces in any of its members, then the optimal solution has been reached

4. Examples for plastic analysis of trusses through the suggested multidimensional method

Fig. 5 shows a truss indeterminate to the first degree for which the maximal allowed loading is to be determined.

The compressive and tensile yielding of all the truss rods is equal to 12000[N].

The initial feasible solution is set as $P=F_1=F_2=F_3=0$.

For the first iteration, there are no saturated rods, thus in the transformed truss, there is no restriction on the sign of the forces in the rods. One of possible solutions is obtained by removing rod '3' from the truss and applying on it a unit external force as is shown in Fig. 6.

The analysis of the determinate truss of Fig. 6 gives the following results:

$$P' = 1[N], \quad F'_1 = 1.756[N], \quad F'_2 = -2.02[N]$$

By Eqs. (6) and (7), this solution is multiplied by the factor: θ=5940.6 and is added to the current solution, thus the current solution becomes:

$$P = 5940.6[N], \quad F_1 = 10431.7[N],$$

$$F_2 = -12000[N], \quad F_3 = 0$$

Rod 2 became saturated by tension, thus it is transferred to the set of admissible constraints, J^-, which includes all the rods that can be lengthened, in the original truss but cannot

Table 4
Solution of the transformed problem

Two-dimensional network terminology	Truss terminology
Find a set of edges capable of conducting an admissible two-dimensional flow in the transformed network. In these edges, find the flow distribution if the source edge conducts a flow of a unit magnitude. (For 1D case-find a directed path between the head and the tail vertices of the source edge in the transformed network)	In the transformed truss, select a subset of truss elements forming a stable determinate truss. Apply a unit loading to the truss in the direction of the external forces and find the forces in the rods of the transformed truss. There exist a variety of algorithms for treating determinate trusses [25]. Such algorithms can be employed to facilitate the execution of this step

Table 5
Augmenting the solution to the current solution of the original problem

Two-dimensional network terminology	Truss terminology
The flows obtained in step 4 are multiplied by the increment coefficient θ $\theta = \mathrm{Min}_{\forall j \notin J}\left[\frac{(\text{capacity of edge } j)-(\text{current flow in edge } j)}{(\text{flow in edge } j \text{ in the transformed network})}\right]$ and are added to the current solution of the problem. In calculation of θ, the type of the capacity (upper/lower) is taken in accordance with the type of the force in the transformed rod	The forces obtained in step 4 are multiplied by the increment coefficient θ $\theta = \mathrm{Min}_{\forall j \notin J}\left[\frac{(\text{yield limit of rod } j)-(\text{current force in rod } j)}{(\text{force in rod } j \text{ in the transformed truss})}\right]$ and are added to the current solution of the original problem. In calculation of θ, the type of the yield limit (compression/tension) is taken in accordance with the type of the force in the transformed rod.

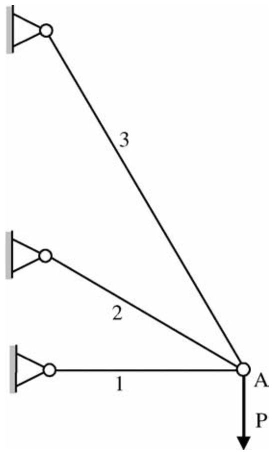


Fig. 5. The truss upon which the method is applied.

sustain tension. In engineering terminology, this rod is replaced with a strut.

The optimal solution to the original problem has not yet been achieved, since in the current original truss there are two non-saturated rods, 1 and 3, that prevents the movement of joint A in the direction of the external force.

In the next transformed problem the force in rod 2 is to be greater or equal to zero, i.e. to be unloaded or in the state of compression. This can be achieved by removing rod 2 from the original truss and applying the unit external force upon the new transformed truss, as shown in Fig. 7.

The analysis of the transformed truss of Fig. 7 gives the

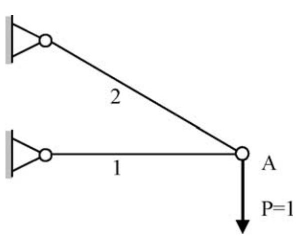


Fig. 6. Transformed problem of the truss presented in Fig. 5.

