Hyperbolic Geometry and Graphs

Yuval Shavitt

Euclid axioms.

1. Every two **points** lie on exactly one **line**.

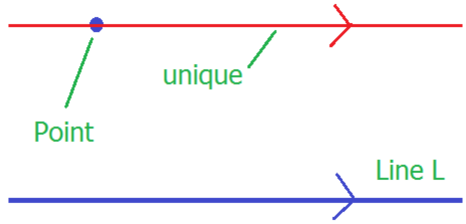
2. Any **line** segment with given endpoints may be continued in either direction.

3. It is possible to construct a **circle** with any **point** as its centre and with a radius of any length.

4. All **right angles** are equal.

5. Given a **line** L and a **point** not on L, there is one and only one **line** which contains the **point** and which is **parallel** to L.

It is this fifth **axiom**,
the **Parallel Postulate**,
that caused
a lot of trouble.



The question was, could this be derived from the other **axioms** and common notions as a **theorem**?

If so, it could be removed from the **list of axioms**, which would then be a smaller yet still (hopefully) **complete** set.

Euclid himself seems to be unsure on this question. He certainly seems to go out of his way to avoid using the **Parallel Postulate** in his opening **theorems**.

Many, many mathematicians attempted to prove the **Parallel Postulate** to be either a necessary **axiom**, or a **theorem**. All failed until the 19th century, when **Gauss, Lobachevsky** and **Bolyai** found a solution.

Carl Friedrich Gauss
(1777-1855)

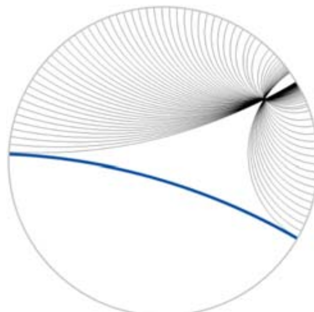
Nikolai Lobachevsky
(1792-1856)

Janos Bolyai
(1802-1860)

They had the idea of replacing the **Parallel Postulate** with another **axiom**, without removing the idea that the **axioms** should be **consistent**.

The revised fifth **axiom** looked like this:

*Given a **line** and a **point** not on that line, there are **infinitely many lines** parallel to the given **line** through the **point**.*



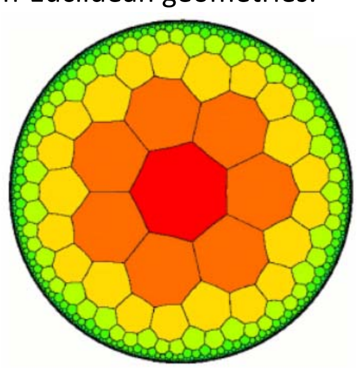
All the other **axioms** were the same. This new geometry was shown to be **consistent**, and so another geometry could stand alongside **Euclid's** as a possible way of modelling the universe.

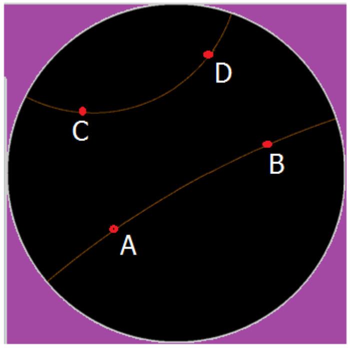
This proved **Euclid** to have been remarkably far-sighted in including his fifth **axiom** in his list. The work above shows that it cannot be derived from the others, and is thus essential.

So what does **Non-Euclidean Geometry** look like?

Well, there are a number of non-Euclidean geometries.

We will look at here hyperbolic geometry, as investigated by **Gauss**, **Lobachevsky** and **Bolyai**, which can be modelled simply by the following *geometry-in-a-circle*.





The picture shows
two **(straight) 'lines'**
in this geometry;
these are arcs of **circles** that
meet the black **circle** edge
at right angles.

A **straight line** in any geometry
is the shortest distance
between two **points**
(the geodesic).
We have an unusual idea
of distance in operation here
that means these circular arcs
are indeed the shortest distance
between the given **points**.

We are free to interpret
the notions of
'point' and **'line'**
in Euclid's axioms
as we wish, as long
as we are consistent.

