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ORIGINAL ARTICLE

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Transforming engineering knowledge through graph representations: transferring the Willis method to linkages and trusses

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Abstract The paper introduces an approach for transforming methods and knowledge between different engineering fields through general discrete mathematical models, called graph representations, which carry engineering knowledge of specific systems. The idea is demonstrated by showing the transformation of the known method in planetary gear trains—the Willis method—to two other engineering systems: linkages and trusses. In doing so, two efficient methods were derived: one for analysing compound linkages, such as those containing tetrads, and another for compound trusses. These new methods were derived from two relations characterising graph representations: a representation that is common to two engineering fields and the duality relation between representations. The new approach underlying these transformations is shown to open new ways of conducting engineering research by enabling a systematic derivation of engineering knowledge through knowledge transformations between the graph representations.

Keywords Graph representations · Knowledge transformation · Linkages · Planetary gear trains · Trusses · Willis method

Introduction

The paper is a part of a general research work, in which different engineering systems are represented by discrete mathematical representations providing a global perspective over these systems. One of the main contributions of this approach is that it enables transforming engineering knowledge between diverse engineering fields.

Till now the approach has been applied to a variety of engineering fields, which demonstrated through the graph representations that known methods in one engineering field are equivalent to other methods in the same or a different field. Some of the important relations that have been obtained in the course of the research are: Betti's theorem and the unit force method in structural mechanics were derived from Tellegen's theorem in electrical theory [1]; the Maxwell-Cremona method for trusses and the image velocity method in linkages were proved to be equivalent [2]; the force and displacement methods from structural mechanics were proved to be derivations of circuit and node methods in electrical circuits, respectively [3]; the conjugate beam theorem was derived from the duality theorem in graph theory [4].

The current paper develops further applications of this approach and employs its ability to transform methods between engineering fields in order to derive new engineering methods. It is shown that the known Willis method in planetary gear systems can be transformed to linkages and then to trusses both times yielding a new method.

The paper employs three graph representations: the potential line graph representation (PLGR), the potential graph representation (PGR) and the flow graph representation (FGR). PLGR is a representation applicable both to planetary systems and linkages [5], yielding a knowledge transfer channel between the two systems, as is depicted in Fig. 1.

Using this relation the Willis method, known in planetary gear trains [6], is translated in the section "Transferring the method to linkages" into the terminology of linkages. The obtained method constitutes a new method for the analysis of compound linkages.

As is shown in the section "The duality relation between linkages and trusses" there is a duality relation between linkages and trusses, stemming from the duality between their corresponding graph representations—PGR and FGR. Such a relation presents an

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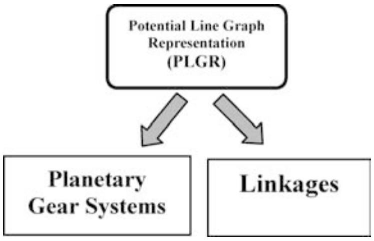


Fig. 1 A knowledge transfer channel between linkages and planetary gear systems

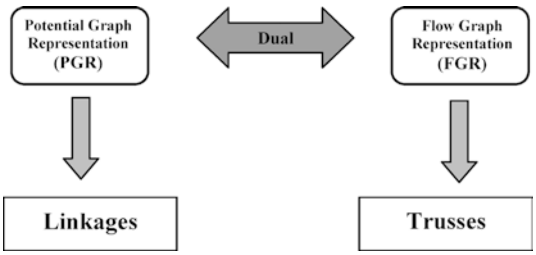


Fig. 2 A knowledge transfer channel between linkages and trusses

additional knowledge transfer channel used in this paper—between linkages and trusses, as is depicted in Fig. 2.

Using this channel, the Willis method for linkages obtained in the section "Transferring the method to linkages" can be further translated to the terminology of trusses, as is done in the section "Transferring the method to trusses", yielding a new method for analysis of compound trusses.

The overall route that is traversed by the Willis method is shown in Fig. 3.

Since the mathematical foundation of the suggested approach is discrete mathematics, specifically graph theory, the process described above is convenient for computerisation.

Graph representations and represented engineering systems

Graph representations

The graph representation [3] is an isomorphic graph-theoretical substitute of an engineering system, which can be used for design, analysis, and other forms of reasoning upon the system. The purely mathematical essence of graph representations makes them convenient for computerisation and enables them to provide tools for a generalised treatment of the engineering systems. Different types of graph representations are characterised by four main parts, as is outlined in the diagram of Fig. 4.

In more detail, the aspects characterising a graph representation constitute the following details:

1. Embedded knowledge—This primary component contains mathematical knowledge underlying the representation, including mathematical laws, theorems and methods. Once an engineering system is associated with a representation, the embedded knowledge becomes available for the engineering system.
2. Relations to other graph representations—Due to the mathematical foundation of the graph representations, it is inevitable that mathematical relations emerge between them. Specifically, there exist different mathematical relations between different types of graphs, which can be formulated in the terminology of relations between the graph representations. This was found to be of a crucial importance since it constitutes the basis for the knowledge transfer between different graph representations. The current paper employs the duality relation between graph representations based on the known graph theory duality [7]. It is used here to transfer knowledge between linkages and trusses through the duality between their corresponding representations.
3. Represented engineering domains—Each graph representation can be applied to represent a number of engineering domains. Choosing a suitable graph representation for an engineering system being treated has an immediate impact on the effectiveness of the subsequent process. This part contains decision rules to determine to which engineering disciplines this representation is applicable.
4. Rules for construction of the graph representation—The algorithmic steps for constructing the graph representation corresponding to an engineering system. In the paper, three different graph representations are employed and their construction rules are listed in Table 1, Table 2 and Table 3.

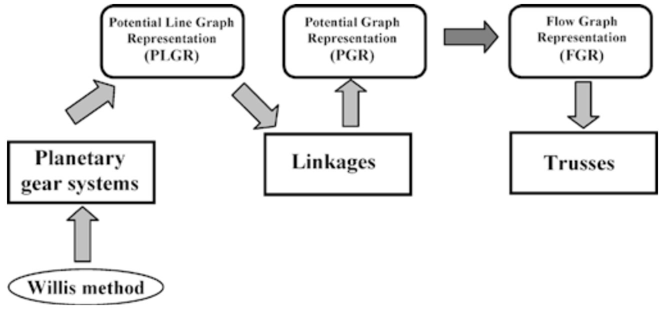


Fig. 3 A knowledge transfer route made in the course of the paper

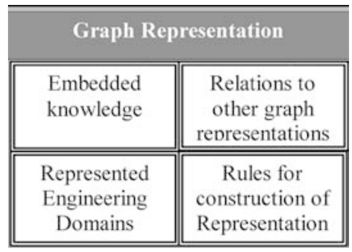


Fig. 4 A schematic structure of a graph representation

Table 1 Correspondence between the terminologies of PGR and linkage

Linkage	Potential graph representation
Regular link	Unknown edge
Driving link	Source edge
Joint in linkage	Graph vertex
Joint velocity	Potential of a vertex
Relative link velocity	Potential difference of an edge

The PGR for linkages

The PGR [8] is used in this paper for representing linkages. The following list describes the basic properties of the representation:

1. The potential graph representation is a directed graph.
2. Each vertex of the graph is associated with a vector, called a "potential".
3. Each edge of the graph is associated with a vector called a "potential difference" that is equal to the difference between the two potentials of the edge end vertices.
4. The edges of the graph are separated into two sets: "unknown edges"—the edges for which the direction of the potential difference is initially given, but the magnitude is unknown, and "source edges"—the edges for which the potential difference is given both in direction and magnitude.
5. For each circuit in the graph, the potential differences of its edges satisfy:

$$\sum_{i \in \mathbf{C}} \overrightarrow{\Delta}_i = 0 \tag{1}$$

where $\overrightarrow{\Delta}_i$ is the potential difference of edge i, and C is the set of edges in a circuit.

In [8] it was proved that PGR is an isomorphic representation of kinematic chains, or linkages. Table 1 summarises the terminology translation rules (construction rules) between a linkage and the corresponding PGR.

The PLGR for planetary gear systems and linkages

The PLGR [5] is a representation similar to the PGR described in the previous section. The main difference is in the way the representation is used to represent engineering systems. As can be seen from Table 1, elements of engineering systems are associated with the edges of the PGR. On the contrary, the elements of engineering systems represented by the PLGR are associated with the vertices of the graph. The list of mathematical properties of the representation [5] follows.

1. The PLGR is a directed graph.
2. Each vertex of the graph is associated with a vector called the "potential".
3. Each edge of a PLGR is associated with a vector called the "potential difference", which is equal to the vector difference between the potential of its tail vertex and the potential of its head vertex.
4. Similarly to the PGR, the edges of the graph are separated into "unknown" and "source" edges.
5. Similarly to the PGR, for each circuit in the graph, the potential differences of its edges satisfy Eq 1.
6. Additionally, all the edges of a PLGR possess the following relation between different components of their potential difference:

$${}^{L}\overrightarrow{\Delta}_i = \overrightarrow{r_i} \times {}^{A}\overrightarrow{\Delta}_i \tag{2}$$

where: ${}^{L}\overrightarrow{\Delta}_i$ is termed the linear component of the potential difference of edge i, and ${}^{A}\overrightarrow{\Delta}_i$ is termed the angular component of the potential difference of edge i, and $\overrightarrow{r}_i$ is a constant vector associated with the edge i.

In this paper, the PLGR is used to represent linkages and planetary gear systems. Table 2 summarises the terminology translation rules between these engineering systems and the representation.

The FGR for trusses

The FGR [3] is used in this paper to represent trusses. The mathematical properties of the representation follow.

1. The FGR is a directed graph.
2. Each edge of the graph is associated with a vector called "flow".
3. The edges of the graph are separated into two sets: "unknown edges"—the edges for which the direction of the flow is initially given, but the magnitude is unknown; and "source edges"—edges for which the flow is given both in angle and magnitude.
4. For each cutset in the graph, the flows through its edges satisfy:

$$\sum_{i \in \mathbf{U}} \overrightarrow{F}_i = 0 \tag{3}$$

where $\overrightarrow{F}_i$ is the flow through edge i, and U is the set of edges in a cutset.

Table 3 summarises the terminology transformation rules between trusses and the FGR.

The duality relation between linkages and trusses

It has been proved in [8] that the PGR and the FGR are dual representations, meaning that there is a mathematical relation between the two representations allowing a transfer of knowledge between them. The most basic relations between the dual representations are summarised in Table 4. From the duality between the

Table 2 Correspondence between the terminologies of linkages, planetary gear trains and PLGR

Linkages	Planetary gear trains	Potential line graph representation (PLGR)
Regular link	Gear or planet carrier	Graph vertex
Kinematical pair	Gear engagement, or gear-carrier interconnection	Graph edge
Angular velocity of a link	Angular velocity of a gear train component	Potential of a vertex
Relative angular velocity in kinematical pair	Relative angular velocity between two engaged gears or gear and its planet carrier	Angular potential difference of an edge
Relative linear velocity between two links at a reference point	Relative linear velocity between two gear train components at a reference point	Linear potential difference of an edge

Table 3 The correspondence between the terminologies of FGR and trusses

Truss	Flow graph representation (FGR)
Truss rod, reaction	Unknown edge
External force	Source edge
Joint in the truss	Graph vertex
Force in the truss element	Flow through an edge

FGR and the PGR one may automatically conclude a duality relation between linkages and trusses, as it was done in [8]. The summary of this relation appears in Table 5. Table 6 provides a number of examples of different trusses and their dual linkages, with the corresponding graph representations.

Transferring the Willis method to linkages and trusses

The Willis method is a well known gear-train analysis method established in 1841 by Willis [9]. The Willis method, sometimes referred to as the formula method, is widely employed and its details appear in almost all the textbooks on machine theory [6]. This method allows to analyse planetary systems by transforming planetary gear trains into regular gear trains. The current section employs graph representations to transfer the Willis method first to plane linkages and then to trusses. The first transformation will be performed by means of a common representation between linkages and planetary gear systems, whereas the second one will be performed through the duality relation between the representation of linkages and the representation of trusses.