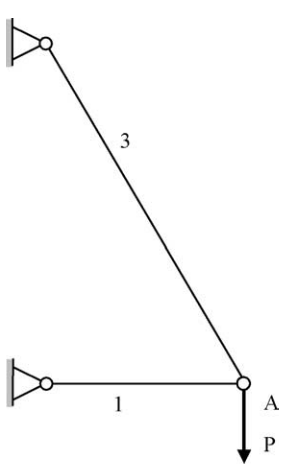


Fig. 7. Solution of the transformed problem in the second iteration.

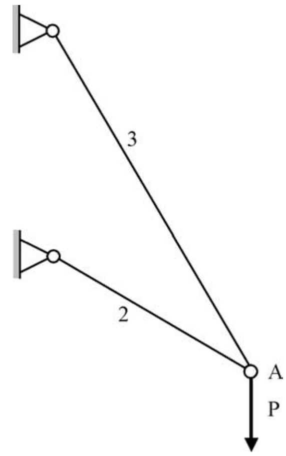


Fig. 8. Solution of the transformed truss of the third iteration.

following results:

$$P' = 1[N], \quad F'_1 = 0.577[N], \quad F'_3 = -1.154[N].$$

By Table 5, this solution is multiplied by the factor: $\theta = 2718$ and is added to the previous solution of the original problem, and the current solution becomes:

$$P = 8658.6[N], \quad F_1 = 12000[N],$$

$$F_2 = -12000[N], \quad F_3 = -3136.6[N]$$

Now, there are two saturated rods: rod 1 has reached the tensile yielding and rod 2 has reached the compressive yielding. Therefore, in the two saturated sets: $J^- = \{2\}$ and $J^+ = \{1\}$.

Turning the two rods into plastic enables movement of joint A. Nevertheless, the optimal solution has not been yet

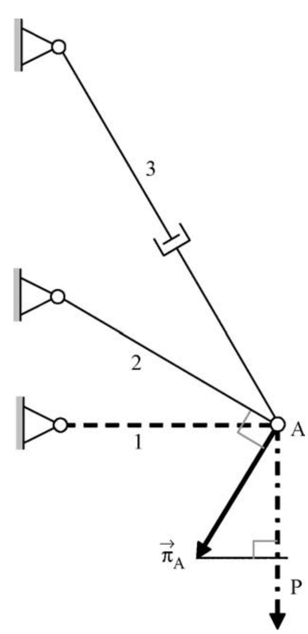
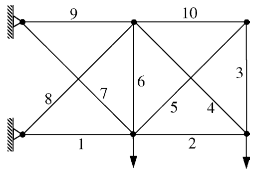
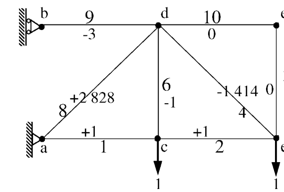
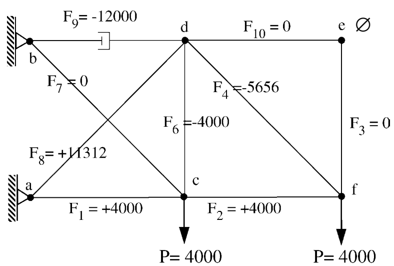
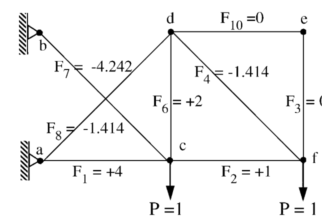
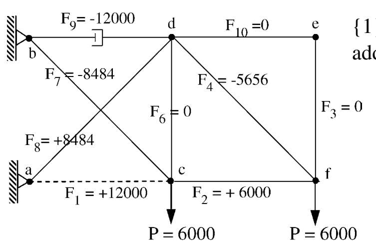


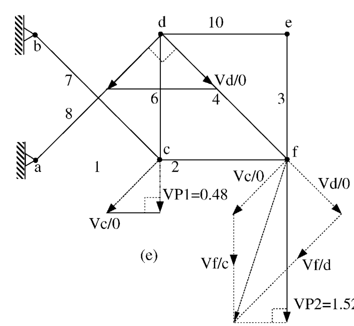
Fig. 9. Cable-strut problem equivalent to the dual transformed problem of the final iteration.

