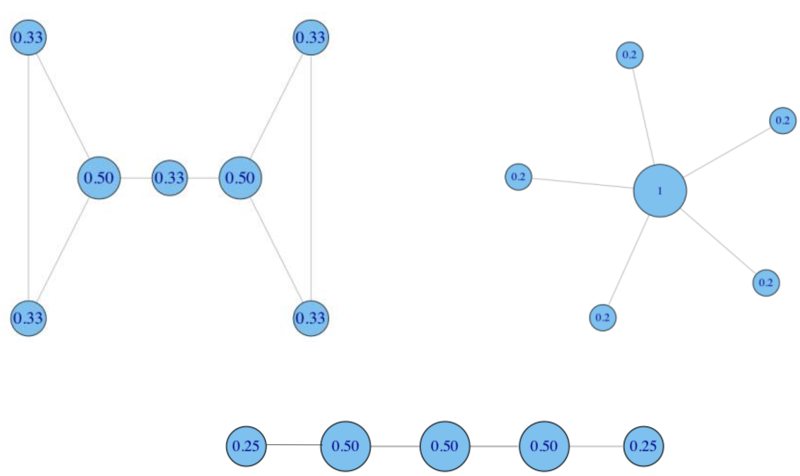
Graph Centrality

- Degree centrality
  - The node degree
- Closeness centrality
  - (the sum of distances to all other nodes)$^{-1}$
- Betweenness centrality
  - The number of shortest path thru a node
- Eigenvector centrality

degree: normalized degree centrality

divide by the max. possible, i.e. (N-1)



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betweenness centrality: definition

betweenness of vertex i

paths between j and k that pass through i

$$C_B(i) = \sum_{j<k} g_{jk}(i) / g_{jk}$$

all paths between j and k

Where g_{jk} = the number of geodesics connecting *j*-*k*, and
$g_{jk}(i)$ = the number that node *i* is on.

Usually normalized by:

$$C_B^{'}(i) = C_B(i) / \boxed{[(n-1)(n-2)/2]}$$

number of pairs of vertices excluding the vertex itself

directed graph: (N-1)*(N-2)

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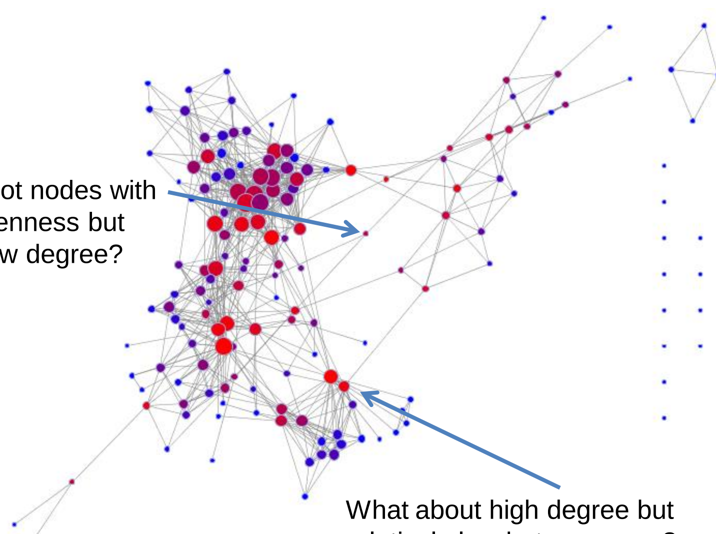
example

Nodes are sized by degree, and colored by betweenness.

Can you spot nodes with high betweenness but relatively low degree?



What about high degree but relatively low betweenness?

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Eigenvector Centrality

Adjacency matrix

$$x_i = \frac{1}{\lambda} \sum_{j \in N_i} x_j = \frac{1}{\lambda} \sum_{j=1}^{N} A_{ij}\, x_j$$

$\boxed{\lambda X = AX}$

X is the vector of Eigenvalues

PageRank is conceptualy similar

The Graph Diameter

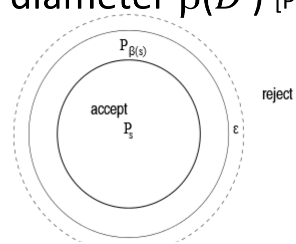
- Diameter = the maximal distance between all pairs of vertices

Unweighted graph

- Shortest path from a node to all others: $\Theta(m)$
- All pair shortest path: $\Theta(n \cdot m)$ (using BFS)
- Using matrix product: $O(n^{2.376} \text{ polylog}(n))$
  - And $\Theta(n^2)$ space [Alon *et al*., FOCS 1992]
- Fast algorithms that use $\Theta(n^2)$ space
  - $\Theta(n^3/\log n)$ for dense graphs [Feder & Motwani, STOC 91]
  - $O(n^2(\log \log n)^2/\log n)$ for sparse graphs [Chan, SODA'06]

Unweighted Graph (cont.)