The Willis method in planetary gear systems

For planetary gear systems the Willis method can be algorithmically summarised as follows:

1. Choose one of the components (either carrier or gear) of the planetary gear train and designate it by s. Its angular velocity would accordingly be designated by $\vec{\omega}_s$.

Table 4 The terminology correspondence between PGR and its dual FGR

Potential graph representation (PGR)	Flow graph representation (FGR)
Edge	Edge
Circuit	Cutset
Potential difference of an edge	Flow through the edge
Potential difference source edge	Flow source edge
Unknown edge	Unknown edge

Table 5 The terminology correspondence between dual linkages and trusses

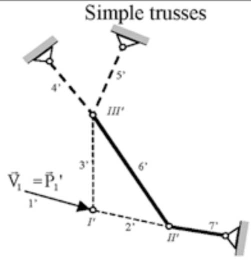
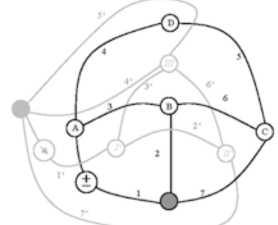
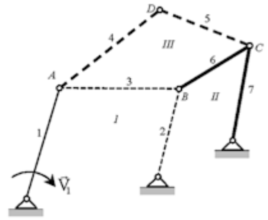
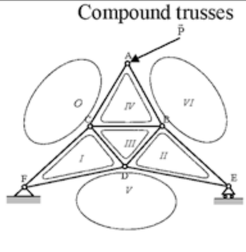
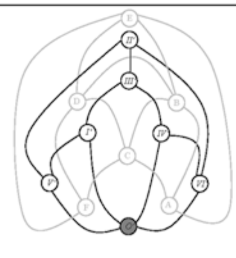
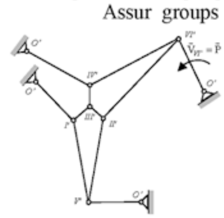
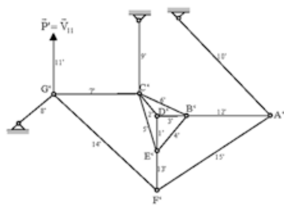
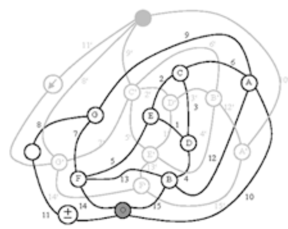
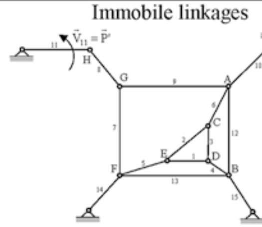
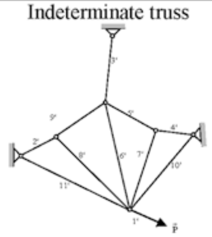
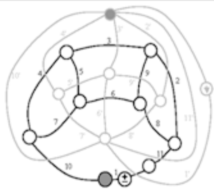
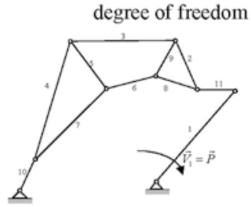
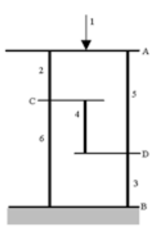
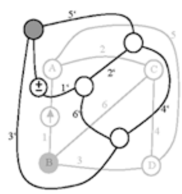
Linkages	Trusses
Regular link	Truss rod
Link relative velocity	Force in the truss element
Driving link	External force

2. Add to all the angular velocities of the gear system a value of $-\vec{\omega}_s$. As a result, we obtain a transformed planetary gear system designated by s, where link s is now fixed while the link that in the original system has been fixed becomes a mobile gear or carrier.
3. Analyse the transformed planetary gear system. If s has been properly chosen the new system becomes easier to analyse. For example, it may become a simple gear train.
4. Use Eq. 4 to obtain the solution of the original system from the solution of the transformed one:

$$\mathrm{i}_{\mathrm{n-m}} = \frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}} = \frac{1 - \mathrm{i}^{(\mathrm{s})}_{\mathrm{n-fix}}}{1 - \mathrm{i}^{(\mathrm{s})}_{\mathrm{n-fix}}/\mathrm{i}^{(\mathrm{s})}_{\mathrm{n-m}}} \qquad (4)$$

   where i and $i^{(s)}$ are transmission ratios in the original and transformed systems respectively, n and m are indices of two arbitrary elements, and fix is the index of the element that is fixed in the original system.
5. End.

Table 6 Duality relations between different types of linkages and trusses

Trusses	Corresponding graph representations	Linkages
Simple trusses 		Linkages decomposable to dyads. 
Compound trusses 		Linkages containing higher order Assur groups 
Unstable trusses 		Immobile linkages 
Indeterminate truss 		Mechanisms with more than one degree of freedom 
One dimensional truss 		Infi mechanism 

As was explained above, every planetary gear system can be represented by a PLGR. This means that the terminology belonging to the planetary systems can be mapped into the terminology of PLGR. Consequently, methods and theorems formulated in the terminology of the one can be translated through this mapping into the terminology of the other.

Using the terminology correspondence rules provided in Table 2, one may translate algorithm 1 to the terminology of the PLGR, and obtain the following algorithm:

1. Choose one of the vertices of the graph and designate it by s. Its potential would accordingly be designated by π_s.
2. Add to all the potentials of the graph a value of $-\pi_s$. As a result we obtain a transformed graph, where vertex s becomes the reference vertex, i.e., the vertex of zero potential. On the other hand, the vertex that has been the reference vertex in the original system becomes a vertex with finite potential in the transformed system.

3. Analyse the transformed graph.
4. Use Eq. 5 to obtain the solution of the original graph from the solution of the transformed one:

$$\mathrm{i}_{\mathrm{n-m}} = \frac{\pi_{\mathrm{n}}}{\pi_{\mathrm{m}}} = \frac{1 - \mathrm{i}^{(s)}_{\mathrm{n-ref}}}{1 - \mathrm{i}^{(s)}_{\mathrm{n-ref}}/\mathrm{i}^{(s)}_{\mathrm{n-m}}} \tag{5}$$

   where i and $i^{(s)}$ are the transmission ratios between the potentials in the original and transformed graphs respectively, n and m are indices of two arbitrary vertices, and *ref* is an index of the vertex that is a reference vertex in the original system.
5. End.

Obtaining algorithm 2 from algorithm 1 completes the first stage of the process, as is highlighted in Fig. 5.

Transferring the method to linkages

A linkage is an additional engineering system that also can be represented by the PLGR. Thus, the method formulated in the terminology of the PLGR can be translated now into the terminology of linkages as well. This step, as highlighted in Fig. 6, is performed by applying the construction rules of Table 2 to algorithm 2. (Fig. 7 shows a linkage consisting of a tetrad).

1. Choose one of the links of the linkage and designate it by s. The angular velocity of the link would accordingly be designated by ω_s.

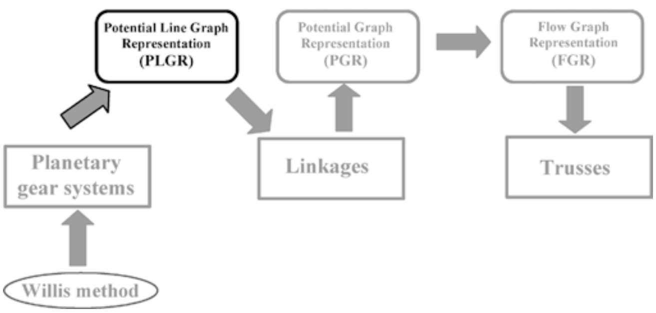


Fig. 5 The first step in the transformation process

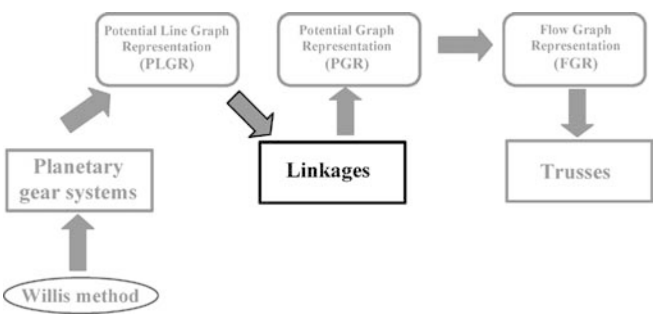


Fig. 6 The second step in the transformation

2. Add to the angular velocities of all the links in the linkage a value of $-\omega_s$. As a result we obtain a transformed linkage designated by s, where link s is the fixed link, while the link that has been fixed in the original system becomes mobile.
3. Analyse the transformed linkage.
4. Use Eq. 6 to obtain the solution of the original linkage from the solution of the transformed one:

$$\mathrm{i}_{\mathrm{n-m}} = \frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}} = \frac{1 - \mathrm{i}^{(s)}_{\mathrm{n-fix}}}{1 - \mathrm{i}^{(s)}_{\mathrm{n-fix}}/\mathrm{i}^{(s)}_{\mathrm{n-m}}} \tag{6}$$

   where i and $i^{(s)}$ are the transmission ratios in original and transformed linkages respectively, n and m are indices of two arbitrary links, and *fix* is an index of the link that is a fixed link in the original system.
5. End.

The validity of the proposed method mathematically follows from the validity of the original Willis method. Thus, the process that was used to obtain this method can be considered to be the proof of the method. Furthermore, the obtained method, similarly to the Willis method in planetary systems, can significantly facilitate the analysis of a linkage if the link s is chosen properly. In the following example the transformation of the linkage makes it decomposable to Assur groups [10] of a lower class, thus reducing the complexity of the analysis procedure.

An example for the analysis of linkage by means of the transformed Willis method

Consider the linkage of Fig. 7, known in the literature as a Stephenson linkage type III [6]. The linkage has one degree of mobility and its structure is composed of the driving link 1 and a tetrad 2,3,4,5,6,7,8,9.

Performing step 1 of algorithm 3 to this linkage, one may choose link 3 to be the link s. Applying step 2 yields the linkage shown in Ffig. 8. As a result of the transformation, the joints connected to link 3 become fixed supports, while the joints that were originally fixed supports become the joints connecting a new link, 0, to the rest of the mechanism.

The linkage of Fig. 8 is a simple linkage that can be separated into a driving link and two dyads. The analysis of such a linkage is rather straightforward and can

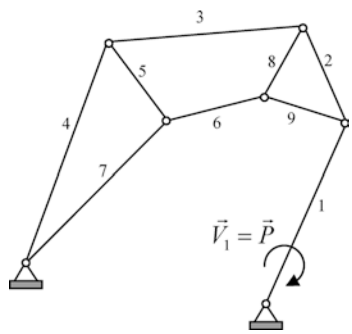


Fig. 7 A linkage consisting of a tetrad

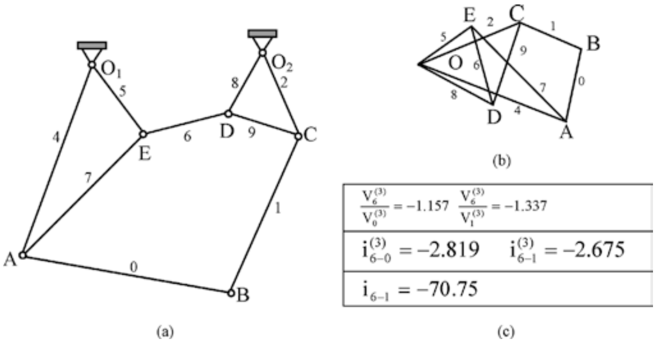


Fig. 8 **a** Linkage of Fig. 7 where link 3 is fixed; **b** The corresponding image velocity diagram; **c** The final calculation of the relative velocity in the link of the original mechanism

be done for example by the image velocity method, as is shown in Fig. 8b.

Now, one can employ the solution obtained in Fig. 8b and Eq. 6 to evaluate the linear velocities of the links in the original linkage. Figure 8c shows an example calculation of the transmission ratio between the angular velocities of link 6 and the driving link of the original linkage.