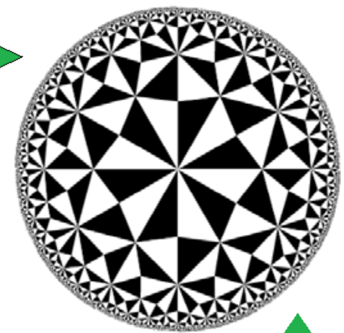
If we regard our black **circle** as the unit **circle**,
centre O, in the Argand diagram,
the **distance** between the points $\mathbf{z_1}$ and $\mathbf{z_2}$ is

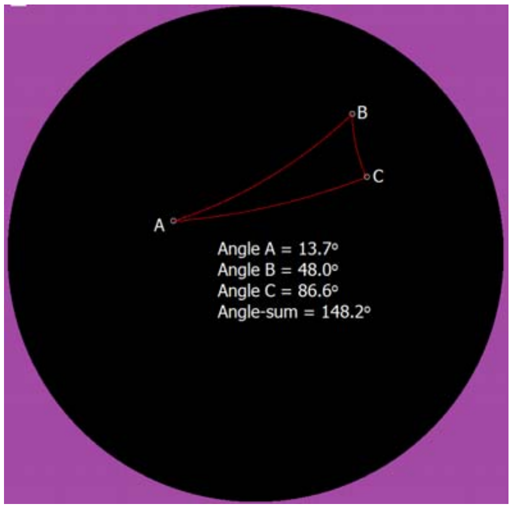
$$\boxed{d(z_1, z_2) = \tanh^{-1}\left(\left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| \right)}$$

If $z_2 = 1$ and $z_1 = -1$, then $d(z_1, z_2) = \tanh^{-1}(1) = \infty$.

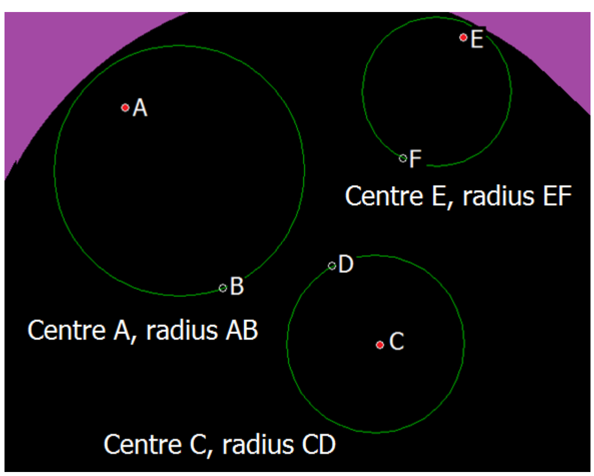
The **distance function (or metric)** here is such that towards the
edge of the circle, **distances** get bigger and bigger.
The **distance** across the entire circle is infinite.

The area of each triangle here, for example, is the same.





What are the angles here? 90°, 45° and 30°, which add to 165°. The angle-sum of a triangle in hyperbolic geometry is less than 180°.

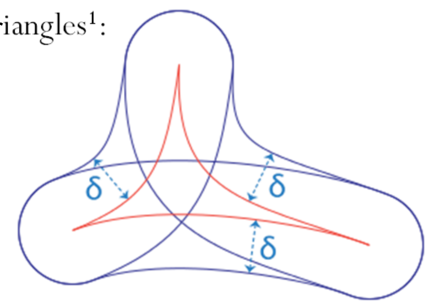


A miraculous fact: **circles** in this geometry, (even with our strange idea of distance), look exactly like **circles** in Euclidean geometry.

Their centres, however, don't: the less central that the centre of the **circle** is, the more it diverges from the Euclidean centre.

δ-Hyperbolic Spaces

- A δ-hyperbolic space is a geodesic metric space in which **every** geodesic triangle is δ-thin.
- Definition by Rips – Thin Triangles[1]:

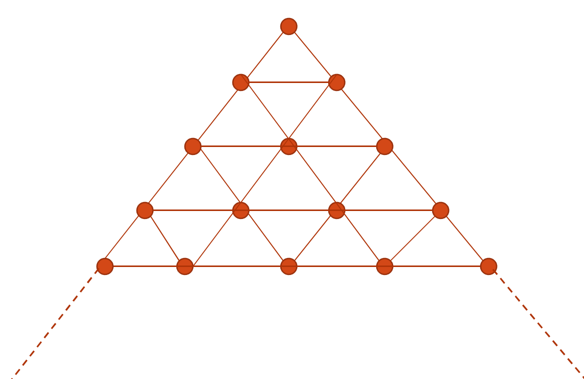


- Obviously a Euclidean metric is ∞-hyperbolic, as δ is not bounded.

1. "δ-hyperbolic space", *Wikipedia*, http://en.wikipedia.org/wiki/%CE%94-hyperbolic_space.

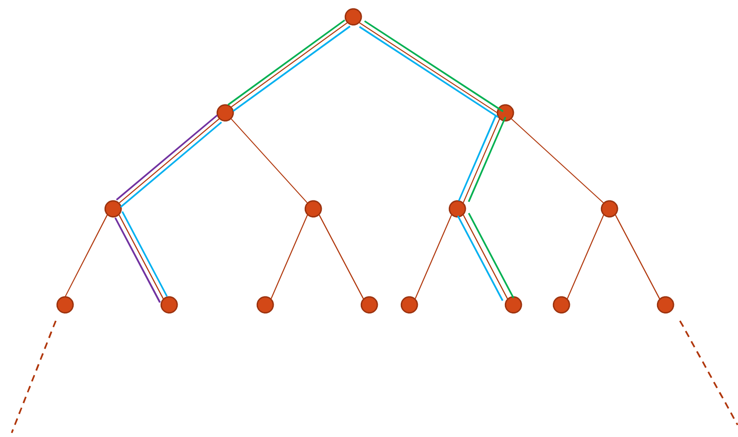
δ-Hyperbolicity of Infinite Lattices

- Infinite lattices may be Euclidean or hyperbolic:
- For an infinite triangle lattice, the distance from the triangle's sides keeps growing & growing as the triangle is growing.



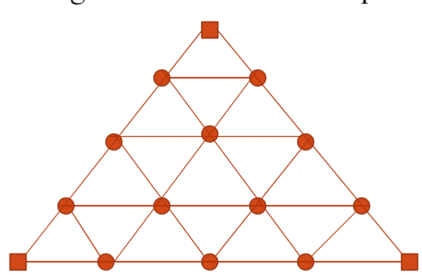
δ-Hyperbolicity of Infinite Lattices

- On the other hand, other types of lattices may be hyperbolic, like a tree (which is 0-hyperbolic):



δ-Hyperbolicity of Finite Graphs

- Finite graphs are (almost) always hyperbolic.
- Let's take a triangular lattice as an example:



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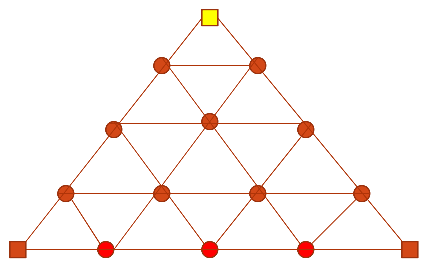

$\delta = 0$

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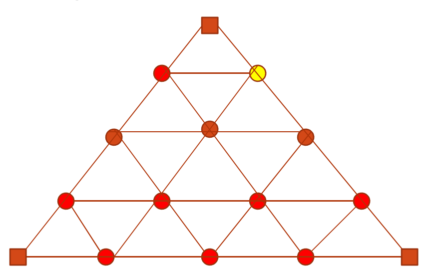

$\delta = 1$

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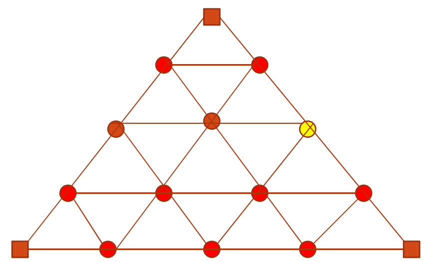

$\delta = 2$

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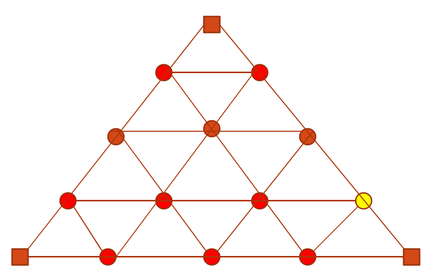