- Estimating D by $\overline{D}$ [Dor *et al.,* 1997]
  - $\overline{D} \le D \le \overline{D} + 2$
  - Time $\Omega(n^2)$
  - Space $\Theta(n^2)$
- Testing if the diameter is below $\overline{D}$ or the graph is ε-far from a graph with diameter $\beta(\overline{D})$ [Parnas & Ron,2002]



Bounds

- Trivial bounds:
  For any vertex *v*: ecc(*v*) ≤D≤2·ecc(*v*)
  - ecc(*v*) is the eccentricity of *v*.
  - Can be computed in Θ(m) time & space
- Double sweep lower bound:
  choose *v* s.t. d(*v,u*)=ecc(*u*) for some *u*.
  - for trees (and other special graphs) D=ecc(*v*)
  - For other graphs it is a tighter lower bound
  - Can be computed in Θ(m) time & space
- Tree upper bound:
  for any spanning tree, the tree diameter is an u.b.
  - Can be computed in Θ(m) time & space

Discussion

- Iterate for different vertices can improve the bounds
- For the tree upper bound, chose the highest degree node
  - Good for power-law graphs
- Iterating may not always help
  - E.g., the tree u.b. for a cycle

Experimnets

- Internet router graph from Skitter (2005)
  n=1,719,037 m=11,095,298
- A web graph (.uk domain, 2005)
  n=39,459,925 m=783,027,125
- Peer to peer graph (eDonkey sharing, 2004)
  n=5,792,297 m=142,038,401
- IP traffic graph
  n=2,250,498 m=19,394,216

Simulation Results

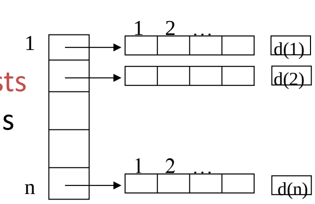
	t.l.b.		d.s.l.b.		h.d.t.u.b.		r.t.u.b.		t.u.b.		iterations
INET	29		**31**		**34**		34		38		10,000
	998	0.0001	2	0.49	26	0.1002	23	0.0633	127	0.0011	
P2P	8		**9**		**10**		10		10		5,000
	210	0.0094	1	0.7005	1	0.039	120	0.0032	3237	0.0001	
WEB	26		**32**		**33**		33		34		2,000
	816	0.001	1	0.985	34	0.0015	46	0.0025	1572	0.0005	
IP	9		**9**		**9**		9		10		10,000
	5331	0.0001	1	0.989	4	0.0543	12	0.0346	6284	0.0001	

Portion of iteration hitting the bound

First iteration to hit the bound

Testing the Graph Diameter

- We assume graphs are represented by the incidence lists of the vertices, where each list is accompanied by its length.
- Allowed queries:
  - what is the degree, d(v), of any vertex v?
  - who is the i'th neighbor of v, for any vertex v and index 1≤ i ≤ d(v)?



Parnas & Ron, 1992

ε-far Definition

Let P_s be a fixed parameterized property, $0<\varepsilon<1$, and m a positive integer. A graph G having at most m edges is ε-far from property P_s (with respect to the bound m), if the number of edges that have to be added and/or removed from G in order to obtain a graph having property P_s, is greater than $\varepsilon \cdot m$.

Otherwise, G is ε-close to P_s.

A Testing Algorithm

A testing algorithm for (parametrized) property P_s, with boundary function $\beta(\cdot)$, is given a (size) parameter $s>0$, a distance parameter $0<\varepsilon<1$, a bound $m>0$, and a query access to an unknown graph G having at most m edges.

The output of the algorithm is *accept* or *reject*.

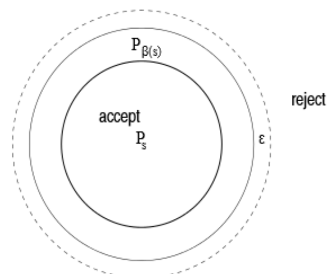
- If G has property P_s, then the algorithm should output accept with prob. at least 2/3
- If G is ε-far from property P_s, then the algorithm output reject with prob. at least 2/3.

Diameter Testing

Testing if a graph diameter is bounded by D.

A family of algorithms which differ in:

- The boundary function β(·)
- Query and time complexity
- The value of ε



Why Only small ε?