Transferring the method to trusses

As was explained in the section "The duality relation between linkages and trusses", in 2001 a duality relation between plane linkages and plane trusses [8] has been established, yielding the result that for each truss there is a corresponding linkage having dual kinematical properties. This idea has opened up a new avenue of research and practical applications since due to this relation, knowledge and methods available for one of these systems can be transformed and employed in the other [2]. Accordingly, it is now proposed to transform the method developed in the previous section and apply it to trusses. Figure 9 shows how the current stage fits into the overall method transformation process.

The only adjustment needed to be performed before commencing the transformation concerns the variables used to describe the system. According to Table 5, the forces acting in the truss rods correspond to the relative linear velocities in the corresponding links of the dual

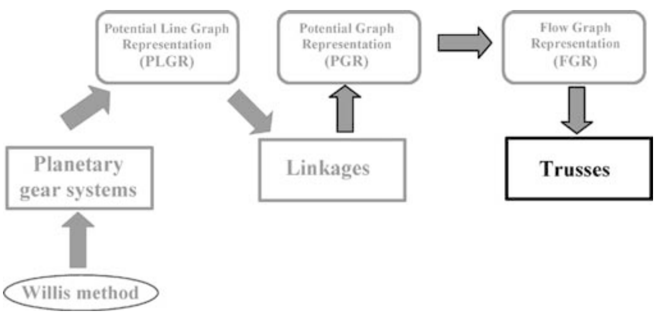


Fig. 9 The final step in the method transferring process

linkage [2,8]. Since the equations developed in this paper involve the angular velocities of the links and not the linear ones, the forces in the truss are to be "weighted". This is done by dividing each force by the length of its corresponding link in the dual linkage, obtaining what we term "weighted forces". In order to distinguish between the two types of forces, the weighted forces will be designated by *f*, in contrast to the regular forces designated by *F*.

The steps for solving a truss by means of the transformed Willis method are as follows:

1. Construct a linkage dual to the truss.
2. Apply the transformation used in Algorithm 3 to the linkage, namely, fix some link *s* and turn the ground into a mobile link.
3. Construct the "transformed truss"—a truss dual to the transformed linkage obtained in 2.
4. Analyse the transformed truss and find the forces in its rods.
5. Calculate the weighted forces of each rod by dividing the magnitude of the force acting in the rod by the length of the corresponding dual link.
6. Use Eq. 6 to transform the transmission ratios between the weighted forces in the transformed truss into the transmission ratios between the weighed forces in the original truss.

$$i_{n-m} = \frac{f_n}{f_m} = \frac{1 - i_{n-fix}^{(S)}}{1 - i_{n-fix}^{(s)}/i_{n-m}^{(s)}} \tag{7}$$

7. The actual forces can now be obtained by multiplying the weighted forces by the corresponding link lengths.

According to the linkage-truss duality [8], the truss corresponding to a linkage consisting of dyads only is a simple truss, namely, a truss constructed by starting with a basic triangular element, and adding two connected rods at a time [11].

On the other hand, linkages composed of higher order modular groups correspond to compound trusses in which all the analysis equations are to be solved simultaneously. Thus, in accordance with the results of the section Transferring the method to linkages, the transformed Willis method in trusses enables to replace a compound truss with a simple one using the proposed technique.

To clarify this idea, the compound truss given in Fig. 10 is analysed using the transformed Willis method.

The truss of Fig. 10 is compound, thus it is reasonable to attempt solving it using the proposed procedure. The linkage dual to this truss is shown in Fig. 11a.

The linkage of Fig. 11a, which actually is the same as the one treated in the section "An example for the analysis of linkage by means of the transformed Willis method" (Fig. 8), is composed of a driving link 1' and a tetrad, thus according to the previous section it would be efficient to fix link 3' and change the driving link to 4', as shown in Fig. 11b. The transformed truss, dual to the transformed linkage of Fig. 11b, appears in Fig. 12a.