$\delta = 1$

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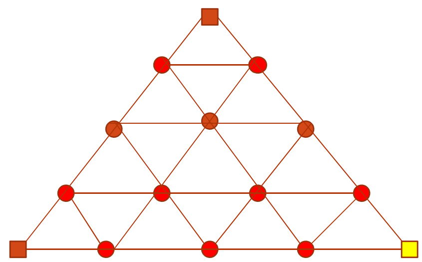

δ = 0

δ-Hyperbolicity of Finite Graphs

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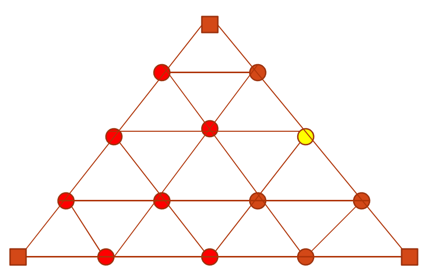
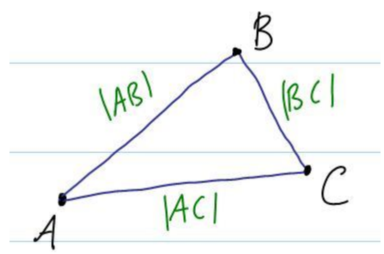


δ = 2

- Therefore in this example, the graph is 2-hyperbolic.
- In the general form, for a side of length l, the graph is $\left\lfloor \frac{l}{2} \right\rfloor$-hyperbolic.

δ-Hyperbolic Spaces

- In a Euclidean space, we can represent three points A, B, C with a triangle[1]:

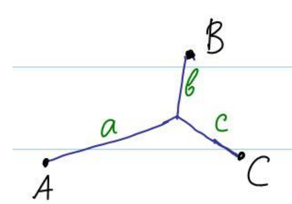


- The shortest distance from A to B is of length |AB|, and so on.

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

- There is another way to represent this[1]:



- This tripod representation yields 3 equations:

$$a + b = |AB|$$
$$a + c = |AC|$$
$$b + c = |BC|$$

- Solving this set of equations yields:

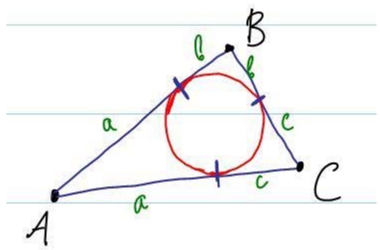
$$a = \frac{1}{2}(|AB| + |AC| - |BC|)$$
$$b = \frac{1}{2}(|AB| + |BC| - |AC|)$$
$$c = \frac{1}{2}(|AC| + |BC| - |AB|)$$

- We denote $a = (B|C)_A$, the **Gromov product** of B and C with respect to A.

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

- The geometric way to arrive to this result is by inscribing a circle in the triangle[1]:

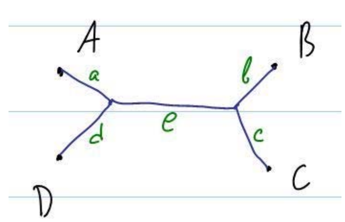


- The circle divides the sides such that we get a, b, and c.

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

- How may we represent 4 points this way[1]?
- At first we may try by analogy:

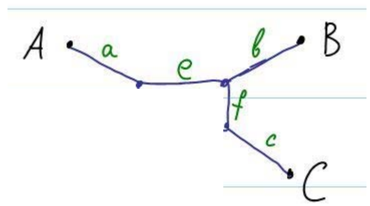


- However, this way we have only 5 degrees of freedom, while we have a set of 6 different distances.
- A solution is not guaranteed. For example:
  - Suppose $|AB|=|BC|=|CD|=|DA|=1$ and $|AC|=|BD|=2$.
  - Both B & D must be on the midpoint between A and C.
  - However, $|BD|$ must be 2, not 0.

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

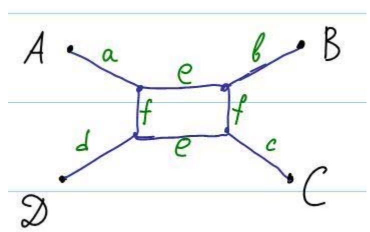
- We add another degree of freedom[1]:



- We can solve 6 equations with 6 variables.
- Even simpler – note that A, B, C form a tripod with side lengths of a+e, b, c+f.
  - This immediately yields: $b = (A\,|\,C)_B$.
  - Similarly, $a = (B\,|\,C)_A$, $c = (A\,|\,B)_C$, $d=(A\,|\,C)_D$.

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

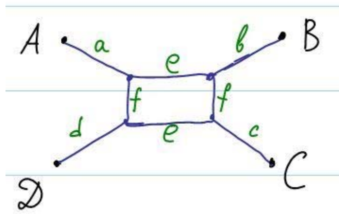


- Consider the perfect matchings between the points A, B, C, D[1]:

$$|AB| + |CD| = (a + b + c + d) + 2e$$
$$|AD| + |BC| = (a + b + c + d) + 2f$$
$$|AC| + |BD| = (a + b + c + d) + 2e + 2f$$

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces

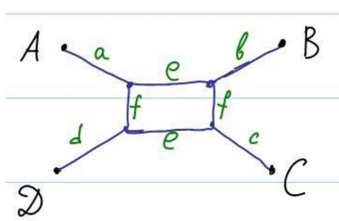


$|AB| + |CD| = (a + b + c + d) + 2e$
$|AD| + |BC| = (a + b + c + d) + 2f$
$|AC| + |BD| = (a + b + c + d) + 2e + 2f$

- 2e and 2f are the amounts by which the longest matching exceeds the other two[1].
- In particular, the rectangle collapses into an edge when the two longest matchings have the same size (as happens in a tree).

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolic Spaces



$|AB| + |CD| = (a + b + c + d) + 2e$
$|AD| + |BC| = (a + b + c + d) + 2f$
$|AC| + |BD| = (a + b + c + d) + 2e + 2f$

- Definition: a metric space is **Gromov hyperbolic** if for every 4 points[1]:

$$\exists \boldsymbol{\delta} \geq \mathbf{0}, \boldsymbol{min}(\boldsymbol{e}, \boldsymbol{f}) \leq \boldsymbol{\delta}$$

- More intuitively – if the rectangle in the middle is not "too fat".

1. "Tight spans and Gromov hyperbolicity", *Calculus VII*, http://calculus7.org/2012/11/11/tight-spans-and-gromov-hyperbolicity/.

δ-Hyperbolicity of Finite Graphs

- An equivalent definition:

Given a graph G = (V, E) and 4 vertices $v_1, v_2, v_3, v_4 \in V$, let:

$$d_1 = d(v_1, v_2) + d(v_3, v_4)$$
$$d_2 = d(v_1, v_3) + d(v_2, v_4)$$
$$d_3 = d(v_1, v_4) + d(v_2, v_3)$$

Without loss of generality, let:

$$d_1 \geq d_2 \geq d_3$$

We define:

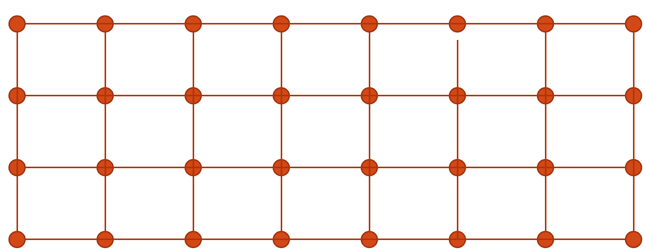
$$\delta(v_1, v_2, v_3, v_4) = \frac{d_1 - d_2}{2}$$

- The hyperbolicity of the graph is therefore defined by:

$$\delta = \max_{v_1, v_2, v_3, v_4 \in V} \delta(v_1, v_2, v_3, v_4)$$

δ-Hyperbolicity of Finite Graphs

- Example on a 4X8 grid:



δ-Hyperbolicity of Finite Graphs

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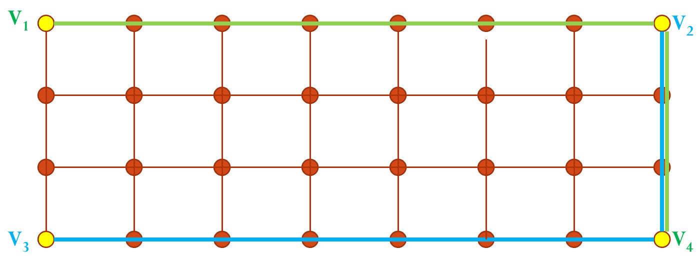

$$d_1 = 10 + 10 = 20$$

δ-Hyperbolicity of Finite Graphs

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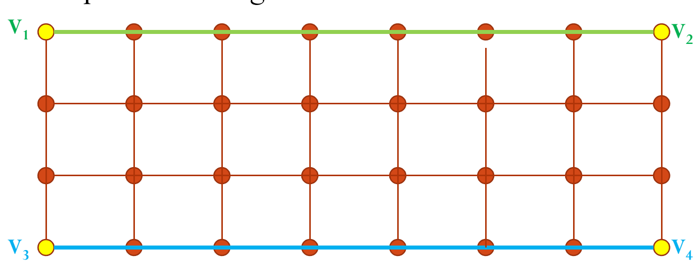

$$d_1 = 10 + 10 = 20$$
$$d_2 = 7 + 7 = 14$$

δ-Hyperbolicity of Finite Graphs

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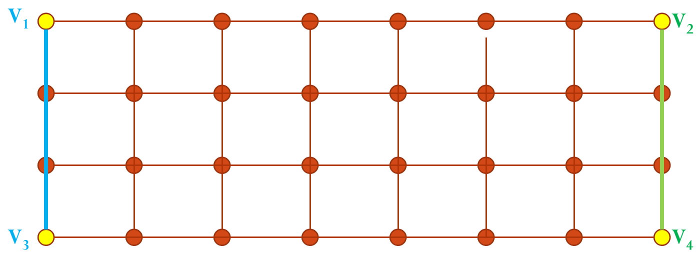

$$d_1 = 10 + 10 = 20$$
$$d_2 = 7 + 7 = 14$$
$$d_3 = 3 + 3 = 6$$

δ-Hyperbolicity of Finite Graphs

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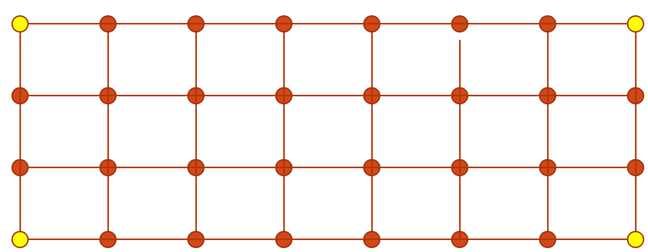

$$d_1 = 10 + 10 = 20$$
$$d_2 = 7 + 7 = 14$$
$$d_3 = 3 + 3 = 6$$
$$\delta = \frac{20 - 14}{2} = 3$$

Applications

- Estimating distances on the Internet by embedding it onto a hyperbolic space[1]:
  - Closest-server.
  - Building an application level multicast tree.
- Label routing[2].
- Classical problems in hyperbolic space[3]:
  - PTAS for TSP problem when cities lie in H^d.
  - Nearest neighbor search data structure with $O(\log n)$ query time, $O(n^2)$ space.
- And more others!

1. Y. Shavitt and T. Tankel, "Hyperbolic Embedding of Internet Graphs for Distance Estimation and Overlay Construction", *IEEE/ACM Transactions on Networking*, 16(1):25—36, 2008.
2. D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguna, "Hyperbolic Geometry of Complex Networks", *Physical Review E*, v.82, 036106, 2010.
3. R. Krauthgamer and J. Lee, "Algorithms on Negatively Curved Spaces", *FOCS '06 Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science*, pp. 119-132, 2006.

The Problem

- Brute force solution for finding a graph's δ-Hyperbolicity includes going over $O(n^4)$ quads of vertices.
- Not feasible for large networks.
- **How may we find δ (or a good estimate) more efficiently?**

- This is your **home assignment**
  - You can assume $O(E) = O(V)$