Table 6
Application of the algorithm for the plastic analysis of a compound truss

<table>
<tr><th rowspan="2">Iteration</th><th colspan="3">Original truss</th><th colspan="2">Transformed truss</th><th>Connection</th></tr>
<tr><th>Solution of the original problem</th><th>J^+_{cables}</th><th>J^-_{struts}</th><th>Solution of the dual transformed problem</th><th>Solution of the transformed problem</th><th>θ</th></tr>
<tr><td>1</td><td>Zero force in all the rods.

</td><td>∅</td><td>∅</td><td>The truss is stable, hence no movement of the joints can occur without causing deformation of truss members.</td><td>Removing rods 5 and 7 and solving the obtained determinate truss.

</td><td>4000</td></tr>
<tr><td>2</td><td>

(d)</td><td>∅</td><td>{9} rod 9 is added</td><td>Even though rod 9 is replaced by a strut, the truss remains stable.</td><td>Removing rods 9 and 5 and solving the obtained determinate truss.


(c)</td><td>2000</td></tr>
<tr><td>3</td><td>

(d)</td><td>{1} rod 1 is added</td><td>{9}</td><td>Joints C and F can do a valid movement in the direction of the external forces without causing deformation in the truss members.


The solution of the dual transformed problem is zero thus the optimal solution to the original problem has been reached.</td><td></td><td></td></tr>
</table>

achieved as from kinematical calculation it is easy to conclude that A cannot move in the direction of the external force when rod 2 is a strut and rod 1 is a cable.

In the third transformed truss, rod 1 is limited to be in tension (cable) and rod 2 is limited to be in compression (strut). Fig. 8 depicts a possible solution of the new transformed truss by removing rod 1 from the original truss and applying a unit external force.

The analysis of the transformed truss of Fig. 8 gives the following results:

$$P' = 1[N], \quad F_2' = 0.991[N], \quad F_3' = -1.721[N]$$

Since the force in rod 2 in the transformed truss is positive, and in the original truss it is strut, hence the constraints of transformed problem have not been violated.

By Eqs. (6) and (7), this solution is multiplied by the factor: $\theta = 5150.14$ and is added to the current solution, which becomes:

$$P = 13808.8[N], \quad F_1 = 12000[N],$$

$$F_2 = -6896.2[N], \quad F_3 = -120[N]$$

One can see, that rod 2 came out from the saturation thus it has left J^- and instead rod 3 has entered J^-.

The dual problem (kinematical) is checking whether joint A in the original truss can move in the direction of the external force so that the deformation of the structure elements is minimal while rod '3' is strut and rod '1' cable. The solution of the problem is shown in Fig. 9: by moving joint 'A' in the direction perpendicular to rod 2 no truss element is deformed and the vector of the displacement of 'A'—has a positive component in the direction of the external force.

Since the transformed problem is solved so that no truss elements are deformed, the value of the objective function of the transformed problem (the work needed to be exerted to move the joint) becomes zero, which guarantees (step 5) that the optimal solution to the original problem has been achieved, and is:

$$P = 13808.8[N], \quad F_1 = 12000[N],$$

$$F_2 = -6896.2[N], \quad F_3 = -120[N]$$

Table 6 shows application of the algorithm for plastic analysis of a more complicated truss system.

5. Conclusions and further research

The paper has expanded the network flow optimization algorithm to a multidimensional case through a systematic process using the primal dual method. The approach was shown to be useful for solving real engineering optimization problems, specifically, it was applied to plastic analysis of trusses. In general, the solution of the original optimization problem is obtained by solving a series of simpler, transformed optimization problems.

The ideas presented in the paper were shown to be derived systematically to enable employing the approach in other applications. In the paper, the algorithm for specific engineering problem—the plastic analysis of trusses, was derived after it was represented by a multidimensional flow network, thus the knowledge existing in the latter could be transformed and applied for solving the former. Although the algorithm was applied only to plastic analysis, it can certainly be applied in various operational research applications where the problem requires treatment of two dependent flows through the network [23].

As it was explained and demonstrated along the paper, the algorithm was developed in a systematic way by applying the primal-dual method from linear programming. This result indicates on the possibility that other combinatorial algorithms for solving specific optimization problems in engineering can be derived by following the same process as the one described in the paper.

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