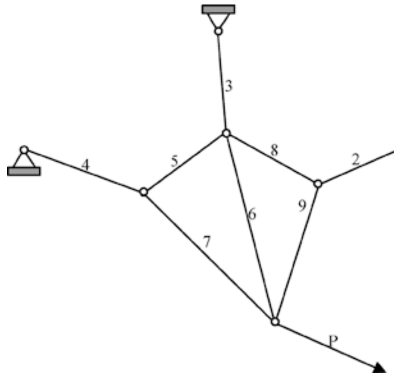


Fig. 10 A compound truss

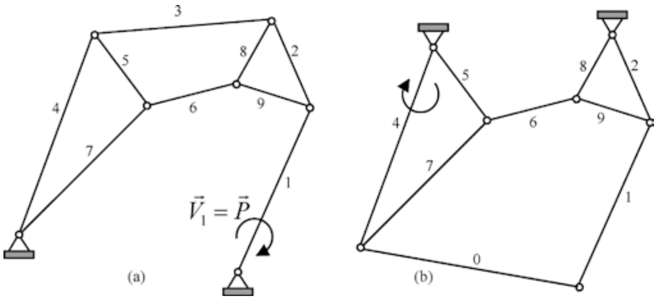


Fig. 11 The linkage dual to the truss of Fig. 4 and its transformation; **a** The dual linkage; **b** The transformed dual linkage

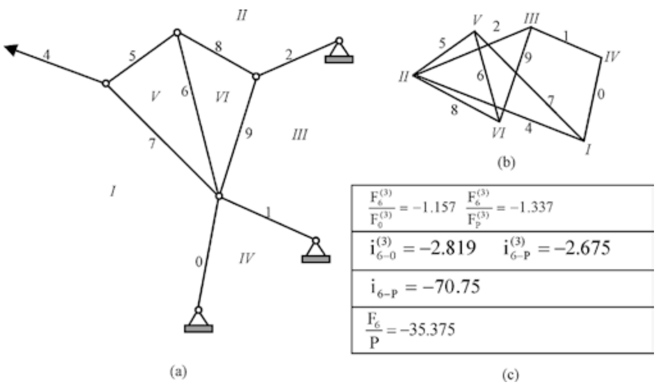


Fig. 12 The corresponding transformed truss; **a** The transformed truss; **b** The corresponding Maxwell-Cremona diagram

The truss of Fig. 12 is a simple truss and thus it can be solved by one of the efficient methods available for solution of such trusses. One of such methods is the well-known graphical method, called Maxwell-Cremona diagram [12], shown in Fig. 12b. One can see that consistently with the results reported in [2], the Maxwell Cremona diagram of the truss of Fig. 12a is identical to the image velocity diagram of its dual mechanism, as appears in Fig. 8.

The solution of the transformed truss can now be substituted into Eq. 7 to yield the ratio between the weighted value of the external force and the weighted forces in the rods of the original truss.

The algebraic manipulations needed to find the force in rod 6 of the original truss (Fig. 10) appear in Fig. 12c.

Conclusions

The paper has introduced an approach for transforming methods between engineering fields through graph representations. It employs the fact that planetary gear trains and linkages have been represented by the same graph representation, to transform the Willis method from planetary gear trains to the terminology of linkages. Since the Willis method is applicable to solve compound gear trains, upon transformation, it became a method suitable for analysis of compound linkages, such as those containing tetrads.

Another type of transformation—a duality transformation—was demonstrated in the paper by employing the duality relation between the PGR and FGR. Since the former has already been applied to represent linkages and the latter to trusses, knowledge from plane linkages can be transformed to plane determinate trusses and vice versa. In the paper, the transformed Willis method for linkages was further transformed to statics and a method for analysis of compound trusses was derived.

Throughout the paper, the knowledge was transformed through a compound route from gear systems to linkages, and then to trusses. The validity of such a transformation, although not intuitive, is mathematically proved through the properties of the graph representations underlying the transformation process.

The process demonstrated in the paper contributes to engineering much beyond developing methods for analysis of trusses and mechanisms. It shows that graph representations may constitute a powerful framework for a systematic derivation of engineering knowledge by transforming knowledge from one engineering field to another.

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