- Every connected graph with *n* vertices can be transformed into a graph with diameter at most *D* by adding at most $n/\lfloor D/2 \rfloor$ edges.

$\Downarrow$

- Every connected graph with *n* vertices and *m* edges is ε-close to having diameter *D* for every $\varepsilon \geq \frac{2}{D} \cdot \frac{n}{m}$

$\Downarrow$

- $\varepsilon_{n,m} \stackrel{\text{def}}{=} \frac{m}{n} \cdot \varepsilon$

Main Results
Testing algorithms for diameter *D*

1. Boundary function $\beta(D) = 4D+2$
   Query time $O(1/\varepsilon_{n,m}{}^3)$
   1-sided error: always accept graphs with diameter at most *D*.
2. Boundary function $\beta(D) = 2D+2$
   Query time $O(\frac{1}{\varepsilon_{n,m}{}^3} \cdot \log^2 \frac{1}{\varepsilon_{n,m}})$
   2-sided error
3. Boundary function $\beta(D) = D(1+\frac{1}{2^i-1})+2$, $2 \le i \le \log(D/2+1)$
   Query time $O(\frac{1}{\varepsilon_{n,m}{}^3} \cdot \log^2 \frac{1}{\varepsilon_{n,m}})$, $\varepsilon = \Omega(\frac{n^{1-\frac{1}{i+2}} \cdot \log n}{(i+2)m})$

$$\varepsilon_{n,m} = \Omega(\frac{n^{-\frac{1}{i+2}} \cdot \log n}{(i+2)})$$

Main Results

	$\beta(D)$	ε	Remarks
1.	2D+2	Any	One Sided Error
2.	$\left(1+\frac{1}{2^i-1}\right) \cdot D+2$	$\Omega\left(\frac{n^{1-\frac{1}{i+2}} \cdot \log n}{(i+2) \cdot m}\right)$	Two Sided Error
	4D/3 + 2	$\tilde{\Omega}\left(n^{-1/4}\right)$	i = 2
	D+4	$\Omega(1/\mathrm{poly}(\log n))$ for D = poly(logn)	i = log(D/2 + 1)

Time and Query Complexity: $\tilde{O}\left(\frac{1}{\varepsilon^3}\right)$

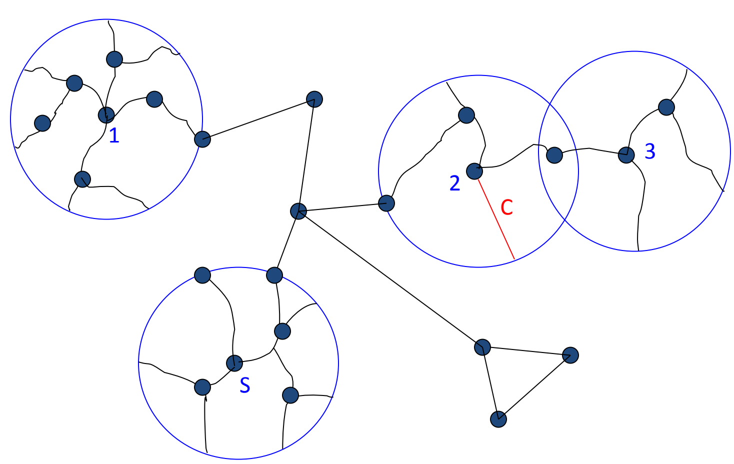
Algorithm

Input: D, n, m, ε.
Parameters: C, k, α.

- Set $\varepsilon_{n,m} = \frac{m}{n} \cdot \varepsilon$
- Uniformly select $S = \Theta(1/\varepsilon_{n,m})$ starting vertices.
- For each starting vertex - perform a BFS to distance at most C until k vertices are reached.
- If at most $\alpha \cdot S$ starting vertices reach < k vertices then accept, otherwise reject.

Time and Query Complexity: $O(k^2 \cdot S) = O(k^2/\varepsilon_{n,m})$

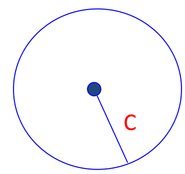
Illustration of the Algorithm



Proof of Correctness

Good Vertex: If C-neighborhood contains ≥ k vertices.

Bad Vertex: If C-neighborhood contains < k vertices.



We Show:

- Diameter ≤ D ⟹ Almost (all) vertices are good.
- Diameter > β(D) ⟹ Many vertices are bad.

Reducing the Diameter

Lemma 1:
If at least (1-1/k)n of the vertices are good, then the graph can be transformed into a graph with diameter at most 4C+2 by adding at most 2n/k edges.

Proof: