THERMODYNAMIC ANALOGIES
IN GRANULAR MATERIALS

Research Thesis

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for the Degree of Doctor of Philosophy

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Abstract

This Thesis investigates possible statistical mechanics descriptions of granular materials. We identify two characteristics of granular materials, which prevent them from being in thermodynamic equilibrium. Since thermal energy is irrelevant for the translational motion of macroscopic grains, granular systems tend to jam in a limited set of states and do not explore their possible states effectively. This limit on ergodicity is even stronger in densely packed systems, where global rearrangements are required in order for single particles to move. The second property preventing even dilute granular systems from being in thermodynamic equilibrium is dissipation, or energy loss in the inelastic collisions between grains. Dissipation breaks the dynamics’ symmetry to time reversal and prohibits detailed-balance. We distinguish between these two factors by considering two extreme systems: one of dense and static packings of grains, where jamming is prominent, and the other of dilute and dynamic granular gases, where dissipation is dominant.

In the case of strongly jammed systems, we investigate the role of friction on compaction and segregation in static granular packings. We construct a mechanical model incorporating the effect of friction on the local volume and entropy of the packing, and together with original mean field geometrical calculations, combine it with Edwards’ thermodynamic hypothesis on the statistical mechanics of static granular packings. Our model predicts that for systems of identical grains, larger friction causes less efficient compaction. Mixtures are found to segregate due to frictional differences between grains. A phase diagram for segregation versus friction coefficients of the two species is generated. Finally, the resulting segregation is related directly to the volume fraction without the explicit use of the yet unclear notion of compactivity.

Inspired by granular gases, we investigate the statistical mechanics of driven dissipative systems by introducing a minimal stochastic model for their dynamics. The model incorporates the essential features of dissipative interactions with maximal randomness. Upon an interaction between two particles, the non-dissipated energy is randomly redistributed between them. We demonstrate various aspects of the non-equilibrium behavior of driven dissipative systems on our model: The energy distribution differs from the equilibrium Boltzmann distribution, however we find that its high energy tail is proportional to the Boltzmann distribution; Time dependent fluctuation-dissipation relations are generally violated, however we show that they asymptotically hold for long waiting times, and exactly hold for simple measurements; Effective temperatures defined by different means on the same system generally differ, however their numerical values are ordered and in some cases we were able to show that effective temperatures exactly coincide. Our stochastic mod-
eling provides a framework for theoretically investigating the statistical mechanics of systems far from equilibrium. Although the exact results we present are obtained for our specific model, all effects we identified are expected to be valid, at least qualitatively, in other driven dissipative systems as well. We provide preliminary explanations for this generality of our results to other systems and suggest continuing the research by seeking underlying principles and general results for system far from equilibrium.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$c$</td>
<td>Scaled velocity</td>
</tr>
<tr>
<td>$C(t), c(t)$</td>
<td>Correlation function</td>
</tr>
<tr>
<td>$d, D$</td>
<td>Diameter</td>
</tr>
<tr>
<td>$d(E)$</td>
<td>Density of states</td>
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<tr>
<td>DOS</td>
<td>Density Of States</td>
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<td>$E, e, e_i$</td>
<td>Energy</td>
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<tr>
<td>$F$</td>
<td>Coupling strength to bath</td>
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<tr>
<td>$F_D$</td>
<td>External field</td>
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<tr>
<td>$F_{DR}$</td>
<td>Fluctuation-Dissipation</td>
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<td>$g$</td>
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<td>$g(\lambda)$</td>
<td>Collision rate in Boltzmann Equation</td>
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<td>$h$</td>
<td>Generating function</td>
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<tr>
<td>$m$</td>
<td>Coupling strength to probe</td>
</tr>
<tr>
<td>$n$</td>
<td>Mass</td>
</tr>
<tr>
<td>$\hat{n}$</td>
<td>Unit vector connecting centers of colliding particles</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of particles</td>
</tr>
<tr>
<td>NESS</td>
<td>Non-Equilibrium Steady State</td>
</tr>
<tr>
<td>$r$</td>
<td>Location</td>
</tr>
<tr>
<td>$R(t), r(t), \chi$</td>
<td>Radius</td>
</tr>
<tr>
<td>$\Delta R(t)$</td>
<td>Location</td>
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<tr>
<td>$R(t), r(t)$</td>
<td>Radius</td>
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<tr>
<td>$T_B$</td>
<td>Bath temperature</td>
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<tr>
<td>$T_C$</td>
<td>Configurational temperature</td>
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<tr>
<td>$T_F, T_{FD}$</td>
<td>Fluctuation temperature</td>
</tr>
<tr>
<td>$T_G$</td>
<td>Granular temperature</td>
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</tbody>
</table>
$T_M$  Moment temperature
$T_R$  Moment ratio temperature
$T_S$  Entropic temperature
$T_K$  Kubo temperature
$v$  Velocity
$v_i$  Contribution of particle $i$ to total volume
$v_n$  Relative normal velocity
$x_i$  Coordinate
$V$  Volume
$X$  Compactivity
$z$  Vertical coordinate

Fraction of energy given to every particle

$Z$  Partition function
$\alpha$  Restitution coefficient
$\epsilon$  Normal restitution coefficient
Average energy per particle

Energy spacing between states

$\gamma$  Ratio of granular temperatures in mixture
$\Gamma$  Normalized acceleration

Interaction rate per particle

$\lambda$  Drag coefficient
$\mu$  Friction coefficient
$\theta$  Angle between contacts

$\rho, \Phi$  Volume fraction
$\sigma$  Root mean square velocity
$\xi$  Random force
1 Introduction

Granular materials, which are receiving growing attention in the physical community [1, 2, 3, 4], are a perfect example of a classical complex system: The mechanical behavior of a single grain is simple and well understood, while the behavior of a large collection of grains is complex and exhibits a rich variety of yet unexplained phenomena. Granular materials appear in nature, ranging from geology and sand at the beach to planetary rings in space. Moreover, they are important in many industrial processes: construction, pharmaceuticals, food, cosmetics, and more. In all cases the complexity originates from the grains being macroscopic compared to atomic length scales.

It is tempting to draw analogy to the statistical mechanics of atoms and molecules with the grains playing the role of microscopic particles. However, the macroscopic nature of the grains gives rise to several crucial differences: First, thermal energy at room temperature is negligible compared to the different possible forms of mechanical energy in granular systems. This leads to jamming of the system in locally stable states, rendering ergodicity questionable. Jamming is even more pronounced in dense packings where the rearrangement of many other grains is required in order to move a single grain. The second difference between macroscopic grains and microscopic particles is that kinetic energy is dissipated upon grain interactions, hence the notion of an isolated system is irrelevant and, moreover, detailed-balance may not be obtained.

Therefore, jamming and dissipation preclude the use of equilibrium statistical mechanics for obtaining the constitutive relations characterizing the coarse-grained material properties. Furthermore, hydrodynamic descriptions of granular materials are problematic, since length scales are typically not separated. Gradients are on similar length scales to particle separation and concepts such as local thermodynamic equilibrium may not be employed.

The kinetic energy of an isolated dissipative system decays with time until the system is static. However, a non-equilibrium steady state with a positive and constant average energy may be maintained by continuously driving the system. Not only does this steady state differ from thermal equilibrium, it is also qualitatively different from other classes of systems with non-equilibrium steady states, which receive much attention [5]: Glasses are out of equilibrium, but are at a very slow transition to thermodynamic equilibrium; Systems which are stirred, or driven by a periodic force changing much faster than their thermalization rate, may be far from equilibrium. However, when not driven they reach equilibrium, and are therefore constantly in a transition to equilibrium. Granular materials, on the contrary, will never reach equilibrium, neither when driven nor when undriven.
To some extent granular systems exhibit similarities (and many dissimilarities) to the solid, liquid and gas states of matter, and may be classified accordingly [1]: *Granular Solids* are static dense packings of grains; *Granular Liquids* are comprised of densely packed moving grains; and *Granular Gases* are dilute systems of grains in motion. Granular solids reflect jamming, and granular gases dissipation, while granular fluids reflect both.

This Thesis addresses the possibility of thermodynamic-like descriptions of both jammed and dissipative systems. It concentrates on the two extreme cases of static systems (that are strongly jammed) and of very dilute (thus unjammed) dynamic systems (where dissipation is prominent). The following sections of this Introduction will overview the phenomena observed and questions raised during the contemporary experimental and theoretical research of granular solids and granular gases. A large portion of this Thesis deals with driven dissipative systems in general, with granular gases being one pedagogical example for them presented in detail in this Introduction. That is, many of our results are relevant to a broader class of driven dissipative system, and we therefore provide in the last section of the Introduction an overview of fluctuation-dissipation relations in various non-equilibrium systems. Our results concerning fluctuation-dissipation relations are relevant to this wide variety of non-equilibrium systems.

### 1.1 Granular Solids

A static packing of grains is a typical example for a system jammed in a small portion of phase space, without the possibility of exploring other energetically allowable (and even preferable) states. Jamming is manifested, for instance, in the compaction due to gravity: even though an ordered crystalline packing has the largest possible *volume fraction* (ratio of the sum of grain volumes to the volume the system occupies), and hence the minimal gravitational energy, such compaction is never achieved. Simply pouring grains brings them to a quite dilute packing known as random loose packing (see Fig. 1.1), and extensive operations [6] (operations such as tapping or shaking which have a statistical nature, rather than Maxwell-demon operations which act on single grains) performed on the system may only compact the system to random close packing (RCP), which still has a smaller volume fraction than an ordered crystalline packing.

Another interesting feature of granular solids is that since grains exert only repulsive forces one on each other, these systems differ in rheological properties from ordinary solids, where attractive forces exist between atoms. Issues of stress distributions [8, 9, 10], stress transmission [10, 11, 12, 13] and sound propagation [14] supply endless questions for research.
1.1.1 Compaction Experiments

Although a granular solid is static, its dynamics are relevant since its geometrical state depends on the history of its preparation. This is most easily demonstrated on the volume fraction of a packing. In a compaction experiment [15, 16, 17], one starts by pouring grains into a container. By tapping the container, the grains rearrange and after many taps typically arrive at a more compact state. When the tapping stops, the system is static, but its volume fraction depends on the number of taps it went through.

This compaction process is logarithmically slow (see Fig. 1.2) and exhibits fluctuations in the volume fraction both between different experimental realizations, and throughout the tapping process. These fluctuations suggest the thermodynamic description of granular solids, described in the following section. Another feature demonstrated in Fig. 1.2 is reversibility, which too suggests the validity of a thermodynamic description. When the tapping amplitude is increased and then decreased, the system eventually reaches a reversible dependence of volume fraction on tapping amplitude.

Specific models [15, 18, 19, 20] explain the logarithmically slow temporal evolution in this compaction process and connect it to jamming, coming from the limited possibility of the system to explore other packings. We are interested in the statistical properties of the obtained packings, and thus turn now to the statistical mechanics of static packings.
1.1.2 Edwards’ Thermodynamic Analogy

Edwards has suggested an analogy [6] between the packing geometry of jammed granular systems (or granular solids) and the statistical mechanics of thermodynamic equilibrium. In this analogy each mechanically stable arrangement of the grains is equivalent to a micro-state in statistical mechanics, and the total volume the grains occupy plays the role of energy. This formalism considers the ensemble of mechanically stable arrangements of the grains prepared by the same “macro” process. Even though the system is static and does not move with time within this ensemble, ergodicity is assumed and equilibrium statistical mechanics considerations are employed for determining the probability of finding the system in any one of its “micro-states” and for calculating average values of “macroscopic” entities. This is analogous to an ordinary solid, which has lost ergodicity upon solidification. The hypothesis requires the existence of an analog of temperature, referred to as compactivity and denoted $X$. This description is similar to glassy systems, the statistical properties of which are partially determined from an effective temperature differing from the environment temperature [21, 22, 23, 24]. Granular solids are an even more extreme example than glasses for systems out of equilibrium, since they have a finite compactivity, while their real thermodynamic temperature is effectively zero.

In analogy with the canonical ensemble of states in thermal equilibrium, the probability for the occurrence of a state with volume $V$ is assumed to be proportional to $\exp(-V/X)$, where we measure $X$ in units of volume so that the analog of Boltzmann’s constant is equal to one. In analogy with thermodynamic equilibrium, where the temperature may be defined as $T \equiv (dS/dE)^{-1}$, with $S$ the entropy
and $E$ the energy, a formal definition of compactivity is given by $X \equiv (dS/dV)^{-1}$. Recent experimental evidence for reversibility in compaction processes [16] has provided justification for this thermodynamic analogy and has proposed a connection between compactivity and experimentally controllable quantities [25]. However, the proposal remains controversial, mostly because it is not clear how compactivity is measured (not to mention controlled) experimentally.

To date, one of the drawbacks of this description was that it was not clear how practical calculations can be performed on it and how these theoretical concepts are to be applied to specific experimental situations. In Section 3 we provide simple geometrical calculations, which enable investigating the statistical mechanics of granular assemblies from the local packing properties. In particular, we investigate the role of friction on local packing geometry and use Edwards’ assumptions for deducing the global behavior of packings of rough grains.

### 1.1.3 Segregation

Granular mixtures tend to segregate. Macroscopic grains do not exert any (repulsive or attractive) forces when not in contact, thus such segregation (similar to phase separation) may not be attributed to energetic considerations.

Granular liquids are probably the most common for exploring segregation (see Fig. 1.3, and 1.4). However, it is a common phenomenon in shaken or tapped granular solids as well (see Fig. 1.5). Segregation is observed for grains varying in size [26], shape [27], friction coefficient [28, 29] and density [30]. Segregation occurs due to vibration [31], tapping [32], rotation [28], pouring [33] and shearing [34]. Experiments are performed for mixtures of many particles of two different species (for a review see [35]) and with single intruder particles in systems of a single species [36]. Vibrated granular systems have also drawn attention due to pattern formation differing from those observed in liquids [37, 38].

Most explanations for segregation are given in terms of the dynamics [29, 40]. However, there have been some attempts [6, 41, 42, 43] to explain them by Edwards’ thermodynamic description in analogy with phase transitions in equilibrium statistical mechanics. In Section 3 we investigate segregation due to friction using Edwards’ description.
Figure 1.3: Segregation in a rotating drum: An initially mixed state (left) evolves with time to a stratified pattern (right) (see [28] for details).

Figure 1.4: Stratification due to pouring: A mixture of different types of grains is poured next to the wall at the left and bands are spontaneously formed (from [33]).

Figure 1.5: Segregation due to periodic vertical shaking: The large grains initially at the bottom (left image) rise with time (images from left to right) (from [39]).
1.2 Granular Gases

A system with a relatively small volume fraction of grains undergoing inelastic collisions, and with a continuous supply of energy, such as a vibrated container of grains, will eventually reach a non-equilibrium steady state manifesting the balance between driving and dissipation. Granular gases are thus used to understand the statistical mechanics of driven dissipative systems. Non-equilibrium phenomena investigated in granular gases are: clustering, non-Boltzmann energy distributions, velocity correlations, breakdown of energy equipartition, and fluctuation-dissipation relations. Below, we overview the experimental, numerical and theoretical investigation of these phenomena in granular gases.

Our research described in Sections 4, 5, 6, and 7 is inspired by granular gases. However, it aims at a more fundamental understanding of driven dissipative systems and achieves that by abstract modeling of such systems.

1.2.1 Experiments

Realizing a granular gas in the lab is problematic due to gravity, which tends to pack the grains at the bottom of the container. However, with vigorous vertical driving (at accelerations significantly larger than the gravitational acceleration) one may fluidize the grains and obtain a relatively homogeneous gas within the container (see Fig. 1.6). This driving is typically realized either by mechanically shaking the container walls [44] or by applying a fluctuating electric field on the grains [45].

![Figure 1.6: Left: Schematic diagram of a granular gas experiment: Grains are marked with circles. The top and bottom walls vibrate vertically. Right: Vertical distribution of the volume fraction in experiments at various driving amplitudes (from [44]).](image)

Granular gases are clearly not in equilibrium with their environment. At room temperature such macroscopic grains would not move at significant velocities (and would not reach a relatively uniform volume fraction) just due to thermal energy. The particles may be tracked and their velocities measured, and it is customary to
characterize the grain’s typical energy by the *granular temperature* defined as the average kinetic energy,

\[
T_G \equiv \langle \frac{mv^2}{2} \rangle
\]  

(1.1)

(we measure temperature in units of energy, and hence set Boltzmann’s constant to one).

As \(T_G\) is significantly higher than room temperature, the system is not in equilibrium with its environment. Nonetheless, one may question whether a driven granular gas is effectively at equilibrium with its driving mechanism. Thus, when comparing the non-equilibrium steady state of a driven dissipative system to thermodynamic equilibrium particle energies are scaled with respect to their average energy, and the normalized energy distribution is compared to the Boltzmann distribution. Namely, the root mean squared velocity is defined \(\sigma^2 \equiv \langle v^2 \rangle \equiv 2T_G/m\), and scaled velocities \(c \equiv v/\sigma\) are used. Granular gas experiments [44, 45] clearly indicate that the normalized velocity distributions deviate from the Gaussian Maxwell-Boltzmann distribution found in equilibrium (see Fig. 1.7). Its high energy tail may be fitted by a stretched exponential,

\[
p(c) \sim \exp(-Ac^n)
\]  

(1.2)

with \(n \approx 3/2\). The form of the high energy tail depends both on the dissipative interactions and on the driving. It should be noted that typical experimental driving does not have a well behaved equilibrium limit. That is, driving elastically colliding grains in such a manner will not bring them to equilibrium. In section 7 we shall see that when the driving is thermal-like (i.e., a conservative system would equilibrate with it) the high energy tail of a dissipative system driven by it is equilibrium-like, and does not exhibit the overpopulated high energy tail typically attributed to granular gases.

Another deviation from equilibrium behavior is found when testing energy equipartition in granular mixtures. In equilibrium two systems in contact eventually reach a common temperature. This holds for a gas of different atoms or molecules in the same container, where the different species may be thought of as separate thermodynamic systems in contact. For non-equilibrium steady states, on the other hand, this does not hold. Experiments of two species of grains differing in their mechanical properties shaken in the same container [46, 47] indicate that each species possesses a different granular temperature (see Fig. 1.8).
Figure 1.7: Velocity distribution from granular gas experiments. Horizontal velocity $v_x$ is scaled by the root mean square velocity $\sigma^2 \equiv \langle v_x^2 \rangle$. (A) Distribution clearly has overpopulated high energy tails compared to the equilibrium Gaussian distribution (dotted line). (B) The high energy tail may be fitted by a stretched exponential $p(v) \sim \exp(-Av^n)$ with $n = 1.52$ (solid line), and differs significantly from an exponential $n = 1$ and a Gaussian $n = 2$ (dotted lines) (from [44]).

Figure 1.8: Vertical profiles of granular temperature in a granular mixture experiment. The dotted and dot-dashed lines represent the granular temperatures of the two species (steel and glass) and the solid line represents the ratio $\gamma \equiv T_{glass}/T_{steel}$ between them (from [46]).
1.2.2 Inelastic Hard Sphere Model

We theoretically describe granular gases as frictionless inelastic hard spheres, moving freely in space between collisions. Particles are hard, hence collision times are infinitely short, and whenever two particles meet their velocities are instantaneously modified conserving momentum and angular momentum, but dissipating energy. In the collision of two particles with centers located at \( \vec{r}_1 \) and \( \vec{r}_2 \), the post-collision velocities \((\vec{v}_1', \vec{v}_2')\) are related to the pre-collision velocities \((\vec{v}_1, \vec{v}_2)\) by

\[
\begin{align*}
\vec{v}_1' &= \vec{v}_1 - \frac{1 + \epsilon}{2} v_n \hat{n} \\
\vec{v}_2' &= \vec{v}_2 + \frac{1 + \epsilon}{2} v_n \hat{n},
\end{align*}
\]

(1.3)

where

\[
\hat{n} \equiv \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}
\]

(1.4)

is the unit vector connecting the centers of the particles,

\[
v_n \equiv (\vec{v}_1 - \vec{v}_2) \cdot \hat{n}
\]

(1.5)

is the relative normal velocity, and \( \epsilon = \epsilon(v_n) \) is the coefficient of normal restitution, which may generally depend on the impact velocity \( v_n \).

The center of mass velocity of the colliding pair does not change, and the relative velocity is changed so that its normal component is multiplied by \(-\epsilon\) and its tangential component is not changed. This assures conservation of linear and angular momentum, and a reduction in energy. The total kinetic energy of the colliding pair

\[
E = \frac{m}{2} |\vec{v}_1|^2 + \frac{m}{2} |\vec{v}_2|^2
\]

(1.6)

may be easily seen from Eq. (1.3) to be modified due to the collision to

\[
E' = \frac{m}{2} |\vec{v}_1'|^2 + \frac{m}{2} |\vec{v}_2'|^2 = E - \frac{1 - \epsilon^2}{4} m v_n^2
\]

(1.7)

For elastic collisions \( \epsilon = 1 \) and energy is conserved, whereas for perfectly plastic collisions \( \epsilon = 0 \) and energy dissipation is maximal. For frictional grains the relative tangential velocity changes in every collision according to the tangential restitution coefficient.

In reality, the normal restitution coefficient \( \epsilon(v_n) \) varies with the impact velocity [48, 49]. This dependence is incorporated in many granular gas simulations [50, 51] in order to prevent the numerical problem of inelastic collapse [52, 53, 54, 55].
For the theoretical analysis presented below we shall assume a constant (velocity independent) restitution coefficient \( \epsilon \).

Freely evolving dissipative systems lose energy with time due to the inelastic interactions between their constituents, and eventually reach a trivial steady state where all particles have zero energy. Such undriven systems are referred to as cooling system, and their state is customarily studied after scaling particle velocities by the momentary root mean square velocity. Heated systems, on the other hand, receive a continuous injection of energy from some driving mechanism, and may thus be maintained in a non-trivial steady state.

The theoretical investigation of driven granular gases includes various models for driving. Following the driving in actual experiments, in some models energy is injected at the system's boundaries. Such models are investigated mostly with numerical simulations. In models amenable to some analytical analysis energy is injected homogeneously throughout the system.

As in the experiments described above, driving at the boundaries may be implemented by vibrating walls [51]. More homogeneous states may be obtained by continuously shearing the container walls [56, 57] with the Lees-Edward boundary condition [58, 59]. A third driving mechanism at the boundaries, which is less realistic experimentally, but is more esthetic from a statistical mechanics point of view, is by thermal walls [60, 61]. In such driving every particle hitting the wall returns with a velocity drawn from an equilibrium distribution at some temperature. This may be thought of as having an infinitely large equilibrium system at the other side of the wall.

Homogeneous driving may be implemented either by a stochastic thermostat [62, 63] or by contact with an ideal gas heat bath [64]. In both cases particles interact inelastically one with the other, and furthermore interact with some omnipresent driving mechanism. For the ideal gas heat bath the dissipative particles occasionally collide with other elastically colliding particles maintained in equilibrium at a temperature \( T_B \). For a stochastic thermostat, on top of inter-particle collisions, each particle undergoes Langevin dynamics,

\[
\frac{d\vec{v}_i}{dt} = \frac{d\vec{v}_i}{dt} \bigg|_{\text{coll}} + \xi_i(t) - \lambda v_i(t),
\]

where \( \xi_i(t) \) is an uncorrelated Gaussian random force with \( \langle \xi_i(t) \rangle = 0, \langle \xi_i(t)\xi_j(t') \rangle = 2\lambda T_B \delta_{ij}\delta(t - t') \), \( \lambda \) is a drag coefficient determining the coupling strength between the system and the drive, and \( T_B \) is the temperature characterizing the driving. A non-dissipative system undergoing either of these two driving mechanisms would equilibrate at a temperature equal to \( T_B \). Therefore, these driving mechanisms are viewed as keeping the dissipative system in contact with an equilibrium bath at
temperature $T_B$.

One last driving mechanism referred to in dissipative systems is the Gaussian thermostat. This thermostat scales the energies of all particles in the system after every dissipative interaction in it, so that the total system energy remains exactly constant. The steady state reached by a dissipative system heated by a Gaussian thermostat is exactly equivalent to that obtained in a cooling system (without a thermostat) after scaling all particle energies to the momentary average energy in the system [65].

In our theoretical investigation of driven dissipative systems presented in sections 4, 6, and 7 we concentrate on systems in contact with a homogeneous equilibrium bath realized either by the stochastic thermostat or by the ideal gas heat bath. In both cases, equilibrium with the bath is obtained as a non-singular limit at vanishing inelasticity.

### 1.2.3 Simulations

Granular gases may be investigated numerically by event-driven [66, 67] molecular dynamics simulations of the inelastic hard sphere model presented above. These simulations follow the locations and velocities of all particles and integrates the equation of motion with time to yield the exact dynamics for any given initial state.

In freely evolving systems the decay with time of the granular temperature may be calculated. As long as the system is homogeneous this decay behaves for asymptotically long times as $T_G \sim t^{-2}$ [68, 55, 69], in agreement with theoretical predictions [70]. Similarly, the steady state granular temperature in a driven system agrees with theoretical predictions (see e.g. [71]). Numerical simulations have successfully managed to demonstrate [72, 73, 74] the breakdown of energy equipartition observed in granular mixture experiments [46, 47], and predicted by kinetic theory calculations [75].

A very visual feature of cooling granular gases is clustering [68, 76] (see Fig. 1.9). The dissipative nature of grain collisions causes the system to be unstable to small fluctuations in the volume fraction. Wherever the volume fraction rises, the collision rate rises, and hence the dissipation rate rises, so that grains are less energetic and are less likely to leave this region, which evolves into a cluster of grains. Clustering has been one of the major obstacles in building a thermodynamic and hydrodynamic description of granular gases using continuous fields. It has, furthermore, been used [77] as a possible explanation for the non-Gaussian velocity distributions found in granular gases.

Theory [78], simulations [79] and experiments [44, 45] indicate that the single particle velocity distribution differs from the equilibrium Gaussian distribution (see Fig. 1.10). The distribution has been found to be different in cooling systems.
and in steady states of heated systems. Kinetic theory calculations agree to some extent with simulations, but only as long as the system remains homogeneous and no correlations appear due to clustering [79]. In the dilute and elastic limit correlations disappear and simulations agree with kinetic theory.

Figure 1.9: Particle locations in a numerical simulation of a freely cooling granular gas (from [68]).

Figure 1.10: Distribution of the normalized velocity \( c \equiv v/\sigma \) (where \( \sigma^2 \equiv \langle v^2 \rangle \)) in numerical simulations of driven granular gases, demonstrating the crossover from \( p(c) \sim \exp(-Ac^2) \) (dashed lines) at low velocities to \( p(c) \sim \exp(-Ac^{3/2}) \) at high velocities (solid lines). \( \epsilon \) marks the restitution coefficient which we denote by \( \epsilon \) (from [79]).
1.2.4 Boltzmann Equation

The first assumption customarily done in order to theoretically analyze a dilute gas is Boltzmann's molecular chaos approximation. We assume that the pre-collisional velocities are uncorrelated, so that we may write the rates of collisions in terms of single particle velocity distributions. Within this mean-field approximation, the collision rate between particles of velocity \( \vec{v}_1 \) and particles of velocity \( \vec{v}_2 \) is proportional to their relative normal velocity \( v_n \) [defined in Eq. (1.5)] multiplied by the probability of two particles having these velocities \( p(\vec{v}_1)p(\vec{v}_2) \). We assume a homogeneous state with the single particle distribution \( p(\vec{v}) \) depending only on velocity and not on location.

The change with time of \( p(\vec{v}_1, t) \) due to collisions contains a loss term due to collisions with particles with any other velocity \( \vec{v}_2 \) impacting at any direction \( \hat{n} \) [defined in Eq. (1.4)], and a gain term due to collisions of particles with pre-collisional velocities \( \vec{v}''_1 \) and \( \vec{v}''_2 \) and impact direction \( \hat{n} \) yielding \( \vec{v}_1 \) and \( \vec{v}_2 \) as post-collisional velocities. These two terms are brought together into the Boltzmann Equation for a dissipative hard sphere gas [78]

\[
\frac{\partial p(\vec{v}_1, t)}{\partial t} \bigg|_{\text{coll}} = g \int d\vec{v}_2 \int d\hat{n} v_n \Theta(v_n) \left[ \frac{1}{\epsilon^2} p(\vec{v}''_1)p(\vec{v}''_2) - p(\vec{v}_1)p(\vec{v}_2) \right].
\]

(1.9)

The step function \( \Theta(v_n) \) assures collisions only between particles approaching one another \( (v_n > 0) \) and prohibits collisions between particles moving away one from the other \( (v_n < 0) \). \( g \) is a proportionality constant depending on the volume fraction, spatial dimension and particle diameter. One \( \epsilon \) factor in the gain term comes from changing \( v''_n \) with \( v_n \) in the collision rate, and the other from changing from \( d\vec{v}_2'' \) to \( d\vec{v}_2 \) as the integration variables [78]. Eq. (1.9) accounts only for the inelastic collisions between particles. For driven systems an additional term should be included to describe driving.

Particle velocities are typically scaled by the (momentary in case of cooling) root mean square velocity in the system \( (\sigma^2 \equiv \langle v^2 \rangle) \), \( c \equiv v/\sigma \) and a scaling solution is assumed,

\[
p(\vec{v}) \sim \tilde{p}\left(\frac{v}{\sigma}\right)
\]

(1.10)

Corrections to the equilibrium Gaussian distribution may be approximated by an expansion in Sonine polynomials. These are used to obtain corrections to the velocity moments which are perturbative in the inelasticity [78]. Moreover, the distribution’s high energy tail may be obtained by asymptotic expansions of the Boltzmann Equation. These yield \( \tilde{p}(c) \sim \exp(-Ac) \) for the cooling state [80], and \( \tilde{p}(c) \sim \exp(-Ac^{3/2}) \) for the heated case [78].
1.2.5 Maxwell Model

An alternative to approximately solving the Boltzmann Equation for the inelastic hard sphere gas is to make the assumption of Maxwell molecules, known also as the Maxwell model [81]. In the gas described by Eq. (1.9) the collision rate between any two particles linearly depends on their normal relative velocity. The Maxwell model takes this collision rate as uniform for all particles in the system, yielding the following Boltzmann Equation,

\[
\frac{\partial p(\vec{v}, t)}{\partial t} \bigg|_{\text{coll}} = g \int d\vec{v_2} \int d\hat{n} \left[ \frac{1}{\epsilon} \frac{p(\vec{v_1})p(\vec{v_2}) - p(\vec{v_1})p(\vec{v_2})}{\epsilon} \right].
\]

(1.11)

In [82] the Maxwell model for inelastically colliding particles has been investigated assuming a constant collision rate independent of the typical velocities in the systems. In [83] the uniform collision rate \( g \) was taken to be linearly proportional to the root mean square velocity in the system \( \sigma \). Both versions of the model yield the same form for the steady state energy distribution for a driven system, but different distributions for a cooling system.

The one-dimensional model was first investigated in [82] for the cooling as well as for the heated case. Recursive relations for all moments of the velocity distribution were obtained. In [83] the model was investigated in arbitrary dimension, and the high velocity tail of the distribution was found to decay as a power law for cooling systems. These power law tails are in odds with the stretched exponential tails found in experiments and simulations of granular gases. Ref. [84] showed that velocity correlations strongly affect the high velocity tail, by demonstrating that placing the Maxwell model on a one-dimensional lattice changes the tail from power law to stretched exponential. Ref. [85] then demonstrated how the collision rate determines the high velocity tail by showing that in the Maxwell model the tail is algebraic for the cooling case and exponential for the driven case.

In Section 4 we present an even simpler model for driven dissipative dynamics, and use it in order to investigate energy distributions, fluctuation dissipation relations (in Section 6) and effective temperatures (in Section 7).
1.3 Fluctuation-Dissipation Relations and Effective Temperatures

Granular systems supply useful realizations for investigating the statistical mechanics of jammed and of dissipative systems. An overview of this research was given in the first two sections of this Introduction. Another issue this Thesis deals with is fluctuation-dissipation relations about non-equilibrium steady states. These are relevant to several granular systems, but also to a broader class of non-equilibrium systems. The last section of the Introduction describes recent work on fluctuation-dissipation relations and definitions of effective temperatures far from thermodynamic equilibrium.

1.3.1 Equilibrium Fluctuation-Dissipation Theorem

The fluctuation-dissipation theorem [86] connects two types of small deviations from thermodynamic equilibrium: spontaneous fluctuations around equilibrium, and responses to external forces taking the system away from equilibrium. We shall concentrate on time dependent measurements, however these are equivalent to frequency dependent measurements, and are similar in principle to spatially dependent measurements.

In time domain, fluctuations in an observable $x$ are characterized by its two-time autocorrelation function,

$$ C^x(t) \equiv \langle x(t')x(t' + t) \rangle - \langle x \rangle^2, \quad (1.12) $$

which in a steady state depends only on the time separation $t$ between the two measurement times $t'$ and $t' + t$. The delayed response function is defined with respect to a sudden change in the field $F$ conjugate to $x$ (so that the Hamiltonian has the form $\mathcal{H} = \mathcal{H}_0 - xF$),

$$ R^x(t) \equiv \frac{\partial \langle x(t' + t) \rangle}{\partial F(t')} \quad (1.13) $$

That is, at time $t'$ the external field is changed to a new value and $R$ then probes the effect of the initial field (before $t'$) on the subsequent behavior of the observable $x$. In a steady state this similarly depends only on the time $t$ elapsed since changing the field $F$ at time $t'$ and measuring the observable $x$ at time $t' + t$.

In equilibrium, the correlation and response functions may be shown (see e.g. [87]) to be related by the fluctuation-dissipation theorem,

$$ C^x(t) = R^x(t)T, \quad (1.14) $$
where $T$ is the system’s temperature.

A more common form of this theorem is obtained by differentiating Eq. (1.14) with respect to time. Then, instead of using the response function $R^x(t)$ to a step function change in the field $F$, it is customary to define the generalized susceptibility $\chi^x(t)$ with respect to an impulsive change in the external field. The response to a general time dependent field $F(t)$ is then given by,

$$x(t) = \langle x \rangle + \int_{-\infty}^{t} dt' \chi^x(t - t') F(t'),$$

and Eq. (1.14) implies (see [87]) that

$$\frac{dC^x(t)}{dt} = -\chi^x(t)T.$$  

The fluctuation-dissipation theorem [Eq. (1.14)] consists of several aspects:

(1) Correlation and response functions of an observable have the same spatial and temporal (or frequency) dependence.

(2) The ratio of correlation to response is equal for every observable and any measurement performed on the system.

(3) The ratio of correlation to response equals the system’s temperature.

The fluctuation-dissipation theorem holds as long as the deviations entering the correlation and response functions are small, and more important, as long as these deviations are about thermodynamic equilibrium. Below we shall describe fluctuation-dissipation measurements about steady states which are far from equilibrium, but will restrict ourselves to small deviations from these states. As long as deviations from the steady state are small one expects to find linear relations between correlation and response. However, as the fluctuation-dissipation theorem relies on the system being in thermodynamic equilibrium, neither aspects (1) nor (2) above are proven, and aspect (3) may not even be considered since a non-equilibrium system does not have a well defined temperature away from equilibrium. One should, therefore, test: (1) whether correlation and response functions of a single observable have the same temporal (or spatial) dependence, and (2) whether correlations and response functions for different observables or different measurements on the same system yield the same ratio. In Section 6 we address these issues on our simple stochastic model.

Even if neither aspects (1) or (2) of the fluctuation-dissipation theorem hold, one may use the ratio of correlation to response to define an effective temperature scaling fluctuations in the system [88, 89],

$$T^x_{\text{f}}(t) \equiv \frac{C^x(t)}{R_x(t)}.$$  

(1.17)
This fluctuation temperature generally depends on the observable \( x \) (or on the measurement performed) and on the time scale \( t \) of the measurement.

### 1.3.2 Effective Temperatures

When building a statistical mechanics analogy for non-equilibrium systems one of the most important goals is to try and identify analogs of temperature. An effective temperature for a non-equilibrium system is an intensive quantity which measures some energy scale in the system in a manner that yields the temperature in equilibrium. Several methods have been suggested for defining effective temperatures in systems far from thermodynamic equilibrium. Here we give their definitions and in the following section we shall give an overview of recent results testing fluctuation-dissipation relations and attempting to relate the different effective temperatures in various non-equilibrium systems.

- In granular gases and liquids it is customary to make an analogy with energy equipartition in ideal gases at equilibrium, and define the \textit{granular temperature} \( T_G \) as the average kinetic energy per particle [Eq. (1.1)].

- In a similar manner to the definition of compactivity for jammed systems, one may alternatively define an \textit{entropic temperature} as:

\[
T_S \equiv \left( \frac{dS}{dE} \right)^{-1}, \tag{1.18}
\]

where the entropy is defined from the energy distribution \( p(e) \) found in the system,

\[
S \equiv - \int p(e) \ln p(e) de. \tag{1.19}
\]

(Note that \( T_S \) differs from the compactivity, \( X \equiv (dS/dV)^{-1} \).)

- Fluctuations have been used to define effective temperatures in different manners. One is by fluctuation-dissipation relations [Eq. (1.17)]. Further definitions for effective temperatures have been proposed based solely on fluctuations. These tacitly assumed the corresponding response function is known and defined effective temperatures from volume fluctuations in granular compaction experiments [17] or from energy fluctuations in simulations of sheared dense foam [90]. The fluctuation theorem [91, 92, 93] was used in order to infer an effective temperature from energy fluctuations in experiments of a vibrated dilute granular gas [94].

- Recent extension of the virial theorem, expressing the equilibrium temperature from ensemble averages of particle locations (without any kinetic information) [95, 96] lead to the definition of a \textit{configurational temperature} \( T_C \) as another measure of effective temperature for non-equilibrium systems [97, 98].
1.3.3 Fluctuation-Dissipation Relations Far From Equilibrium

Most of the study of fluctuation-dissipation relations far from equilibrium is by experiments or numerical simulations.

Simulations of granular gases [99] have been used to measure fluctuation-dissipation relations in two different measurements (see Fig. 1.11). The emphasis was not on the temporal dependence, however both measurements yielded a fluctuation temperature coinciding up to numerical accuracy with the granular temperature. In mixtures, where each species possesses a different granular temperature the fluctuation temperature of each species was found to coincide with its granular temperature [100].

Figure 1.11: Parametric plots of response vs. correlation for two different fluctuation-dissipation measurements in simulations of driven granular gases. Ratios of correlations to response fit a slope corresponding to the granular temperature (dashed lines) and are significantly different the bath temperature (solid line). $r$ marks the restitution coefficient which we denote by $\epsilon$ (from [99]).

Kinetic theory indicates that in a granular gas the fluctuation temperature is expected to be slightly larger than the granular temperature. This result has first been obtained for a cooling gas [101], where a significant effect is due to fluctuation dissipation relations measuring characteristics of the system’s history, when typical energies were larger due to the cooling. Subsequent calculations [74] predicted small deviations between the fluctuation temperature and the granular temperature even in heated gases. These calculations did not attempt to calculate time dependence of correlation and response functions, and to test aspect (1) of the fluctuation-dissipation theorem. Moreover, these calculations are approximate and we suspect that deviations between the fluctuation temperature and the granular temperature in them may be an artifact of the accuracy of the approximations made.

Experimental measurements of fluctuation-dissipation relations in granular sys-
tem have been undertaken only for denser systems. Some dependence on measurement frequency has been found for a vertically shaken system probed by a torsion oscillator [102]. Variance between the fluctuation temperature and the granular temperature has been found in a Couette geometry shear cell [103].

Numerical simulations of a model for granular compaction [104] calculated the fluctuation temperature for long measurement times and showed it coincided with the compactivity. A time-dependent measurement of another compaction model [105] showed that the fluctuation temperature depends on the measurement time and coincides with the bath temperature for short times and with the compactivity for long times.

Numerical simulations of sheared foam indicate that time-independent fluctuation temperatures of different observables all coincide with the entropic temperature [90], and that the fluctuation temperature depends on the measurement time [106]. Fluctuation-dissipation relations have been measured experimentally using passive probe particles in active bacterial baths. The resulting fluctuation temperature was shown to be larger than the environment temperature [107], and to depend on the measurement frequency [108].

Experiments of single particles in an optical trap showed that the fluctuation temperature differs from the environment temperature [109, 110]. Time dependence of the fluctuation temperature has also been observed in numerical simulations of models for molecular motors [111, 112]. The fluctuation temperature in these simulations deviated from the environment temperature [113]. This deviation has been related to the rate of energy flow into the system [114].

Fluctuation-dissipation relations are also investigated in glasses [89, 5]. Simulations [23] indicate that the fluctuation temperature depends on the measurement time and coincides with the bath temperature for short times and with the entropic temperature for long times. Fluctuation temperatures have been shown in simulations to be equal for fluctuations of many different physical quantities [115]. Dependence of the fluctuation temperature on observable was found in simulations of another model [116]. Theoretical solutions of a glass model showed that the fluctuation temperature depends on measurement time and on the observable used [117]. Experiments show that the fluctuation temperature depends on the measurement time scale [118].

To conclude, several systems exhibit significant differences between different effective temperatures, while in others these seem to coincide. Moreover, some systems exhibit temporal dependence of fluctuation dissipation relations, however in most cases this may be explained to be a result of fast modes thermalizing with the environment, while slower modes are more strongly coupled to the driving mechanism (or remember the system’s history in glasses). All these results for time dependent
fluctuation-dissipation relations are experimental and numerical, and to our knowledge there are no theoretical results for time dependent correlation and response function in systems far from equilibrium.

In Section 6 we exactly calculate time dependent fluctuation-dissipation relations in our driven dissipative model and in Sections 4, 6, and 7 demonstrate variance as well as coincidence between different effective temperatures.
2 Outline

The research described in this Thesis is presented in the following sections. Each section consists of a paper written throughout the course of the research. The papers are given by the chronological order in which the work described in them was performed. Before presenting the papers, we provide here a short overview of them and of the relations between them within the larger context of this Thesis.

The research began with the question of compaction and segregation in static granular packings. The first paper (Section 3 [119]) presents a geometrical model for describing granular packings in terms of local characteristics, and a mechanical model for incorporating friction into these packings. It then combines these models with Edwards’ thermodynamic description [6] to yield predictions about the effect of friction on compaction and about segregation due to differences in friction between grains.

The focus then shifted to dynamical systems, and in particular to granular gases, where dissipation is prominent and jamming is unimportant. We investigated granular gases, but also driven dissipative system in general, with a simple stochastic model we introduced. The second paper (Section 4 [120]) presents the model, demonstrates its similarities to granular gases, and provides several exact calculations that can be performed on it. The steady state energy distribution is calculated. Then, a fluctuation-dissipation measurement on it is constructed and used in order to demonstrate that the fluctuation temperature generally differs from the granular temperature.

When further investigating our stochastic dissipative model we realized it may be generalized to include general densities of states for the particles, and not only the uniform density of states characteristic of the simple model of [120]. We then encountered a paper [121], which investigated the non-dissipative limit of our model with a non-uniform density of states, and claimed for observing non-equilibrium behavior. The third paper incorporated in this Thesis (Section 5 [122]) identifies that the model investigated in [121] is in fact in equilibrium, and emphasizes the true non-equilibrium nature of our dissipative model [120].

Following the calculation of static fluctuation-dissipation relations in driven dissipative systems in [120] we turned to time dependent relations in non-equilibrium systems in general. We exactly solved the time dependent correlation and response functions in our stochastic model, and present these results in the fourth paper (Section 6 [123]). We demonstrate that correlation and response functions of a single observable generally have different temporal dependences, and that fluctuation temperatures obtained in different measurements generally differ. These results are given for our driven dissipative model, and we provide some explanations for their
origin and speculate on their generality to other non-equilibrium systems.

The last paper in this Thesis (Section 7 [124]) includes a detailed description of the model and results presented in [120], together with further results on the model. We calculate the temporal evolution of the system’s energy distribution to its steady state, and investigate several other effective temperatures that are customarily defined on non-equilibrium systems: We investigate the hierarchy of effective temperatures defined from moments of the non-Boltzmann energy distribution, and relate them to the configurational temperature [95, 96, 97, 98]. We calculate the entropic temperature for our continuous energy model as well as for its discrete energy version.

The work presented in these five papers uses simple models in order to investigate the statistical mechanics of systems far from equilibrium, either due to strong jamming [119], or due to dissipation [120, 123, 124]. In all cases we provide exact theoretical results for the specific models under consideration, and use these results in order to deepen our understanding of these general classes of non-equilibrium systems. In the first paper [119] a thermodynamic-like description [6] is assumed, and the results derived from it may be used to test its validity experimentally. In the subsequent papers statistical properties are deduced from the dynamical rules governing the temporal evolution, and thermodynamic-like descriptions are tested. In three cases [120, 123, 124] non-equilibrium phenomena are identified, while in another [122] an alleged non-equilibrium model is found to behave as in equilibrium.
3 Role of Friction in Compaction and Segregation of Granular Materials

Yair Srebro and Dov Levine

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We investigate the role of friction in compaction and segregation of granular materials by combining Edwards thermodynamic hypothesis with a simple mechanical model and mean-field based geometrical calculations. Systems of single species with large friction coefficients are found to compact less. Binary mixtures of grains differing in frictional properties are found to segregate at high compactivities, contrary to granular mixtures differing in size, which segregate at low compactivities. A phase diagram for segregation versus friction coefficients of the two species is generated. Finally, the characteristics of segregation are related directly to the volume fraction without the explicit use of the yet unclear notion of compactivity.
Role of friction in compaction and segregation of granular materials

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We investigate the role of friction in compaction and segregation of granular materials by combining Edwards’ thermodynamic hypothesis with a simple mechanical model and mean-field based geometrical calculations. Systems of single species with large friction coefficients are found to compact less. Binary mixtures of grains differing in frictional properties are found to segregate at high compactivities, contrary to granular mixtures differing in size, which segregate at low compactivities. A phase diagram for segregation versus friction coefficients of the two species is generated. Finally, the characteristics of segregation are related directly to the volume fraction without the explicit use of the yet unclear notion of compactivity.

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1. INTRODUCTION

Friction plays a significant role in packings of grains in a granular material, however due to the nonequilibrium nature of such materials their macroscopic properties do not trivially result from the microscopic physics of their constituents. Compaction and segregation are two macroscopic phenomena which occur in granular materials undergoing series of extensive operations [1] (operations of a statistical nature rather than “Maxwell-demon” operations which act on single grains). Packings of identical hard spheres compacted by extensive operations reach a state of random close packing (RCP), rather than an ordered crystalline packing, which has a higher volume fraction (defined as the ratio of the sum of grain volumes to the volume the system occupies). Understanding the process leading to RCP, its geometrical properties, and whether higher volume fractions may be obtained is of great importance for physics and for engineering. The tendency towards segregation in granular mixtures comprising grains with various mechanical properties is interesting for the physicist and disturbing for the engineer, for whom a homogeneous mixture is often an industrial need (for a review see Ref. [2]).

Segregation in granular materials has received much attention in recent years (for a review see Ref. [3]). The phenomenon is observed for grains varying in size [4], shape [5], friction coefficient [6], and density [7]. Segregation occurs due to vibration [8], tapping [9], rotation [6], pouring [10], and shearing [11]. Experiments are performed for mixtures of many particles of two different species (for a review see Ref. [12]) and with single intruder particles in systems of a single species [13]. Existing theoretical modeling of segregation due to rotation [6,14,15] and pouring [10,16] is based on kinetic phenomena: segregation is explained as a result of different flow properties of the different species. Monte Carlo simulations of dynamic phenomena in vibrated systems give insight into segregation [17] as well as compaction dynamics of single species systems [18]. Similar dynamic phenomena have been captured analytically in models based on free volume considerations [19–21].

This paper deals with the role of friction in compaction and segregation through the analysis of the static properties of granular materials. The results obtained may be used in order to verify experimentally the validity of the models used for describing these static properties. One such proposal for the description of static granular materials is the analogy to the statistical mechanics of thermodynamic equilibrium proposed by Edwards [1], a more detailed description of which will be given in the following section. The central idea behind it is that even though the system is static and does not move with time within the ensemble of mechanically stable arrangement of the grains, we may assume ergodicity and employ statistical mechanics considerations for the probability of finding the system in any one of its states. This model requires the existence of an analog of temperature, referred to as “compactivity” (other effective temperatures may be defined for jammed granular materials [22,23], and attempts have been done to connect them to the compactivity [24,25]). Recent experimental evidence for reversibility in compaction processes [26] has provided justification for this thermodynamic analogy and have proposed a connection between compactivity and experimentally controllable quantities [27], however the proposal remains controversial.

Edwards’ hypothesis is that an analog of the free energy $F = V - XS$ is minimized, where $V$ is the system volume, $S$ is the entropy, and $X$ is the compactivity. Our expectation that this formalism may predict frictional segregation, as will be shown rigorously and discussed in the following sections, is based on the following argument. Consider two systems of identical grains, such that the friction coefficient of the first system, $\mu_1$, is greater than that of the second system, $\mu_2$. Then $S_1 > S_2$ because every configuration available to the second system may be identified realized to the first system, while the converse is not true. This suggests that under certain circumstances a mixture of grains with different friction coefficients may prefer to segregate in order that its entropy be maximized, competing with the preference of the entropy of mixing to be maximized in the homogeneously mixed state.

The statistical hypothesis has been used to investigate segregation in binary mixtures of species differing in size by mapping them to the Ising model, resulting in segregation below some critical compactivity [1,28]. Recent geometrical calculations have enabled relating the ideas of the statistical proposal to actual densities of granular systems [29].

In Sec. II we use Edwards’ statistical hypothesis together
with a simple mechanical model for the quantitative description of friction in two-dimensional (2D) and 3D single species granular materials. This is combined with simple calculations of Voronoi cell volumes for general coordination numbers resulting in the dependence of volume fraction on friction coefficient and compactivity. Section III uses the mean-field approximation in order to describe segregation in a binary mixture of grains differing in frictional properties. Unlike mixtures of grains differing in size, which may be mapped to the Ising model [1,28], frictional differences between grains result in larger entropy for the rougher grains. Therefore, these systems may not be mapped exactly onto the Ising model and segregation occurs above a critical compactivity and not below it. We then generate a phase diagram for segregation versus friction coefficients of the two species. Finally, the dependence of the results on compactivity is eliminated by using the volume fraction as a measure for compactivity. Section IV concludes with a summary of the results.

II. SINGLE SPECIES

A. Statistical model

We use the statistical mechanics hypothesis proposed by Edwards [1] for the description of jammed granular systems. In this formalism each mechanically stable arrangement of the grains is equivalent to a microscopic state in statistical mechanics, and the total volume the grains occupy plays the role of the energy. The analog of temperature is assumed to exist, is denoted \( X \), and is called the compactivity. We measure \( X \) in units of grain volume so that the analog of the Boltzmann constant is equal to unity and \( X \) is dimensionless. In analogy with the canonical ensemble for thermal systems, the probability for the occurrence of a state with volume \( V \) is assumed to be proportional to \( e^{-V/X} \).

We consider a system of \( N \) identical spherical (in 3D) or circular (in 2D) grains. For every arrangement of the grains, the total volume of the system may be written as the sum over all grains of the Voronoi cell volume around each grain, \( v_i \):

\[
V = \sum_{j=1}^{N} v_j .
\]  

Average volumes of Voronoi cells, calculated neglecting spatial correlations between locations of grains, have been shown to agree with exact calculations [30]. Moreover, correlations have been shown to have a small effect on the dependence of total volume on compactivity [29]. Therefore, we use a mean-field approximation and assume that the volume of every Voronoi cell is evenly distributed between a minimal and a maximal volume, \( v_{\text{min}} \) and \( v_{\text{max}} \). The geometrical and mechanical considerations determining these volumes will be presented in the following section, and at this stage it is only assumed that \( v_{\text{min}} \) and \( v_{\text{max}} \) are uniform for all grains in the system.

Following the analogy with the canonical ensemble, we may calculate the partition function:

\[
Z = \int_{v_{\text{min}}}^{v_{\text{max}}} \cdots \int_{v_{\text{min}}}^{v_{\text{max}}} e^{-\sum v_i /X} dv_1 \cdots dv_N
= \left[ \int_{v_{\text{min}}}^{v_{\text{max}}} e^{-v_i /X} dv_i \right]^N
= \left[ 2X e^{-v_{\text{mid}} /X \sinh (\Delta v /X)} \right]^N .
\]  

where we have introduced the notations \( v_{\text{mid}} = (v_{\text{min}} + v_{\text{max}})/2 \) and \( \Delta v = (v_{\text{max}} - v_{\text{min}})/2 \).

The average volume per grain may easily be derived from the partition function as

\[
\langle v \rangle = \frac{\langle V \rangle}{N} = \frac{1}{N} \frac{\partial \ln Z}{\partial X} = v_{\text{mid}} + X - \Delta v \coth \left( \frac{\Delta v /X}{2} \right) .
\]  

This expression has been derived in Ref. [1], and it is clearly seen that as \( X \to 0 \), \( \langle v \rangle \to v_{\text{max}} \), and as \( X \to \infty \), \( \langle v \rangle \to v_{\text{mid}} \).

This is analogous to thermal systems, where at low temperatures the system is most probable to be found in its ground state, and as the temperature is increased exited states are occupied with increasing probability until the limit of infinite temperature, where all states are occupied with an equal probability, and the system’s energy is the average energy of all these states.

B. Mechanical model

We would now like to introduce mechanical and geometrical considerations to estimate \( v_{\text{min}} \) and \( v_{\text{max}} \), which must be known in order to evaluate the expression in Eq. (3).

The minimal volume is achieved for hexagonal packing in 2D and face centered cubic or hexagonal close packing in 3D. The corresponding Voronoi cell volumes are \( v_{\text{min}}^{2D} = \sqrt{3}/2 r^2 \) and \( v_{\text{min}}^{3D} = \sqrt{3}/2 r^3 \), where \( r \) is the grain radius. The resulting volume fractions are \( \Phi_{\text{min}}^{2D} = \pi r^2 /4 \sqrt{3} = \pi / \sqrt{12} \approx 0.91 \) and \( \Phi_{\text{min}}^{3D} = (4/3) \pi r^3 / \sqrt{3} = \pi / \sqrt{18} = 0.74 \).

Although purely geometric considerations determine \( v_{\text{min}} \), the frictional forces between the grains manifest themselves in \( v_{\text{max}} \). The idea behind this is that friction at grain contacts allows for the formation of arcs and for a gradual decrease in the number of contacts per grain [31], which in turn increases the volume of the Voronoi cell around every grain.

In order to estimate the effect of friction in granular materials we will consider “toy systems” consisting of a small number of grains (three in 2D and four in 3D), calculate the effect of friction there, and use the result in order to obtain an approximate prediction for the dependence of \( v_{\text{max}} \) on the friction coefficient \( \mu \). This, in turn, may be inserted together with \( v_{\text{min}} \) into Eq. (3) in order to obtain an approximate expression for the dependence of the total volume of the system, and hence of the volume fraction, on \( \mu \). The results will depend on the compactivity, whose physical significance still requires elucidation. However, even without understanding its significance, a few predictions for experimental results may be drawn from the model presented here.
The 2D toy system consists of two grains lying on top of a horizontal plane and a third grain lying on top of these (see Fig. 1). All grains and the horizontal plane are assumed to be hard, and frictional forces with an equal coefficient of friction, $\mu$, act at all four contacts. A uniform gravitational force acts downwards on all grains. The condition for mechanical equilibrium is that forces and torques acting on all grains vanish. It can easily be seen that this is satisfied whenever

$$\sin \theta = \frac{1}{1 + \cos \theta} \leq \mu,$$

(4)

where $\theta$ is half the angle between contacts of the top grain with the bottom grains (see Fig. 1). For $\mu \geq 1$ Eq. (4) is always satisfied, and any state with $\pi/6 \leq \theta \leq \pi/2$ is mechanically stable. For smaller values of $\mu$ only states with $\theta = \theta_{\text{max}}$ are mechanically stable, where $\theta_{\text{max}}$ is determined from

$$\sin \theta_{\text{max}} = \frac{1}{1 + \cos \theta_{\text{max}}} = \mu.$$

(5)

If $\mu \leq (2 + \sqrt{3})^{-1} \approx 0.3$, the frictional forces cannot hold the top grain on top of the two bottom ones, and these simple considerations may not be used in order to determine $v_{\text{max}}$. In this case the volume of the Voronoi cell around every grain is set in the model to $v_{\text{min}}$. Substituting this into Eq. (3) yields $v(v) = v_{\text{min}}$, which corresponds to the maximal volume fraction quoted earlier, and hence, to crystallization. Therefore, this model may not be used for frictionless systems, since it predicts crystallization at every compactivity.

In 3D we consider three grains in an equilateral triangle lying on top of a horizontal plane and a fourth grain lying on top of them [see Fig. 2(a)]. As in the 2D case, all grains and the horizontal plane are assumed to be hard, frictional forces with an equal coefficient of friction, $\mu$, act at all six contacts, and a uniform gravitational force acts downwards on all grains. Again, the condition for mechanical equilibrium can easily be seen to be given by Eq. (4), however now $\theta$ is the angle between the vertical direction and the line connecting the top grain with any one of the bottom grains [see Fig. 2(b)]. As in 2D, for small values of $\mu$, and specifically for frictionless systems, these considerations cannot be used to estimate the volume fraction, since this model predicts crystallization regardless of compactivity, while actual 3D granular systems do not fully crystallize, but fall into a RCP state.

We would now like to use the results of these simple considerations to evaluate the maximal volume of the Voronoi cell around every grain in a granular material. In such systems frictional forces allow the existence of large angles between contacts of every grain with its surrounding grains. We assume that the maximal angle between contacts in a granular material depends on the friction coefficient in a similar manner to its dependence in the toy systems. In order to calculate the Voronoi cell volume around every grain we first assume that the grain is far from the system’s boundaries, since we are seeking a description of average bulk properties of the packing. Moreover, we assume that the angles between adjacent contacts of this grain with its surrounding grains are uniform and are given by $\theta$ as in the toy systems. Note that since we allow $\theta$ to vary continuously, it may not necessarily be physically possible to build these packings, even locally. In 2D it is possible only if $\theta$ is an integer fraction of $\pi$, while in 3D it is possible only if the coordination number is 4 or 6, which correspond to $\theta \approx 1.23$ and $\theta \approx 0.96$, respectively. The general expression for the Voronoi cell volume is (see the Appendix)

$$v(\theta) = \begin{cases} \frac{\pi r^2 \tan \theta}{\theta} & (2D) \\ \frac{\pi r^2 \sqrt{3} \sin \theta \tan \theta}{(1 + 3 \cos^2 \theta)^{\tan^{-1} \left( \sqrt{\frac{3}{4} \sin^2 \tan^{-1} \left( \frac{\sqrt{3}}{2} \frac{\alpha}{4} \right)} \right)}} & (3D) \end{cases},$$

(6)

where $\alpha = \cos^{-1}[(3 \cos 2\theta + 1)/4]$ is the angle between two adjacent contacts (see Fig. 6).

C. Results

We combine Eqs. (5) and (6) to get the dependence of $v_{\text{max}}$ on $\mu$. The resulting volume fraction $\Phi$ is now calculated using Eq. (3) and plotted in Fig. 3 as a function of friction coefficient and compactivity. Compaction depends on friction only in the region $0.3 \leq \mu < 1$, where the mechanical model used here is relevant. As $\mu \to 0$, $\Phi$ approaches its maximal value (determined from geometrical considerations), and as $\mu \to \infty$, it approaches a value larger than its minimal value (determined from mechanical considerations), since all possible volumes between the minimal
III. TWO SPECIES: SEGREGATION

A. Mean-field model

We would now like to describe a system consisting of two species of grains differing only in frictional properties. The central question we wish to address in such systems is whether the two species mix homogeneously, or segregate into separate domains. For a system of two species denoted A and B, we shall assume we know the compactivity X and the friction coefficients \( \mu_{AA}, \mu_{AB}, \) and \( \mu_{BB} \) between two grains of type A, between a grain of type A and a grain of type B, and between two grains of type B, respectively. The considerations presented in the preceding section may be used in order to calculate the maximal volumes \( v_{AA} = v_{\text{max}}(\mu_{AA}), \) \( v_{AB} = v_{\text{max}}(\mu_{AB}), \) and \( v_{BB} = v_{\text{max}}(\mu_{BB}) \) of the Voronoi cells around an A grain surrounded by A grains, an A grain surrounded by B grains (or a B grain surrounded by A grains), and a B grain surrounded by B grains, respectively. We describe a monodisperse system, hence the corresponding minimal volumes are identical for all types of grains and are denoted here by \( v_{\text{min}}. \)

We would now like to write the partition function \( Z \) for two species and to derive from it the analog of free energy, \( Y = -X \ln(Z) \), as a function of the concentration \( f = N_A/N \), where \( N \) is the total number of grains and \( N_A \) is the number of A grains. The constraint on the total number of grains of each species in the system causes us to view \( f \) as a local concentration which varies throughout the system. As in the mean-field description of a binary alloy, which is equivalent to the Ising model, a single minimum of \( Y(f) \) means the two species tend to get mixed homogeneously at a concentration equal to the global concentration determined from the number of grains of each species. The existence of two minima of \( Y(f) \) means that the system tends to separate into domains with two different concentrations. The percentage of the system with each of these two minimizing local concentrations is determined from the global concentration according to the Maxwell construction (see, e.g., Ref. [32]).

The number of A grains is \( N_A = fN \) and the number of B grains is \( N_B = (1-f)N \). In the mean-field approximation the number of A-A contacts is \( N_{AA} = f^2N z/N_z \), the number of A-B contacts is \( N_{AB} = f(1-f)N_z \), and the number of B-B contacts is \( N_{BB} = (1-f)^2N_z/N_z \), where \( z \) is the average number of neighbors per grain, or the average coordination number, which is assumed to be uniform for all types of grains. The contribution of every contact to the total volume is limited according to its type between 2\( v_{\text{max}}/z \) and 2\( v_{\text{max}}/z \), with \( i \) and \( j \) denoting A or B for the types of the two grains in contact. The partition function is

\[
Z = \frac{N!}{(fN)!(1-f)N!} \left( \int_{v_{\text{min}}}^{v_{\text{max}}} e^{-vX} dv \right)^{fN} \times \left( \int_{v_{\text{max}}}^{v_{\text{max}}} e^{-(1-f)X} dv \right)^{(1-f)N}.
\]

(7)

Not only does \( Z \) not depend on \( z \) when \( z \) is assumed to be uniform for all types of grains, it can easily be seen that when different values of \( z \) are assigned to the different types of grains, the same expression is obtained. This is an important result in the analogy between the configurational statistical mechanics of a granular mixture and the Ising model or a binary alloy. In the Ising model spins are arranged on an ordered lattice, all spins have the same number of nearest neighbors, \( z \), and the total energy is determined by the states of the spins on all the lattice sites. In a granular system, on the other hand, the disordered spatial configuration determines the volume of the system and every type of grain may have a different number of nearest neighbors, or a different coordination number, which is related to the volume of the Voronoi cell around it, or to the volume it occupies. Due to the different friction coefficients, \( z \) varies between the different types of grains, as is indicated in Ref. [31] where the friction dependence of the average number of contacts is investigated. The number of contacts of every type and the contribution of every contact to the total volume depend inversely on \( z \), and since only their product enters the total volume of the system, the resulting partition function does not depend on the values of \( z \) for the different types of grains [34]. We will soon see where such physical differences between the granular mixture and the Ising model do affect the resulting behavior of the system.

Using the partition function we now evaluate the free energy:

\[
Y = -X \ln(Z).
\]
where we have used the notations \( R_{ij}(X) = \ln(e^{v_{min}/X} - e^{-v_{min}/X}) \) and \( R(X) = [R_{AA} + R_{BB}] / 2 - R_{AB} \), and the Stirling formula has been used to evaluate \( \ln N! \) for large \( N \). Defining \( \Delta v_{ij} = v_{ij} - v_{min} \), we see that

\[
R(X) = \ln \left( \frac{\sqrt{1 - e^{-\Delta v_{AA}/X}}(1 - e^{-\Delta v_{BB}/X})}{(1 - e^{-\Delta v_{AB}/X})} \right).
\]

We would now like to minimize \( Y \) for given overall compositions \( N_{A} \) and \( N_{B} \) and for a given compactivity \( X \). Equation (8) has been derived under the mean-field approximation, therefore it describes the free energy of a region with uniform concentration, \( f \). Formally we should now define the local concentration as a spatially dependent coarse grained function \( f(\mathbf{r}) \) and the free energy as its functional, \( Y[f(\mathbf{r})] = \int f(\mathbf{r})d\mathbf{r} \), and require that \( f(\mathbf{r}) \) minimize \( Y \) under the constraint that \( \langle 1/V \rangle f(\mathbf{r})d\mathbf{r} = N_{A}/N \). The spatial integral over all the system of the last three terms in Eq. (8), which are linear in \( f \), is independent of the function \( f(\mathbf{r}) \). Therefore these terms do not contribute to the minimization of \( Y \), and we need only consider the contribution of the first three terms to \( Y \). We thus obtain an expression similar to the mean-field free energy of an Ising model or of a binary alloy [33].

The “equilibrium” concentration is now determined from

\[
\frac{\partial Y}{\partial f} = XN \ln \left( \frac{f}{1-f} \right) + 2(1-2f)R(X) = 0,
\]

which is equivalent to

\[
2f - 1 = \tanh[R(X)/(2f - 1)].
\]

For \( R(X) < 1 \) the only solution to this equation is \( f = 0.5 \), which corresponds to mixing, while for \( R(X) > 1 \) two different solutions exist and the systems segregates into regions with these two minimizing concentrations. Therefore, the condition for segregation is that \( R(X) > 1 \), where \( R(X) \) is given by Eq. (9).

Since \( R(X=0) = 0 \), no segregation occurs in the limit of low compactivities. Contrast this to the behavior of the Ising model and binary alloys, where phase separation exists at low temperatures, and specifically in the limit of zero temperature [33]. Here there is a minimal critical compactivity, above which segregation occurs, rather than the maximal critical temperature, below which phase separation occurs in the Ising model.

The basic difference between the model presented here for granular materials and the Ising model is as follows. In the Ising model the energy of the system is determined only from its topological state, namely, once the topology of which element is the nearest neighbor of which other elements is specified, the total energy of the system is determined. In the mean-field approximation these topological states are described by the concentration, hence the energy depends only on it. In the granular model presented here a topological state, specifying the types of contacts around every grain, allows for a range of volumes for the Voronoi cells around every grain and therefore for a range of values for the overall volume of the system. As in the Ising model, the topological state is specified by the concentration in the mean-field approximation, however the total volume of the system depends both on the concentration and on the compactivity \( X \).

The reason for segregation only at high compactivities may be understood to stem from the range of possible volumes in the following way. All the Voronoi cells around the grains have the same minimal volume \( v_{min} \) determined from geometrical considerations. The probability for finding a Voronoi cell in an exited state with a larger volume, \( v > v_{min} \), decays as \( e^{-v/v_{min}} \). Therefore at the limit \( X=0 \) all Voronoi cells are expected to be found in their ground state, namely, to have a volume \( v = v_{min} \), independent of the friction of the gran, which only determines the maximal Voronoi cell volume, \( v_{max}(\mu) \). Hence, at low compactivities the differences between grains vanish and the two species mix homogeneously. Segregation may occur only at high compactivities, where exited states, which exist due to friction, have a greater “thermodynamic” weight. This occurrence of segregation at high compactivities rather than at low compactivities may explain the frequent appearance of segregation in granular systems, which typically have significant compactivities, since \( X=0 \) corresponds to a crystalline packing.

Since for \( R>0 \), \( R(X) \) is a monotonic function [as can be seen from Eq. (9)] which begins at zero, and since the condition for segregation is that \( R>1 \), there is only a minimal critical compactivity for segregation (and no upper bound above which there is no segregation) so we may determine whether segregation is possible by investigating the high compactivity limit of \( R(X) \). This limit may easily be evaluated to be equal to

\[
R_{c} = R(X \to \infty) = \ln \left( \frac{\Delta v_{AA} \Delta v_{BB}}{\Delta v_{AB}} \right),
\]

and the condition for the existence of a critical compactivity for segregation for given values of the friction coefficients is that \( R_{c} > 1 \).

This condition may be understood in the following way. The free energy, \( Y = V - XS \), includes a volume \( V \) term and an entropy \( S \) term. At high compactivities the entropy dominates and minimizing the free energy is equivalent to maximizing the entropy. The entropy includes two factors, a “combinatorial entropy” related to the topological state of the system, namely, which grain is in contact with which other grains, and a “geometric” entropy related to the variety of volumes the Voronoi cell around every grain may have within one specific topological state (this is not the case for the Ising model, where the topological state determines the state of the system, and the total entropy includes only the combinatoric entropy, or the entropy of mixing [32]).

The entropy’s dependence on concentration at the limit \( X \to \infty \) is given by
The first two terms are the combinatoric entropy or entropy of mixing, while the last term is the geometric entropy. The geometric entropy plays a role similar to the role of the energy in the Ising model, as can be seen in the expression for the free energy in the Ising model \[ F = N/kT [f \ln f + (1-f) \ln (1-f)] - 2f(1-f)JZ, \] where \( k \) is the Boltzmann constant, \( T \) is temperature, \( J \) is the interaction energy between neighboring spins, and \( Z \) is the number of nearest neighbors per site. Ising systems exhibit no segregation in intermediate values of \( J \), with segregation occurring at \( J > J_c \), which may now be seen to be completely analogous to the condition \( R_s > 1 \) for segregation in granular binary mixtures at high compactivities.

B. Phase diagram

We would now like to generate a phase diagram for segregation and to check for which values of \( \mu_{AA}, \mu_{BB}, \) and \( \mu_{AB} \) segregation is possible. From mechanical considerations we would expect \( \mu_{AB} \) to lie between \( \mu_{AA} \) and \( \mu_{BB} \). Since \( u_{mix}(\mu) \) is monotonic, \( \Delta u(\mu) \) is monotonic as well, and \( \Delta v_{AB} \) is bounded between \( \Delta v_{AA} \) and \( \Delta v_{BB} \). First we notice that if \( \mu_{AB} = \min(\mu_{AA}, \mu_{BB}), \) then no segregation is expected. For given values of \( \mu_{AA} \) and \( \mu_{BB}, \) \( R_s \) varies monotonically with \( \mu_{AB} \), and a necessary condition for segregation in intermediate values of \( \mu_{AB} \) is that there is segregation at \( \mu_{AB} = \min(\mu_{AA}, \mu_{BB}) \). In this case

\[ R_s = \ln \left[ \frac{\max(\Delta v_{AA}, \Delta v_{BB})}{\min(\Delta v_{AA}, \Delta v_{BB})} \right]. \]

Figure 4 displays the regions in the \( \mu_{AA} - \mu_{BB} \) plane where \( R_s > 1 \), and in which segregation may occur. When \( R_s < 1 \) the two species are always expected to be mixed. When \( R_s > 1 \) segregation occurs at the limits of \( \mu_{AB} = \min(\mu_{AA}, \mu_{BB}) \) and \( X = \infty \). Segregation also occurs at larger values of \( \mu_{AB} \) and at finite values of \( X \). The existence of segregation may be verified for the general case by checking whether the expression for \( R(X) \) in Eq. (9) is greater than one. Here we only demonstrate our model results for the limits described above. As will be shown in the following section, in some cases segregation occurs for every compactivity. The phase diagram shows that mixtures of grains with close friction coefficients will mix and what difference in friction coefficients is needed in order to achieve segregation.

Within the framework of the mechanical model used here, frictional forces are relevant only for \( 0.3 < \mu < 1 \). If \( \mu_{AA} \equiv 0.3 \) or \( \mu_{BB} \equiv 0.3 \) it is questionable whether this mechanical model is valid. If both \( \mu_{AA} \equiv 1 \) and \( \mu_{BB} \equiv 1 \) the friction has an identical effect no matter what the values of \( \mu_{AA} \) and \( \mu_{BB} \) are, therefore the two species mix. If only one of \( \mu_{AA} \) or \( \mu_{BB} \) is greater than one, the maximal volume per grain of the corresponding species is unbounded, while for the second species it is bounded, leading to \( R_s > 1 \), and hence to segregation. These regions are designated as well in Fig. 4.

C. Eliminating the compactivity

Given the mechanical properties of the grains, namely, \( \mu_{AA}, \mu_{BB}, \) and \( \mu_{AB} \), we have employed a mean-field approximation in order to solve the Edwards statistical mechanics description of a granular mixture and to obtain the concentration (or concentrations in case of segregation) the system will be found at as a function of compactivity. Even though there is evidence that it may be controlled experimentally [26,27], the compactivity is still not a measurable quantity, and we would like to reach a description of segregation independent of the notion of compactivity.

For a single species we have seen that the total volume of the system, and correspondingly its volume fraction, depend on compactivity. We shall now derive the compactivity dependence of the volume fraction in a binary mixture, and use it in order to eliminate the compactivity from the description of segregation. This will result in a relation between concentrations and volume fraction, which may be measured experimentally.

The average volume per grain may easily be derived from the partition function [Eq. (7)] as

\[ \langle v \rangle = \frac{\langle V \rangle}{N} = \frac{1}{N} X^2 \frac{\partial \ln Z}{\partial X} = X + \langle f \rangle Q_{AB}(X) + 2f(1-f)Q_{AA}(X) + (1-f)^2Q_{BB}(X), \]

where

\[ Q_{ij}(X) = X^2 \frac{\partial R_{ij}(X)}{\partial X} = \frac{\partial}{\partial X} \left[ \frac{\mu_{ij} e^{-v_{min}/X} - v_{ij} e^{-v_{ij}/X}}{e^{-v_{min}/X} - e^{-v_{ij}/X}} \right]. \]
of friction in 2D and 3D granular materials. For a single species this describes the decrease in volume fraction with increasing friction coefficient and with increasing compactivity. An experimental test of the ideas presented is most easily interpreted for systems differing in friction coefficient but with the same compactivity. It is intriguing to consider whether identical preparation would lead to equal compactivities for systems differing only in friction coefficients.

In addition, the model has been used in order to investigate segregation in binary mixtures of grains differing in frictional properties. Unlike mixtures of grains differing in size, which may be mapped to the Ising model, frictional differences between grains result in larger entropy for the rougher grains; therefore, these systems may not be mapped exactly onto the Ising model, we find that segregation occurs above a critical compactivity and not below it. A phase diagram for segregation versus friction coefficients of the two species, which may be tested experimentally, has been generated. By eliminating the compactivity, we have also provided a relation between the volume fraction and the nature of mixing or segregation. This relation both provides an option for experimental validation of the model and the statistical hypothesis and allows one to identify mixtures which are expected to segregate at every compactivity.

The geometrical and mechanical models used to describe friction in this paper are more qualitative than quantitative, and more sophisticated models may be suggested. However, the qualitative results obtained here do not depend on their details but only on basic properties which all such models should have: the minimal volume does not depend on friction and the maximal volume and the number of possible states increase with friction. The experimental validation or invalidation of the aforementioned results may shed light on the validity of the statistical mechanics proposal of Edwards.

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APPENDIX: VORONOI CELL VOLUMES

This appendix describes the calculations leading from the mechanical models in 2D and 3D to the volume of the corresponding Voronoi cells around the grains, given in Eq. (6). In 2D the Voronoi cell around a grain is formed by the tangents to it at the contact points with the surrounding grains. The segment of the Voronoi cell lying between two contacts is the shaded rhombus in Fig. 1, which has an area of $r^2 \tan \theta$, where $r$ is the grain radius and $\theta$ is half the angle between two adjacent contacts. If all angles between adjacent contacts are equal to $2\theta$, the Voronoi cell comprises of $\pi/\theta$ such segments. Even though this is an integer number only for discrete values of $\theta$, we use this for continuous values of $\theta$.

IV. CONCLUSIONS

We have used Edwards’ statistical mechanics hypothesis together with a simple mechanical model to describe the role
The resulting area of the Voronoi cell is \( V_{vor}^D = \pi r^2 \tan \theta \).

In 3D we use a triangulation of the contact points on the surface of every grain, and in analogy to the 2D case, we consider the segment of the Voronoi cell bounded between three contacts (see Fig. 6). We assume that the three contact points \( A, B, \) and \( C \) form an equilateral triangle, therefore \( \angle AOG = \angle BOC = \angle COA = \theta \), where \( O \) is the center of the grain and \( G \) lies on the line directed from \( O \) to the center of the triangle \( ABC \). The segment of the Voronoi cell is the region bounded between the three directions \( OA, OB, \) and \( OC \) (or the planes \( AOB, BOC, \) and \( COA \)) and the tangent planes \( ADGF, BEG, \) and \( CFG \) (which due to symmetry meet at the point \( G \), which is located along the direction from \( O \) to the center of the triangle \( \Delta ABC \)). The segment of the Voronoi cell may be divided into six tetrahedra of equal volume all having the common edge \( OG \): \( OADG, OAFG, OBDG, OBEG, OCEG, \) and \( OCFG \). Its volume is hence given by

\[
V_{vor}^D = 6V_{OCEG} = 6 \frac{V_{OCEG}H_E}{3} = \frac{r^2 \tan \theta}{1 + 3\cos^2 \theta} \sqrt{3\sin \theta}.
\]

where \( V_{OCEG} \) is the area of the triangle \( OCG \) and \( H_E \) is the distance of \( E \) from the plane of \( \Delta OCG \). In order to calculate the total volume of the Voronoi cell we calculate the number of segments it comprises through the solid angle every such segment occupies. The angle \( \angle AOB = \angle BOC = \angle COA = \alpha \) may be calculated from the scalar product of any two of the vectors \( OA, OB, \) and \( OC \) and is given by \( \cos \alpha = (3\cos 2\theta + 1)/4 \). The solid angle formed by these three vectors is \( \Omega = 4\tan^{-1}\left(\sqrt{\tan(3\alpha/4)\tan(\alpha/4)}\right) \). Therefore the Voronoi cell volume is

\[
V_{vor}^D = \frac{4\pi}{\Omega} = \frac{\pi r^3 \sqrt{3\sin \theta \tan \theta}}{1 + 3\cos^2 \theta} \tan^{-1}\left(\sqrt{\frac{3\cos^2 \theta - \tan^2 \theta + \frac{3\alpha}{4}}{4}}\right).
\]
An alternative way of deriving the partition function is to obtain the total volume of the system from summing over the contributions of all grains to the volume and not by summing over the contributions of the contacts. In order to do this we assume all the grains surrounding every grain are of a single species. Under this assumption and in the mean-field approximation, the number of A grains surrounded by A grains is $f_2^A N$, the number of A grains surrounded by B grains is $f(1 - f) N$, the number of B grains surrounded by A grains is $f(1 - f) N$, and the number of B grains surrounded by B grains is $(1 - f)^2 N$. Since the volume of the Voronoi cell around every $i$ grain surrounded by $j$ grains is bounded between $v_{\text{min}}$ and $v_{ij}$, the expression in Eq. (7) immediately follows.
4 Exactly Solvable Model for Driven Dissipative Systems

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We introduce a solvable stochastic model inspired by granular gases for driven dissipative systems. We characterize far from equilibrium steady states of such systems through the non-Boltzmann energy distribution and compare different measures of effective temperatures. As an example we demonstrate that fluctuation-dissipation relations hold, however, with an effective temperature differing from the effective temperature defined from the average energy.
Dissipative many-particle systems are far from thermodynamic equilibrium, and a general theoretical description of their statistical mechanics is lacking, in contrast to systems in equilibrium, for which there is a well established theory. In this Letter we propose a simple exactly solvable model, which provides a context in which certain questions concerning driven dissipative systems may be resolved unambiguously, such as whether different proposed “definitions” of temperature give the same value, as is the case for thermal equilibrium.

Granular materials have in recent years been considered a paradigm for dissipative open systems [1]. In such systems composed of macroscopic particles, energy is dissipated via interactions, being transferred from macroscopic degrees of freedom (motion of grains) into microscopic degrees of freedom, and cannot be transformed back. Continuous driving is needed in order to maintain such a system in a dynamic state. This driving may be realized, for instance, by gravity-driven flow down an incline [2], by continuous avalanches in a rotating drum [3], or in a vibrated container [4].

An oft-studied system is a homogeneously heated granular gas [5]: a collection of hard spheres (or disks in 2D) undergoing inelastic collisions and driven by a stochastic thermostat. This yields Langevin dynamics including a random uncorrelated force \( \mathbf{F}(t) \), satisfying \( \langle \mathbf{F} \rangle = 0 \) and \( \langle F_i(t) F_j(t') \rangle = 2\gamma T_B \delta(t-t')\delta_{ij} \), and a drag force \( -\gamma \mathbf{v} \). This may be interpreted as coupling to a heat bath of temperature \( T_B \) with a coupling strength \( \gamma \). Were the system nondissipative, it would reach equilibrium with temperature \( T_B \), irrespective of the details of the coupling, that is, independent of \( \gamma \).

A driven dissipative system reaches a nonequilibrium steady state, for which a granular temperature \( T_G \) may be defined as the average kinetic energy of the grains. Not only is \( T_G \) always smaller than \( T_B \), but its value depends on the details of the coupling with the bath [6,7]. This steady state behaves statistically differently than an equilibrium state at the same effective temperature. The energy distribution deviates from the Boltzmann distribution, exhibiting overpopulated high energy tails [1]. Surprisingly, even though driven dissipative systems are not in equilibrium, fluctuation-dissipation (FD) relations often hold and may serve to define an effective temperature, \( T_{FD} \) [8], which has been found in numerical experiments to coincide with \( T_G \) [9].

Granular gases may be described theoretically by the Boltzmann equation of kinetic theory; however, solutions exist only in terms of approximations valid for small deviations from the equilibrium Boltzmann distribution, that is, for small inelasticity and low volume fraction [1]. A dissipative model for which some exact results have been found is the one-dimensional Maxwell model [10], in which particles collide inelastically with a uniform collision rate, independent of velocity or location. Energy distributions have also been investigated numerically in a two-dimensional version of this model assuming a random impact parameter in every collision [11].

In this Letter we present a novel exactly solvable dissipative model, in which interactions occur randomly and redistribute energy randomly between the interacting particles. Our main results are (i) all moments of the model may be computed exactly, (ii) in the maximally dissipative limit, the generating function of the model may be solved exactly, and (iii) FD relations hold with \( T_{FD} > T_G \).

Our model consists of a collection of \( N \) particles having energies \( E_i \), with a constant interaction rate between any two particles in the system. In every interaction two particles from the system are chosen at random and their energies are summed. In the case of conservative dynamics (analogous to elastic collisions) this total energy is repartitioned randomly with a uniform distribution between the two interacting particles [12], while for dissipative dynamics (as for inelastic collisions with a constant restitution coefficient) only a fraction \( \alpha \) of the total energy is repartitioned between the particles and the rest is dissipated out of the system. Additionally, we couple the system to a heat bath so that it reaches a steady state.

The simplicity of this model results from the fact that every particle in it is described only by its energy, as opposed, for example, to the 2d degrees of freedom per particle in a \( d \)-dimensional frictionless hard sphere gas.
By eliminating the momentum and spatial variables and using only the energy, we turn the vectorial collisions between particles into scalar interactions, and preclude spatial correlations.

For conservative dynamics ($\alpha = 1$) the system reaches an equilibrium state with the exponential Boltzmann distribution for the particles’ energies, $p(E) = \frac{1}{\Gamma} \exp(-E/\Gamma)$, where the temperature (measured in units of energy) equals the average energy in the system $T = \langle E \rangle$. Dissipative dynamics ($\alpha < 1$) cause energy to decay, and in order to maintain the system in a steady state, we keep it in contact with a heat bath, which is constructed as an infinitely large system of particles obeying the conservative dynamics described above, kept in equilibrium at a temperature $T_B$. The coupling of the dissipative system to the bath is through conservative interactions between a particle chosen at random from the system and a particle chosen at random from the bath, and is characterized by a coupling strength $f$, which is defined as the fraction of the particle’s interaction that is with the bath out of all its interactions (with the bath and with other particles in the dissipative system). We see that the steady state of the dissipative system depends on the bath through both $T_B$ and $f$ [13].

The stochastic equation of motion for the evolution of the energy of particle $i$ during an infinitesimal time step $dt$ is hence given by

$$E_i(t+dt) = \begin{cases} \text{value:} & E_i(t), \\ \text{probability:} & \frac{\Gamma - \Gamma dt}{\Gamma - f dt}, \end{cases} \left[ \begin{array}{c} \delta[E_i(t) + E_{ij}(t)] \\ \langle E_i(t) + E_{ij}(t) \rangle \end{array} \right] \exp(-E_B/T_B).$$

where $\Gamma$ is the interaction rate per particle per unit time (which does not affect the steady state but only the rate of approach to it); $j \in \{1, \ldots, N\}$ is the index of the particle with which particle $i$ may interact, chosen randomly at every interaction; $\delta \in [0,1]$ is the fraction of partitioned energy given to particle $i$ in the interaction, chosen randomly with a uniform distribution at every interaction; $E_B$ is the energy of the particle from the bath with which particle $i$ may interact, which at every interaction is chosen randomly from the equilibrium distribution in the bath, $p_B(E_B) = \frac{1}{\Gamma} \exp(-E_B/T_B)$.

We first demonstrate how all moments of $p(E)$ may be evaluated exactly from the dynamical rule for general restitution coefficient $\alpha$ and coupling strength $f$. Later, we shall consider the generating function of $p(E)$, using a procedure which is formally possible for the general case, but which will be fully solved only for limiting cases.

The first moment of $p(E)$ is the average energy, which in analogy with granular materials is denoted as the granular temperature, $T_G = \langle E \rangle$. This is evaluated by averaging Eq. (1) over the whole system:

$$T_G(t+dt) = (1 - \Gamma dt) T_G(t) + f \Gamma dt \left\{\langle E_i(t) + E_{ij}(t) \rangle \exp(-E_B/T_B)\right\}.$$

In the steady state $T_G(t+dt) = T_G(t)$, and therefore

$$T_G = T_B/[2\alpha - 1 + 2(1 - \alpha)/f].$$

While the model described above is extremely simple, it is interesting to note that it captures, at least qualitatively, some aspects of an actual driven granular gas. In Eq. (3) we see that $T_G$ is always smaller than $T_B$ and depends not only on the restitution through the restitution coefficient $\alpha$, but also on the details of the coupling to the bath through the coupling strength $f$. $T_G$ coincides with $T_B$ only for the two nondissipative limiting cases: conservative interactions ($\alpha = 1$) and interactions only with the bath ($f = 1$). To see this for a real granular gas (in 2D, for convenience), we estimate the ratio $T_G/T_B$ by the following mean-field energy balance calculation (see also [7]). Consider a gas of grains of mass $m$, diameter $D$, and restitution coefficient $\epsilon$, at volume fraction $\Phi$. The mean time between collisions for a grain of energy $E$ is $\frac{1}{T_B} \frac{\Phi}{\sqrt{\pi m}} E^{3/2}$ and the mean energy dissipated per collision is proportional to $(1 - \epsilon) E T_G$. Consequently, the average energy loss rate due to collisions is $\epsilon (1 - \epsilon) E T_G^{3/2}$. This is balanced in the steady state by gain due to driving by, and loss due to friction.
with the bath, 2γ(T_B - T_G), yielding \( T_B - T_G - AT_B^{1/2}T_G^{1/2} = 0 \), which may be solved for \( T_G/T_B \) as a function of the single parameter \( A(e, \Phi, C) = \frac{\Phi^2}{\gamma D_0(e/\gamma)} \).

We note that results of molecular dynamics simulations agree with this estimate, as can be seen in Fig. 1(a). This granular gas energy balance is more complicated than the calculation leading to Eq. (3), since the rate of dissipative interactions in a granular gas depends not only on the coupling strength \( C \) and volume fraction \( \Phi \), but is dynamically determined by the typical energy \( T_G \) through the grain velocities. Moreover, density and energy correlations in granular gases likely break the Boltzmann distribution, \( g(x) \), which may be solved for \( \langle x \rangle \) and its susceptibility with respect to changes in \( F \), \( \chi = \frac{\langle x^2 \rangle - \langle x \rangle^2}{F} \). We consider the relation between the fluctuation \( \langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \), and its susceptibility with respect to changes in \( F \), \( \chi = \frac{\langle x^2 \rangle - \langle x \rangle^2}{F} \).

We use FD relations in order to demonstrate one observable deviation from equilibrium behavior in dissipative systems due to their non-Boltzmann energy distribution. In order to measure FD relations in their simplest form we introduce an internal degree of freedom per particle, \( \lambda_i \), which is coupled to a uniform external field, \( F \), so that the total energy of every particle is \( U_i = E_i + \lambda_i F \), where \( E_i \) is now referred to as the kinetic energy. We consider the relation between the fluctuation \( \lambda_i \) and its susceptibility with respect to changes in \( F \), \( \chi = \frac{\langle \lambda_i^2 \rangle - \langle \lambda_i \rangle^2}{F} \).

We assume driven dissipative dynamics for the kinetic energy together with a nondissipative exchange of kinetic energy.

\[
\langle \lambda_i \rangle = \langle \lambda_i \rangle - f = \frac{1}{F} \int_0^f \text{d}E \int_0^{1/e} \text{d}E' \int_0^{1/e} \text{d}E'' \int_0^{1/e} \text{d}E''' \langle \lambda_i \rangle dh dh' dh'' dh''' \int_0^{1/e} \text{d}E'''' \int_0^{1/e} \text{d}E''''' \langle \lambda_i \rangle dh'''' dh''' dh'' dh' dh'''' dh''' dh'' dh dh
\]

where we have used the generating function of the exponential energy distribution in the bath, \( g(\lambda) = \int_0^{1/e} e^{-\lambda E} \text{d}E \). We transform the integral Eq. (6) into a differential equation by a change of variables to \( \lambda \) in both integrals, multiplication by \( \lambda \), and differentiation by \( \lambda \), yielding

\[
\lambda \frac{dg(\lambda)}{d\lambda} = (1 - f)g(\alpha \lambda) + \left( \frac{f}{\alpha \lambda T_B + 1} - 1 \right)g(\lambda).
\]

We first note that the generating function of the Boltzmann distribution, \( g(\lambda) \), solves Eq. (7) for the two nondissipative limiting cases \( \alpha = 1 \) and \( f = 1 \). Furthermore, the solution for the limit of maximal dissipation (\( \alpha = 0 \)) is \( g(\lambda) = \mathcal{F}(1, 2, 2 - f, -\lambda T_B)(\lambda T_B + 1) \), where \( \mathcal{F} \) is the Gauss hypergeometric function. The theoretical results of these two limiting cases (\( \alpha = 0 \) and \( \alpha = 1 \)) are plotted in Fig. 2 together with results for \( 0 \leq \alpha \leq 1 \) obtained from a Monte Carlo numerical simulation of the model with 1000 particles. The numerical results indicate that the theoretical results known for \( \alpha = 0 \) and \( \alpha = 1 \) bound the family of solutions for all intermediate values \( 0 < \alpha < 1 \). Note that the characteristic form of the energy distribution in this exactly solvable model qualitatively resembles that of granular gases [see Fig. 1(b)], for which only approximate solutions exist.

We now use FD relations in order to demonstrate one observable deviation from equilibrium behavior in dissipative systems due to their non-Boltzmann energy distribution. In order to measure FD relations in their simplest form we introduce an internal degree of freedom per particle, \( \lambda_i \), which is coupled to a uniform external field, \( F \), so that the total energy of every particle is \( U_i = E_i + \lambda_i F \), where \( E_i \) is now referred to as the kinetic energy. We consider the relation between the fluctuation \( \lambda_i \) and its susceptibility with respect to changes in \( F \), \( \chi = \frac{\langle \lambda_i^2 \rangle - \langle \lambda_i \rangle^2}{F} \). We assume driven dissipative dynamics for the kinetic energy together with a nondissipative exchange of kinetic energy.
Energy and internal energy, as described by the following equations of motion:

\[
E(t + dt) = \begin{cases}
\text{value} : & \text{probability} : \\
E_0(t), & 1 - h dt, \\
\left[\zeta E_0(t) + E_i(t)\right], & (1 - f) h dt, \\
\left[\zeta E_0(t) + E_i(t) - x_i F\right], & f h dt,
\end{cases}
\]

(8a)

\[
x(t + dt) = \begin{cases}
\text{value} : & \text{probability} : \\
x(t), & 1 - f \Phi dt, \\
f x(t) - \frac{F}{\Phi}, & f \Phi dt,
\end{cases}
\]

(8b)

where \( h \) is a parameter introduced to describe the ratio between interactions with the bath and interactions between the two types of degrees of freedom. By calculating the first two moments of Eq. (8b) one sees that \( \chi = \frac{\left\langle \Delta x^2 \right\rangle}{\left\langle x_i \right\rangle^2} \) and \( \left\langle \Delta x^2 \right\rangle = \frac{\left\langle \Delta x_i^2 \right\rangle}{\left\langle x_i \right\rangle^2} \). That is, the FD relation, \( \left\langle \Delta x_i^2 \right\rangle = \chi T_{FD} \), is satisfied with an effective temperature \( T_{FD} = \frac{\left\langle E_i \right\rangle}{\left\langle x_i \right\rangle^2} \), which probes the distribution of the kinetic energy \( E \), determined from Eq. (8a). Note that \( \chi \) and \( \left\langle \Delta x_i^2 \right\rangle \) diverge as \( F \to 0 \) (where \( T_{FD} \) is normally defined); however, their ratio is finite and independent of \( F \).

\( T_{FD} \), which characterizes fluctuations, results from the second moment of the energy distribution, and is generally different from \( T_G \), which describes the first moment. \( T_{FD} \) and \( T_G \) coincide only if \( \left\langle E_i^2 \right\rangle = 2 \left\langle E_i \right\rangle^2 \), which is the case for the Boltzmann distribution. For dissipative systems far from equilibrium, \( T_{FD} \) is larger than \( T_G \), and their ratio is given for our model in the limit \( h \to 0 \) by

\[
\frac{T_{FD}}{T_G} = \frac{\left\langle E_i^2 \right\rangle}{\left\langle E_i \right\rangle^2} = \frac{2(2 - f) - 2a(4(1 - f) + a^2(4 - 3f))}{(1 - f)} \left[ \Phi[3 - f - 2a^2(1 - f)] \right].
\]

(9)

In dissipative systems with strong coupling (\( f = 1 \)) and large restitution coefficient (\( a \approx 1 \)) the energy distribution is close to exponential; hence the values of \( T_{FD} \) and \( T_G \) are similar (but not identical). Generally, the ratio \( T_{FD}/T_G \) may reach any value larger than 1, and diverges for vanishing coupling strength (\( f \to 0 \)). As has recently been predicted by kinetic theory [14], we expect \( T_{FD} \) to be larger than \( T_G \) in granular gases as well, where the energy distribution is nonexponential. In the cases studied numerically [9] the energy distributions were only slightly nonexponential, resulting in small differences between \( T_{FD} \) and \( T_G \), which explains their seeming coincidence.

In conclusion, we have presented a simple dissipative model, solved it in terms of all energy moments in the general case, and obtained an exact expression for the generating function in the maximally dissipative limit. Although our model is inspired by granular gases, we believe it may have relevance to a broader class of driven dissipative systems. A dissipative system coupled to a heat bath is very different from a conservative system coupled to the same bath. Not only is the granular temperature lower than the bath temperature, but it also depends on the coupling details. In addition, FD relations hold with an effective temperature \( T_{FD} \) which characterizes the second moment of the energy distribution, and is larger than the granular temperature, \( T_G \). These exactly coincide in equilibrium, where the energy distribution is exponential, however, generally differ in dissipative systems. We expect that careful numerical studies of granular gases will show this hitherto unobserved difference.

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5 Comment on “Temperature in Nonequilibrium Systems with Conserved Energy”

Yair Srebro and Dov Levine


A recent Letter [Phys. Rev. Lett. 93, 230601 (2004)] presents a model with stochastic energy conserving dynamics, and claims that it exhibits non-equilibrium behavior. The results presented as evidence for this claim are (ostensible) deviations from the microcanonical and canonical distributions, and (ostensible) violation of fluctuation-dissipation relations. In this Comment we show that these results are insufficient for proving non-equilibrium behavior, and demonstrate that in fact the same results are obtained assuming equilibrium behavior for the same system and dynamics.
Comment on “Temperature in Nonequilibrium Systems with Conserved Energy”

In a recent Letter [1], Bertin, Dauchot, and Droz (BDD) presented a model with stochastic energy conserving dynamics. BDD’s central claim is that their model exhibits nonequilibrium behavior. The results BDD present as evidence for this claim are (ostensible) deviations from the microcanonical and canonical distributions, and (ostensible) violation of fluctuation-dissipation (FD) relations. In this Comment we show that these results are insufficient for proving nonequilibrium behavior, and demonstrate that in fact the same results are obtained assuming equilibrium behavior for the same system and dynamics defined in [1].

BDD define a system together with its dynamics, and the resulting distributions, Eqs. (5) and (9) in [1] (henceforth (B5) and (B9)), differ from the microcanonical and canonical equilibrium distributions. Were the model’s dynamics to stem from some Hamiltonian description, the corresponding generalized coordinates would define the system’s phase space, and one could compare the distributions in that space to the corresponding equilibrium distributions. BDD do not define a Hamiltonian which yields the system’s dynamics, hence one cannot know what phase space is. Thus, one cannot infer whether the calculated distributions imply any nonequilibrium behavior.

To make this point more explicit, we note that there is a description for BDD’s model, within which (B5) and (B9) are obtained as the equilibrium distributions. For BDD’s dynamics (B3) to be understood as dynamics of an equilibrium system, all final states consistent with energy conservation should be chosen at every time step with equal probability [2]. The density of states (DOS) for BDD’s model to sample phase space in such a uniform fashion is $d(E) = E^{n-1} |d(E_i) - E_i^{n-1}|$ for the inhomogeneous system defined by (B12). This DOS may be thought of as arising from some “hidden coordinates” $\{s_i\}$, such that $|x_i| = s_i^{1/\eta}$, where the $s_i$’s have a flat DOS, and thus define phase space. Since the energy of each particle is given by $E_i = x_i^2/2$, the DOS as a function of $x$ is $|x|^{n-1}$, and the state defined by $|x_i|$ is a collection of $d(x_i) = \prod_{i=1}^{N} |x_i|^{n-1}$ actual states defined by $\{s_i\}$. Therefore, the probabilities for the occurrence of the state $|x_i|$ in (B5) and (B9) are the equilibrium microcanonical and canonical distributions of the actual states $\{s_i\}$ multiplied by the DOS.

In calculating FD relations, it is necessary to know the measure of phase space. The FD measurement that is performed in [1] tacitly assumes a flat DOS for $x_i$, which results in their obtaining $T_{FD} = 2\epsilon$, rather than $T_{FD} = T_{th}$, where $T_{th}$ is the actual equilibrium temperature defined in (B8), and $\epsilon$ is the average per particle energy. Since no DOS may be inferred from the model, it is impossible to construct a correct FD measurement: one cannot prescribe the DOS (flat or otherwise) independent of the dynamics. If the DOS is not flat, using (B3) (or (B12)) does not modify the DOS as a function of $E$ and hence effectively changes the DOS as a function of $x$. Introducing an external field should not affect the structure of the system as reflected in its DOS $d(x)$, but merely change the relation between $x$ and $E$ (see [3] for an example of modeling FD relations in stochastic models). When assuming the DOS $d(x) = x^{n-1}$, proposed above, one obtains $T_{FD} = T_{th}$, irrespective of $\eta$.

Extending this to inhomogeneous systems, BDD’s $T_{FD}$ characterizes the average energy of each type of particle according to its DOS (or value of $\eta$), and is thus spatially nonuniform, while $T_{th}$ is the actual temperature, which is uniform throughout the system. A proper FD measurement would yield a uniform $T_{FD}$, coinciding with $T_{th}$. This is not the reason for the different effective temperatures observed in granular mixtures [4], where species differ in their mechanical properties which determine energy dissipation, rather than in their densities of states. Moreover, FD measurements in granular gases are on the position and not on the momentum (the equivalent of $x$ in BDD’s model), and measure FD relations, which yield the actual temperature in the equilibrium limit, and not the average energy as is the case in BDD’s procedure. Even though $T_{FD}$ does not strictly measure the average energy in granular gases, its value is similar to that of the average energy because the energy distribution differs only slightly from the Boltzmann distribution [3,5].

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6 Fluctuation-Dissipation Relations in Driven Dissipative Systems

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Exact theoretical results for the violation of time-dependent fluctuation-dissipation relations in driven dissipative systems are presented. The ratio of the correlation to delayed response in the stochastic model introduced in [Phys. Rev. Lett. 93, 240601 (2004)] is shown to depend on measurement time. The fluctuation temperature defined by this ratio differs both from the temperature of the environment performing the driving, and from other effective temperatures of the system, such as the average energy (or “granular temperature”). General explanations are given for the time independence of the fluctuation temperature for simple measurements or long measurement times.
Fluctuation-dissipation relations in driven dissipative systems

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Exact theoretical results for the violation of time-dependent fluctuation-dissipation relations in driven dissipative systems are presented. The ratio of the correlation to delayed response in the stochastic model introduced in [Phys. Rev. Lett. 93, 240601 (2004)] is shown to depend on measurement time. The fluctuation temperature defined by this ratio differs both from the temperature of the environment performing the driving, and from other effective temperatures of the system, such as the average energy (or “granular temperature”). General explanations are given for the time independence of the fluctuation temperature for simple measurements or long measurement times.

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The fluctuation-dissipation relation (FDR) [1] provides a fundamental connection between two small deviations from thermodynamic equilibrium—the autocorrelation function of an observable, and the response function of that observable to changes in its conjugate field. In equilibrium, there are two aspects to the FDR: (i) these functions have the same spatial and temporal (or frequency) dependence, and (ii) for every observable their ratio equals the system’s temperature.

Away from equilibrium, there is no such theorem connecting correlation and response. This having been said, there is much theoretical [2–13] and experimental [14,15] interest as to what is the relation between these quantities. In particular, one may ask whether either aspect (i) or (ii) holds for small deviations from nonequilibrium steady states (NESS). Aspect (ii) has been tested extensively for low frequency measurements in various systems, and the effective temperature $T_F$ defined by the ratio of the correlation to response, clearly differs from the environment temperature $T$. Other hand, numerical simulations indicate that much theoretical or frequency-dependent correlations and response in granular packings [8], sheared dissipative foams [5], dense granular packings [6], and driven granular gases [7].

Regarding aspect (i), different frequency dependences of the correlation and response functions have been observed in simulations of glasses [2], sheared foam [8], and models of driven systems [9], as well as in bacterial bath experiments [17]. In all these cases, fast modes thermalize, while slower modes manifest a higher (usually frequency-independent) temperature reflecting driving or system history (e.g., temperature quench). Experiments on vibrated granular systems [15] have shown a weak dependence of $T_F$ on measurement frequency, which may not obviously be attributed to thermalization of fast modes with the environment.

This work aims at understanding the relation between time- (or frequency-) dependent correlations and response in NESS. To our knowledge, these have not previously been calculated exactly for any driven dissipative system. Here we calculate the autocorrelation and response functions in the context of a simple stochastic model of a dissipative system. The model introduced in [13] is constructed about the essential features of any driven dissipative system: energy loss upon interaction between particles (or modes) and energy input through an external driving mechanism. As such, we have reason to hope that our results will be relevant to a broad class of driven dissipative systems [18].

Our main results are: (1) There are certain observables for which the correlation and response have identical temporal dependence, however, (2) for a general measurement, they have different temporal dependences. This notwithstanding, (3) for asymptotically long times, the time dependence of the correlation and response coincides. Spatial FDRs in our model are investigated in [19].

The model consists of $N$ particles, each with two degrees of freedom (DOF), one “kinetic,” $e_i$, and one “internal,” $x_i$. The energy of particle $i$ is $e_i+1/2F_i$, where $F_i$ is an external field on particle $i$, which may depend on time. The $\{x_i\}$ may be thought of as positions or any other coordinates used to measure FDR. In our notation we emphasize the difference between microscopic DOF’s $\{e_i\}$ and $\{x_i\}$ and macroscopic quantities $E=\sum_{i=1}^N e_i$ and $X=\sum_{i=1}^N x_i$. The system dissipates energy through interactions and is maintained in a steady state by being coupled to a thermal bath.

In every interaction two DOF’s are randomly chosen and their energy is stochastically redistributed between them. In interactions between two “kinetic” DOF’s $\langle e\rangle$ there is dissipation: only a fraction $0<\alpha<1$ ($\alpha$ is like a restitution coefficient) of the energy is conserved and the remaining is dissipated out of the system. Interactions between a “kinetic” and an “internal” DOF ($e\sim\alpha$), and between a “kinetic” DOF and a DOF from the bath $\langle e\rangle\sim\alpha$ conserve energy. When all interactions are elastic ($\alpha=1$), detailed balance holds and the system reaches thermodynamic equilibrium with the bath. Rates of interactions are determined by the per-particle interaction rate $\Gamma$, and the dimensionless coupling strengths $0\lesssim<\Gamma,\lesssim\Gamma$ of the “kinetic” DOF’s to the bath and to the “internal” DOF’s, respectively.

The resulting stochastic equation of motion for particle $i$ reads,

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where \( j, k \in \{1, \ldots, N\} \) are indices of particles with which particle \( i \) may interact, chosen randomly at every interaction (we do not allow \( \varepsilon_i = \varepsilon_j \) interactions, therefore \( k \neq i \); \( z \in [0,1] \) is the fraction of repartitioned energy given to particle \( i \) in the interaction, chosen randomly from a uniform distribution at every interaction; \( e_g \) is the energy of a bath DOF chosen randomly at every interaction from the equilibrium distribution \( p_{eq}(E) = T_B e^{-E/2T_B} \) at the environment temperature \( T_B \).

In the steady state we define one-particle and \( N \)-particle autocorrelation functions, \( c(t) = \langle \varepsilon(i_1) \ldots \varepsilon(i_N) \rangle - \langle \varepsilon(i) \rangle^N \) and \( C(t) = \langle \varepsilon(i) \varepsilon(i+1) \rangle - \langle \varepsilon(i) \rangle^2 \), respectively, and corresponding delayed response functions \( r(t) = \langle \delta(x(t)) \rangle / \langle \delta(t) \rangle \) and \( R(t) = \partial \langle X(t) \rangle / \partial \langle F(t) \rangle \). The responses \( r(t) \) and \( R(t) \) are to a sudden change at time \( t = 0 \) of the field on the single particle \( i \) or of the uniform field on all particles, respectively. \( \cdot \) denotes the steady state ensemble average over possible states at the beginning of the measurement \((t=0)\). In equilibrium \((\alpha=1\) or \(f = 1\)), \( c(t) = r(t) T_B \) for any \( x \), and \( C(t) = R(t) T_B \) for \( X \) [20]; we will be concerned with the nonequilibrium cases.

For one-particle FDR we consider a system evolving from a given state at \( t=0 \) until a later time \( t \). We average Eq. (1b) over the stochasticity in the dynamics for this initial state, and denote this averaging by an overline,
\[
\frac{\partial}{\partial t} \langle \delta x(t) \rangle = -f h \langle \delta x(t) \delta F(t) \rangle. \tag{2}
\]
By the central limit theorem, for large systems \((N \gg 1)\) the relative fluctuation \( \langle x(t) \rangle / \langle x(t) \rangle \) of \( x \) is much larger than the relative fluctuation \( \langle E(t) \rangle / \langle E(t) \rangle \), and thus the steady state average \( \langle E(t) \rangle \) may be substituted for the instantaneous value \( \langle E(t) \rangle \) in Eq. (2). For a constant field, noting that \( \langle e \rangle = \langle E \rangle / N = -\langle x \rangle F \) [21], this yields,
\[
\langle x(t) \rangle = \langle x \rangle + \langle x(t) \rangle e^{-\gamma t}, \tag{3}
\]
with \( \gamma = f h / 2 \). The correlation is obtained by multiplying Eq. (3) by \( x(t) \) and averaging over the steady state distribution for all possible initial \((t=0)\) states of the system, which yields \( c(t) = \langle x(t)^2 \rangle - \langle x(t) \rangle^2 e^{-\gamma t} \) [22].

The one-particle response is obtained by maintaining the system in a steady state with some external field, then, at time \( t = 0 \), changing the field \( F \), acting on particle \( i \) from \( F_0 \) to \( F \). We then follow \( x(t) \) and average over the system’s steady state distribution at time \( t = 0 \) to yield \( \langle x(t) \rangle \). The steady state solution of Eq. (2) yields \( -\langle x(t) \rangle F_0 = -\langle x(t) \rangle F = \langle x(t) \rangle \), thus \( \langle x(t) \rangle = -\langle e \rangle [F^{-1} + (F_0^{-1} - F^{-1})] e^{-\gamma t} \). After differentiating with respect to \( F_0 \) and taking the limit \( F_0 \to F \), we get \( r(t) = \langle e \rangle F^{-1} e^{-\gamma t} \).

We thus see that the one-particle correlation and response have the same temporal dependence and aspect (i) of the FDR exactly holds, with the time-independent fluctuation temperature [22],
\[
T^2_\gamma = \frac{c(t)}{r(t)} = T_B \frac{\langle x^2 \rangle - \langle x \rangle^2}{\langle x^2 \rangle}. \tag{4}
\]
Aspect (ii), on the other hand, is violated [13], since \( T^2_\gamma \) generally differs both from the environment temperature \( T_B \) and from the granular temperature, defined as the average energy per DOF, \( T_g = \langle e \rangle \). Only in the equilibrium limits \((\alpha = 1, f \to 1)\) is the distribution of \( x \), the exponential Boltzmann distribution and \( T^2_\gamma = T_c = T_B \).

The FDR is only proven for equilibrium, so one might expect the correlation and response to depend differently on time in a NESS. However, the correlation represents the system’s return to steady state after deviating from it due to spontaneous fluctuations, and the response represents its return to steady state after being moved away from it by some external force. Both are governed by the same physical processes and thus generally possess the same time scales. In simple cases (as the one-particle measurement solved above) there is only one time scale and the correlation and response are not rich enough to have different time dependencies. In order to observe violations of aspect (i) in the FDR we now consider a measurement on a macroscopic quantity. This possesses two time scales—just enough to exhibit a different temporal dependence for the correlation and response.

We assume uniform \( F \) and sum Eq. (2) over particles,
\[
\frac{2}{\Gamma} \frac{d\langle X(t) \rangle}{dt} = -f h \frac{\langle X(t) \rangle + \langle E(t) \rangle}{\langle F(t) \rangle}. \tag{5}
\]
Now, fluctuations in \( E \) are not negligible compared to fluctuations in \( X \), so Eq. (5) must be solved in conjunction with the equation similarly derived from Eq. (1a),
\[
\frac{2}{\Gamma} \frac{d\langle E(t) \rangle}{dt} = -f h \langle \delta X(t) \delta F(t) \rangle - A(e) \langle E(t) \rangle + f(1-h) N T_B. \tag{6}
\]
where \( A(e) = n + (1-f) (1-2 e) \). The average evolution of the system from a state with \( X(0) \) and \( E(0) \) is hence given by the simultaneous solution of Eqs. (5) and (6):
\[
\langle X(t) \rangle = \langle X \rangle + \sum_{\ell=1}^{2} \left[ a_\ell \langle X(0) - \langle X \rangle \rangle + b_\ell \frac{E(0) - \langle E \rangle}{F} \right] e^{-\gamma_{\ell} t}, \tag{7}
\]
where we denote \( k = (A^2 - 2 A_f h + 5h^2)^{1/2}, \) \( a_1 = (1 - A_f + h f) / (h f / 2 k), \) \( a_2 = (1 + A_f - h f) / (2 k), \) \( b_1 = f h / k, \) \( b_2 = -f h / k, \) \( \gamma_1 = (A_f + h f + k) / 4, \) and \( \gamma_2 = (A_f + h f - k) / 4, \) and the steady state values are given by \( \langle X(0) \rangle = \langle X \rangle e^{(1-h) NT_B / (A_f - h f)} \).

To compute the correlation we multiply \( \langle X(t) \rangle \) by \( X(0) \) and average over the steady state distribution of the initial states [that is, over \( X(0) \) and \( E(0) \)] [22]:

\[046132-2\]
For the response we change the field from $F_0$ to $F$ at $t=0$, and after averaging Eq. (7) over the initial states taken from the steady state corresponding to the field $F_0$, differentiating with respect to $F_0$ and taking the limit $F_0\to F$ we have

$$R(t) = \frac{\langle E \rangle}{F} 2 \sum_{i=1}^{N} a_i e^{-\gamma t}. \quad (9)$$

In this $N$-particle measurement $C(t)$ and $R(t)$ share the rates $\gamma_1$ and $\gamma_2$ however, with different prefactors, leading to violation of aspect (i) in the FDR [23]. If one insists on defining a fluctuation temperature $T_F(t) = C(t)/R(t)$, it will depend on the measurement time, as can be seen in Fig. 1, which also demonstrates that the one-particle and $N$-particle measurements yield fluctuation temperatures differing both one from the other as well as from the granular temperature $T_G$ and bath temperature $T_b$.

As the correlation and response generally share the same time scales, for measurement times longer than the maximal time scale of the system (1/2$\gamma_2$ for the $N$-particle measurement in our model) this time scale dominates both correlation and response and aspect (i) of the FDR asymptotically holds for long times (see Fig. 1). It is interesting to speculate that this behavior may hold for general dissipative systems.

Although the FDR is generally not valid in NESS, it is hard to observe its violations in simulations or experiments on dissipative systems. One reason for this is that dissipative systems often have nearly Boltzmann distributions with some effective temperature, and deviations between different effective temperatures of the system (like $T_G$ and $T_i$) are too small to be observed [10,13].

Another reason is that some measurements on systems as a whole do not exhibit at all the FDR violations presented above. To demonstrate this we now show that in the inelastic Maxwell model [24] the frequency-dependent Kubo formula [25] holds exactly even without the effective temperature $T_c$ and the frequency $\gamma$ that we have introduced above. We shall first show that for our model $T_K$ is frequency-independent and equal to $T_G$, which generally differs from $T_c$ and subsequently show that for the Maxwell model $T_K = T_G$.

To make contact with the standard Kubo relations we will imagine the DOF’s $\{x_i\}$ in our model as positions; thus the autocorrelation of the single-particle “velocity” $v_i = dx_i/dt$ is obtained by twice differentiating $c(t)$ calculated above. The relation $c(t) = (r(t))_T^2$ with $T^2$ independent of $t$, as obtained for the one-particle FDR solved above, may be twice differentiated with respect to time and transformed to frequency domain. The Kubo formula then immediately follows with $T_K = T_G$. We have also verified this general result by explicitly calculating $D(w)$ and $\mu(w)$ in our model and obtained $D(w) = \mu(w)T_G^2 = (v^2 - (\gamma)^2)/(\gamma + i\omega)$. Our method may now be applied to the one-dimensional (1D) inelastic Maxwell model (the 2D and 3D versions follow trivially): During a finite time $dt > 0$, short compared to the collision rate $\Gamma$ and to the coupling rate $A$ to the thermostat, the effect of the fluctuating force in the Langevin dynamics each particle undergoes is proportional to $\sqrt{dt}$ (see, e.g., [27]). Thus the velocity of particle $i$ evolves as,

$$v_i(t + dt) = \begin{cases} \frac{1}{\Gamma dt} [1 - \alpha dt] v_i(t) + \frac{1}{\Gamma dt} [1 + \gamma dt] \phi_i(t) \quad &\text{Value:} \\ 1 - \alpha dt v_i(t) + \frac{1}{2} v_i(t) \quad &\text{Probability:} \end{cases}$$

(10)

For stochastic thermostating, $\phi_i(t)$ is an uncorrelated random force with $\langle \phi^2 \rangle = 2 \lambda T_b$, while for Gaussian thermostating, $\phi_i(t) = 0$ and $\lambda = (\sigma^2 - 1)^{1/2}$. Averaging Eq. (10) yields $\langle v_i(t) \rangle = \langle v_i(t) \rangle e^{-\gamma t}$, thus $D(w) = \langle v^2 \rangle/(\gamma + i\omega)$. In the mobility measurement, a periodic acceleration $g(t) = \xi e^{i\omega t}$ is added to a single particle, and from Eq. (10) we have

$$\frac{dv_i(t)}{dt} = -\xi \frac{e^{i\omega t}}{(\gamma + i\omega)}. \quad (11)$$

This has the steady solution $\langle v_i(t) \rangle = \xi e^{i\omega t}/(\gamma + i\omega)$, thus $\mu(w) = 1/(\gamma + i\omega)$, and the Kubo formula holds with $T_K = T_G$.
The difference in $T_F$ between our model and the Maxwell model derives from the fact that $T_F$ and $T_G$ measure fluctuations, expressed in second moments of DOF’s. In the Maxwell model $T_F$ is defined from $\langle \phi^2 \rangle$, which is the second moment of a DOF, while in our model $T_G = \langle \phi \rangle$ is defined from the first moment of a DOF. Only in equilibrium do all moments of the exponential Boltzmann distribution yield a single energy scale and these measurements in both models give the actual temperature.

To our knowledge, we have presented the first exact results for time-dependent violations of FDR in driven dissipative systems by demonstrating that correlation and response functions have different temporal behaviors. This violation, like differences between different effective temperatures of a system, may be small, requiring sensitivity in simulations and experiments. We identified situations relevant to general driven dissipative systems where the correlation and response have the same time dependence: In sufficiently simple measurements the correlation and response share a single time scale and so have the same dependence on time, thus time-dependent FDR violations may be observed only in systems or measurements exhibiting multiple time scales. Finally, since the correlation and response typically share the same time scales, their temporal behavior for measurements with long waiting times asymptotically coincides.

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[16] Dependence of $T_F$ on observable was found in glasses [11], while kinetic theory approximations for granular gases [10] as well as exact solutions of a model system [13] showed $T_F$ differs from the granular temperature.
[17] T. C. Lubensky (private communication).
[18] For example, our model qualitatively reproduces the energy distribution of driven granular gases [13].
[20] The more common FDR is obtained by time differentiating this [see e.g., D. Chandler, Instruction to Modern Statistical Mechanics (Oxford University Press, New York, 1987)].
[21] $\langle \chi \rangle$, $\langle \bar{r} \rangle$, $\langle \tau \rangle$, and $R(t)$ all diverge as $F \to 0$; however $T_F$ and $T_G$ are finite and independent of $F$ [13], as is the case when $s$ is height and $F$ a gravitational field.
[22] The procedure used in [13] to calculate all moments of $r_i$ has been used to obtain the steady state averaged second moments, which for $N=1$ reduce to

$$\langle \chi^2 \rangle - \langle \chi \rangle^2 = \langle f(1-b)^2 + 2f(1-b) + 2f(1-b)^2 \rangle A_0 + \langle f(1-b)^3 \rangle 2A_1$$

$$\langle X(E) - \langle X \rangle|E \rangle = - \frac{N2F^2h(1-h)(4A_1^2(3A_1 + 3f(2 - 9h) - A_2 + 2f(15 - 27h - f(5 - 12h - 3h^2) - (5 - 9h)A_1) - f(18A_2 - 1) - h))}{(2 - 21h - 3h^2) - f(18A_2 + 2f(2 - 5h) - 3h^2)^2 - f(18A_2 + 2f(2 - 5h) - 3h^2)^2 - f(18A_2 + 2f(2 - 5h) - 3h^2)^2}$$

$$\langle \chi^2 \rangle - \langle \chi \rangle^2 = \frac{N2F^2E^3}{F} + \langle X(E) - \langle X \rangle|E \rangle/F$$

with $A_0 = n*(1-f)(1/2a)^n$.
[23] This FDR violation is different in nature and origin from that observed in glasses [2] and driven systems [8,9,17], where fast modes thermalize while slower modes do not.
[26] The Einstein relations, derived here from the Kubo formula with $\alpha=0$, were previously shown to hold for the 1D Maxwell model with a Gaussian thermostat [12].
7 Energy Distribution and Effective Temperatures in a Driven Dissipative Model

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We investigate non-equilibrium behavior of driven dissipative systems, using the model presented in [Phys. Rev. Lett. 93, 240601 (2004)]. We solve the non-Boltzmann steady state energy distribution and the temporal evolution to it, and find its high energy tail to behave exponentially. We demonstrate that various measures of effective temperatures generally differ. We discuss infinite hierarchies of effective temperatures defined from moments of the non-exponential energy distribution, and relate them to the “configurational temperature”, measured directly from instantaneous particle locations without any kinetic information. We calculate the “granular temperature”, characterizing the average energy in the system, two different “fluctuation temperatures”, scaling fluctuation-dissipation relations, and the “entropic temperature”, defined from differentiating the entropy with respect to energy.
Energy distribution and effective temperatures in a driven dissipative model

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We investigate non-equilibrium behavior of driven dissipative systems, using the model presented in [Phys. Rev. Lett. 93, 240601 (2004)]. We solve the non-Boltzmann steady state energy distribution and the temporal evolution to it, and find its high energy tail to behave exponentially. We demonstrate that various measures of effective temperatures generally differ. We discuss infinite hierarchies of effective temperatures defined from moments of the non-exponential energy distribution, and relate them to the “configurational temperature”, measured directly from instantaneous particle locations without any kinetic information. We calculate the “granular temperature”, characterizing the average energy in the system, two different “fluctuation temperatures”, scaling fluctuation-dissipation relations, and the “entropic temperature”, defined from differentiating the entropy with respect to energy.

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I. INTRODUCTION

Systems of many particles interacting dissipatively are far from thermodynamic equilibrium, and a general theoretical description of their statistical mechanics is lacking, in contrast to systems in equilibrium, for which there is a well established theory. In this Paper we use exact solutions of a simple stochastic model in order to explore aspects of dissipative systems. In particular, we are interested in steady states: their non-Boltzmann energy distribution, the way in which a system arrives at its steady state, and various proposed definitions of temperature. Previous theoretical research on energy distributions and effective temperatures in various driven dissipative systems (granular materials [1–9], foams [10–12], and glasses [13–15]) has been mostly numerical or approximate. Therefore the resulting distributions and temperatures may agree with an effective equilibrium behavior due to numerical error.

In systems comprised of macroscopic particles, energy is dissipated via interactions, being transferred from macroscopic degrees of freedom (such as motion of particles) into microscopic degrees of freedom (heat), and can not be transformed back. Continuous driving is needed in order to maintain such a system in a dynamic state. One way to model this driving is by holding the system in contact with a bath, or large energy reservoir. As in [16], we concentrate on driving mechanisms where this bath is in equilibrium at some temperature $T_B$ (this driving bath is not necessarily in thermal equilibrium). A non-dissipative system driven by a thermal bath would reach thermodynamic equilibrium with it, where the energy distribution is given by the exponential Boltzmann distribution, and the system temperature is equal to the bath temperature.

The temperature of an equilibrium system is manifested in various measurements that can be performed on it. A non-equilibrium system does not, a-priori, have a unique well-defined temperature, and each such measurement inspires the definition of a corresponding effective temperature. For example, the entropic temperature $T_E$ is the inverse of the derivative of entropy with respect to energy [11, 13, 17, 18], in analogy with the definition of temperature in statistical mechanics. The principle of energy equipartition in thermal equilibrium motivates defining the granular temperature $T_G$ as the average energy per degree of freedom [19]; The equilibrium fluctuation-dissipation theorem suggests defining the fluctuation temperature $T_F$ as the ratio of fluctuation to response [20, 21]; Recent results expressing the equilibrium temperature from ensemble averages of particle locations (without any kinetic information) [22, 23] lead to the definition of a configurational temperature $T_C$ as another measure of effective temperature for non-equilibrium systems [24, 25]. All these definitions yield the same value in equilibrium.

In steady states of driven dissipative systems all effective temperatures are generally much smaller than $T_B$ ($T_B$ should not be confused with the actual temperature of the environment which is typically much lower), and, unlike thermal equilibrium, their values depend on the details of the coupling with the bath. Although these systems are far from equilibrium, and their energy distributions differ significantly from the Boltzmann distribution, there is evidence for coincidence of different effective temperatures. Different $T_F$s of the same system, obtained from correlations and response of different variables, have been found in numerical experiments to coincide in glasses [13–15], and to coincide with $T_G$ in granular gases [7, 8] and with $T_E$ in sheared foam [11].

In this Paper we study an exactly solvable dissipative model, in which interactions occur randomly and redistribute energy stochastically between the interacting particles. In [26] we first introduced the model, discussed its similarities to granular gases, calculated all moments of its steady-state energy distribution, and showed that $T_F$...
differs from $T_B$. Here we present these results in more detail and provide further results on the model: we define the model in Sec. II and investigate its energy distribution in Sec. III. We use the energy scales defined by high moments of the energy distribution to show that the high energy tail is exponential with a decay rate corresponding to the bath temperature $T_B$, and solve the temporal evolution to the steady state. Section IV deals with various effective temperatures in our model. We show that spontaneous fluctuations of different quantities are scaled by different fluctuation temperatures. We demonstrate that for systems with a single energy scale and with smooth energy distributions, the entropic temperature coincides with the granular temperature, and then investigate cases where they differ. We relate the infinite hierarchies of different effective temperatures defined from the energy distribution to generalizations of the configurational temperature. All results are calculated exactly in the context of our model, and in Sec. V we discuss the generality of these results to other driven dissipative systems.

II. MODEL DEFINITION

At its most basic, a dissipative system consists of a set of degrees of freedom, or modes of excitation, which interact among themselves and with the external environment. When two modes interact with each other, there is energy exchange, with some of the energy being lost to the environment. External driving may be thought of as the injection of energy into the system’s modes from the environment. Our model is constructed as a minimal model including these essential features. When two particles (we shall refer to the modes as particles for simplicity) interact they lose some of their energy and exchange what remains, and when a particle interacts with the environment it is more likely to gain energy rather than to lose.

Our model consists of $N$ particles with energies $\{e_i\}$, with a constant interaction rate between any two particles in the system. In every interaction two particles from the system are chosen at random and their energies are summed. In the case of conservative dynamics (analogous to elastic collisions) this total energy is repartitioned randomly between the two interacting particles (as in [27]). For dissipative dynamics with inelastic collisions, only a fraction $0 \leq \alpha < 1$ of the total energy is repartitioned between the particles and the rest is dissipated out of the system. Thus, $\alpha$ is analogous to a restitution coefficient. The system is coupled to a heat bath so that it may be maintained in a nontrivial steady state. The interactions are shown diagrammatically in Fig. 1, and described in further detail below.

For conservative dynamics ($\alpha = 1$) an isolated system (i.e., not in contact with the driving bath) reaches thermodynamic equilibrium with the exponential Boltzmann distribution for each particle’s energy, $\rho(e) = \frac{e^e}{\sum_{e'} e'^e}$. Where $e$ is the energy of the bath particle with which particle $i$ may interact (chosen randomly at every interaction), $z \in [0, 1]$ is the fraction of repartitioned energy given to particle $j$ in the interaction (chosen randomly with a uniform distribution at every interaction), and $e_B$ is the energy of the bath particle with which particle $i$ may interact, which at every interaction is chosen randomly from the equilibrium distribution in the bath: $\rho_B(e_B) = T_B^{-1} \exp(-e_B/T_B)$.

The simplicity of our model derives from the fact that every particle in it is described only by its energy, as opposed, for example, to the $2d$ degrees of freedom per particle in a $d$-dimensional frictionless hard sphere gas. By eliminating the momentum and spatial variables and using only the energy, we replace the vectorial collisions between particles by scalar interactions. Furthermore, since any two particles may interact, there are no spatial correlations.

FIG. 1: The possible interactions in our model: dissipative two particle interaction (left) and conservative system-bath interaction (right).

$$T^{-1} \exp(-e/T), \text{ where the temperature equals the average energy per particle } T = \langle e \rangle \text{ (we measure temperature in units of energy and set Boltzmann’s constant to one).}$$

Dissipative dynamics ($0 \leq \alpha < 1$) cause energy to decay, therefore we drive the system by attaching it to a heat bath, constructed as an infinitely large system of particles obeying the conservative dynamics described above, kept in equilibrium at a temperature $T_B$. The coupling of the dissipative system to the bath is through conservative interactions between a particle chosen at random from the system and a particle chosen at random from the bath (the system-bath interactions are taken as conservative for simplicity, however dissipative interactions may as well be considered, yielding qualitatively similar results). This contact is characterized by a coupling strength, $0 < f \leq 1$, defined as the fraction of every particle’s interactions with the bath out of all its interactions.

Unlike thermodynamic equilibrium, the dissipative system’s steady state depends on the bath through both $T_B$ and $f$ (see [5] which emphasizes the importance of coupling details).

The stochastic evolution of the energy of particle $i$ during an infinitesimal time step $dt$ is hence given by

$$e_i(t + dt) = \begin{cases} \text{Value:} & e_i(t) \\ \text{Probability:} & \frac{1 - \Gamma dt}{1} & 1 - \Gamma dt \end{cases}$$

where $\Gamma$ is the interaction rate per particle per unit time, $j \in \{1, ..., N\}$ ($j \neq i$) is the index of the particle with which particle $i$ may interact (chosen randomly at every interaction), $z \in [0, 1]$ is the fraction of repartitioned energy given to particle $i$ in the interaction (chosen randomly with a uniform distribution at every interaction), and $e_B$ is the energy of the bath particle with which particle $i$ may interact, which at every interaction is chosen randomly from the equilibrium distribution in the bath: $\rho_B(e_B) = T_B^{-1} \exp(-e_B/T_B)$.
III. ENERGY DISTRIBUTION

In this section we investigate our model’s single-particle energy distribution $p(e)$. We calculate from Eq. (1) the temporal evolution of any moment of $p(e)$. This is used to obtain the temporal evolution of $p(e)$, as well as its form in steady state. We use the energy moments to define two hierarchies of energy scales, whose asymptotic behavior is then used to characterize the high energy tail of $p(e)$. In particular, we shall show that although $p(e)$ differs from the Boltzmann distribution, its high energy tail is exponential.

A. Average Energy

The most direct way to characterize the system’s state is by the average energy per degree of freedom. It is solved by averaging Eq. (1) over the stochasticity in the dynamics and over all particles:

$$
\langle e(t + dt) \rangle = (1 - \Gamma dt) \langle e(t) \rangle + (1 - f) \Gamma dt \langle e(t) \rangle + f \Gamma dt \frac{1}{2} [\langle e(t) \rangle + T_B].
$$

Hence $\langle e(t) \rangle$ satisfies the differential equation

$$
2 \frac{d \langle e(t) \rangle}{dt} = -A_1 \langle e(t) \rangle + f T_B, \tag{3}
$$

with $A_1(\alpha, f) \equiv 1 + (1 - f)(1 - 2\alpha) > 0$. This has the steady state solution

$$
\langle e \rangle = \frac{f T_B}{A_1} = \frac{T_B}{2\alpha - 1 + 2(1 - \alpha)f}. \tag{4}
$$

In analogy with granular materials, this is denoted as the granular temperature $T_G \equiv \langle e \rangle$. It is plotted vs. $\alpha$ and $f$ in Fig. 2.

It is interesting to note that despite the simplicity of our model, it captures, at least qualitatively, some aspects of actual driven dissipative systems \([28, 29]\). For instance, the stationary value of $\langle e \rangle$ is always smaller than $T_B$ and depends not only on the dissipation through the restitution coefficient $\alpha$, but also on the details of the coupling to the bath through the coupling strength $f$. $\langle e \rangle$ coincides with $T_B$ only in the two non-dissipative limits: conservative interactions ($\alpha = 1$) and strong coupling to the bath ($f = 1$).

B. Energy Fluctuations

We now consider the energy distribution around the average energy $\langle e \rangle$. This tests whether the system is equivalent to an equilibrium system at an effective temperature $T_G \equiv \langle e \rangle$. That is, whether the effect of the dissipation is to modify the Boltzmann distribution only by changing its characteristic temperature from $T_B$ to $T_G$. However, this is not the case: we find that the energy distribution is clearly non-exponential, in qualitative agreement with realistic driven dissipative systems, such as granular gases \([1, 3, 4, 28]\) and colloidal suspensions \([30–33]\).

The first moment of $p(e)$ is the average energy calculated above. Higher moments are obtained by taking the average of the $n$th power of Eq. (1). This yields the following differential equation for $\langle e^n \rangle$ in terms of all lower moments and the moments of the energy distribution in the bath (for which $\langle e^n_B \rangle = n! T_B^n$).

$$
\frac{n + 1}{f} \frac{d \langle e^n \rangle}{dt} = -A_n \langle e^n \rangle + \sum_{m=1}^{n-1} \binom{n}{m} \langle e^m \rangle [ (1 - f) \alpha^n \langle e^{n-m} \rangle + f \langle e^{n-m}_B \rangle ]
$$

$$
+ f \langle e^n_B \rangle, \tag{5}
$$

where

$$
A_n(\alpha, f) \equiv n + (1 - f)(1 - 2\alpha^n). \tag{6}
$$

Any initial distribution will evolve with time to the steady state distribution given by

$$
\langle e^n \rangle = \left( \sum_{m=1}^{n-1} \binom{n}{m} \langle e^m \rangle [ (1 - f) \alpha^n \langle e^{n-m} \rangle + f \langle e^{n-m}_B \rangle ]
$$

$$
+ f \langle e^n_B \rangle \right) / A_n. \tag{7}
$$

These expressions for all energy moments are exact arbitrarily far from equilibrium (for general values of $\alpha$ and $f$) and contain information about the entire energy distribution. In the equilibrium limits ($\alpha = 1$ or $f = 1$) Eq.
(7) yields the moments of the exponential Boltzmann distribution, for which \( \langle e^n \rangle = n! T_B^\gamma \).

The steady state energy distribution with moments given by Eq. (7) is shown in Fig. 3. At low energies (of the order of several times \( T_G \)) the distribution is roughly exponential with a decay rate corresponding to the average energy \( T_G \). At intermediate energies the distribution seems to exhibit an overpopulated high energy tail, decaying slower than \( \exp(-e/T_G) \). However, for higher energies (\( e \gtrsim 10 T_G = ST_B \) for the parameters in Fig. 3), the distribution exhibits an exponential decay of the form

\[
p(e) \sim \exp(-e/T_B).
\]

The overpopulation of the high energy tail is only with respect to scaling the energy with the average energy in the system and considering energies comparable to \( \langle e \rangle \), as is customarily done in granular gases [1, 3–6, 28]. Interactions with the bath dominate the high energy tail, since a particle is much more likely to arrive at such high energies due to a conservative interaction with the bath rather than due to a dissipative interaction within the system (and the fraction of very high energy particles in the bath is larger than in the system). In the next section, we use the asymptotic behavior of energy scales defined by the moments to show that the tail is indeed exponential.

![Fig. 3: Steady state energy distribution for restitution coefficient \( \alpha = 0.5 \) and coupling strength \( f = 0.5 \) (resulting in \( T_G = T_B/2 \)) obtained in a numerical simulation of the model (solid line). Exponential distributions at temperatures \( T_R \) and \( T_G \) are given for reference (dashed lines).](image)

**C. High Energy Tail**

The exponential energy distribution of a system in equilibrium contains a single energy scale - the temperature of that system. For a non-equilibrium system, with a non-exponential energy distribution, infinite hierarchies of effective temperatures may be defined from the energy distribution. We define two such hierarchies, \( T_B^{(n)} \) from the ratios of succeeding energy moments, and \( T_M^{(n)} \) by scaling the moments themselves:

\[
T_R^{(n)} \equiv \frac{\langle e^n \rangle}{n! \langle e \rangle^{n-1}} \quad \text{and} \quad T_M^{(n)} \equiv \left( \frac{\langle e^n \rangle}{n!} \right)^{1/n}.
\]

Both reduce in equilibrium to the system’s temperature for any \( n \). Away from equilibrium they typically differ and depend on \( n \) (see Fig. 4), and for \( n = 1 \) both reduce to the granular temperature \( T_G \equiv \langle e \rangle \).

The large \( n \) behavior of \( T_B^{(n)} \) and \( T_M^{(n)} \) reflects the distribution’s high energy tail. For an exponential tail \( p(e) \sim \exp(-e/T_B) \), \( T_B^{(\infty)} = T_B^{(\infty)} = T_B \). For a stretched exponential tail, \( p(e) \sim \exp(-\alpha e^n) \), \( T_B^{(n)} = T_B^{(\infty)} = 0 \) if \( \alpha > 1 \), and \( T_B^{(\infty)} = T_B^{(\infty)} = \infty \) if \( \alpha < 1 \); For a power-law tail \( T_B^{(n)} \sim \infty = \text{while} \ T_B^{(\infty)} \) is undefined.

We now show that the energy moments in our model are consistent with an exponential high energy tail with a decay constant corresponding to the bath temperature \( T_B \). Since \( 0 \leq \alpha \leq 1 \) for \( n \gg 1 \), \( A_n \approx n \) [see Eq. (6)]. Furthermore, for \( 0 < f \leq 1 \), since \( 0 < \alpha < 1 \), we have \( (1-f)n^n \langle e^{n-m} \rangle \ll \langle e^n B \rangle \), and Eq. (7) reduces to

\[
\langle e^n \rangle \approx f(n-1)! \sum_{m=0}^{n-1} \frac{\langle e^m \rangle}{m!} T_B^{(n-m)},
\]

where the term \( f\langle e^m \rangle \) in Eq. (7) has been incorporated as the \( m = 0 \) term in the summation in Eq. (10). The solution for large \( n \) may be approximated by taking the...
continuum limit, where the sum in Eq. (10) transform to an integral, and the resulting equation may be solved to yield

$$\langle e^n \rangle \approx Cn!T_B^n n^{-1}$$

with $C$ a dimensionless constant independent of $n$. Therefore, in the large $n$ limit $T_R^{(n)}$ and $T_M^{(n)}$ both converge to $T_B$,

$$T_R^{(n)} = T_B \left( \frac{n}{n-1} \right)^{f-1} \to T_B$$

$$T_M^{(n)} = T_B (Cn^{f-1})^{1/n} \to T_B$$

as can also be seen in Fig. 4. This supports the observation that the high energy tail behaves as $\exp(-e/T_B)$.

It is intriguing to speculate on the generality of this result, that for very high energies the distribution behaves as an equilibrium distribution with a temperature equal to the bath temperature $T_B$ (see also [16] where similar results have been found for a granular gas driven by an ideal gas heat bath).

D. Approach to Steady State

We solve Eq. (5) recursively with $n$, and find that the time-dependent solution is of the form

$$\langle e^n(t) \rangle = \langle e^n \rangle + \sum_{m=1}^{\infty} C_{n,m} \exp \left( -\frac{A_m \Gamma}{n+1} \right),$$

where $\langle e^n \rangle$ are the steady state moments given by Eq. (7), and $\{C_{n,m}\}$ are constants depending on the initial distribution. $A_m$ is discrete and increases monotonically with $m$, therefore the slowest exponential decay with time, $\exp(-A_1 \Gamma/2)$, dominates the long time behavior of all moments. When scaling high moments to units of energy, one has

$$T_M^{(n)} \sim \langle e^n(t) \rangle^{1/n} \sim \exp \left( -\frac{A_1 \Gamma}{2n} \right).$$

Thus the high energy components of the distribution approach their steady values slower than the low energy ones. It is interesting to note that this is similar to the analysis of [34] for the approach to equilibrium in plasmas.

Figure 5 provides the temporal evolution of the energy distribution starting from two different initial conditions. In both cases the system started with all particles having the same energy, $e_0 = 0$ in one case, and $e_0 = 5T_B$ in the other, and reaches its stationary distribution within several interactions per particle. For $e_0 = 0$ this approach is uniform and faster. For $e_0 = 5T_B$ the system needs a slightly longer time until it arrives to the steady state, and the distribution behaves differently on both sides of $e_0$. For $e > e_0$ single particle interactions with the bath dominate, and the system exhibits the bath dominated exponential tail. For $e < e_0$, on the other hand, the two-body dissipative interactions require multiple collisions in order to change the shape of the distribution continuously from the initial delta function to the smooth steady state distribution.

IV. EFFECTIVE TEMPERATURES

Various definitions of effective temperatures are used to characterize systems far from equilibrium. One of the important questions to be answered in this context is to what extent do effective temperatures defined by different measurements on a single system yield the same numerical value. In Sec. III we first defined the granular temperature as the average energy per degree of freedom $T_G \equiv \langle e \rangle$, and then used the energy moments to define two generalizations of $T_G$ to the hierarchies $T_R$ and $T_M$ of effective temperatures [Eq. (9)]. For a non-exponential energy distribution these generally differ and moreover depend on $n$ (see Fig. 4). In this section we investigate the interrelations between three additional definitions of effective temperatures - the fluctuation temperature $T_F$, the entropic temperature $T_S$, and the configurational temperature $T_C$. 

FIG. 5: (Color online) Temporal evolution to the steady state energy distribution (dashed line) for restitution coefficient $\alpha = 0.5$ and coupling strength $f = 0.5$ obtained in a numerical simulation of the model, starting with all particles having energy $e_0 = 0$ (ascending lines after $N$ and $2N$ interactions in the system), or $e_0 = 5T_B$ (descending lines after $N$, $2N$, $3N$, $4N$, and $5N$ interactions).
A. Fluctuation Temperature

In equilibrium the fluctuation-dissipation theorem may be used to deduce a system’s temperature from the spontaneous fluctuations of any of its physical quantities. The theorem assures that such a fluctuation is equal to the temperature multiplied by the corresponding response function or susceptibility. As such, fluctuations and susceptibilities of various quantities in non-equilibrium systems may be similarly used to define effective temperatures. Here we calculate two such fluctuation temperatures and show that they generally differ one from the other as well as from the granular temperature $T_G$ and bath temperature $T_B$.

To measure fluctuation-dissipation relations, we add degrees of freedom $\{x_i\}$ to our model, whose correlations may be measured, and upon which a response measurement may be performed. To this end we couple the $\{x_i\}$ to an external field $F_i$. That is, we modify the total energy of particle $i$ to $e_i - x_i F_i$, and refer to $\{e_i\}$ as “kinetic” energies (or the system) and to $\{-x_i F(t)\}$ as “internal” energies (or the probe). (This probe is added only for measuring fluctuations and susceptibilities as described in this section; all other sections of the Paper deal with the model defined in Sec. II without the “internal” degrees of freedom.)

We assume the driven dissipative dynamics defined in Sec. II for the system together with non-dissipative exchange of “kinetic” and “internal” energy, as described by Fig. 6. This yields the following stochastic equations of motion:

$$e_i(t + dt) = \begin{cases} \text{Value:} & e_i(t) \\ \text{Probability:} & \frac{1}{1 - \Gamma dt} (15a) \end{cases}$$

$$x_i(t + dt) = \begin{cases} \text{Value:} & x_i(t) \\ \text{Probability:} & \frac{1}{1 - \Gamma dt} \frac{f}{h\Gamma dt} (15b) \end{cases}$$

where $h$ is a parameter introduced to describe the coupling strength between the system and the probe: $0 < h < 1$ is the fraction of the system’s interactions with the probe out of all its non-dissipative interactions (with the bath and with the probe), and $k \in \{1, \ldots, N\}$ is the index of the particle with which particle $i$ may interact in a “kinetic”–“internal” interaction [this is denoted differently from the index $j$ of the second particle in a “kinetic”–“kinetic” interaction, since $e_i$ may not interact with $e_j$ (thus $j \neq i$), while $e_i$ and $x_i$ may interact ($k$ may take the value $i$)].

We examine two measurements testing the relation between steady state fluctuation and response. First, we consider the fluctuation $\langle \Delta x^2 \rangle \equiv \langle x_i^2 \rangle - \langle x_i \rangle^2$ of a single particle’s $x_i$, and its response $r$ with respect to a change in $F_i$, $r \equiv \partial \langle x_i \rangle / \partial F_i$. Second, we define the total system’s $X \equiv \sum_{i=1}^{N} x_i$ and consider the relation between its fluctuation $\langle \Delta X^2 \rangle \equiv \langle X^2 \rangle - \langle X \rangle^2$, and its response $R$ with respect to a change in the uniform field $F, R \equiv \partial \langle X \rangle / \partial F$. (In analogy with spin systems, $x_i$ may be thought of as a single site magnetization and $X$ as the total system magnetization, with $\langle \Delta x^2 \rangle$, $\langle \Delta X^2 \rangle$, $r$ and $R$ the corresponding fluctuations and susceptibilities.)

The fluctuation-dissipation theorem relates these in equilibrium by $\langle \Delta x^2 \rangle = r \cdot T$ and $\langle \Delta X^2 \rangle = R \cdot T$, and inspires the definition in non-equilibrium systems of effective fluctuation temperatures $T_F^{(1)} \equiv \langle \Delta x^2 \rangle / r$ and $T_F^{(N)} \equiv \langle \Delta X^2 \rangle / R$ for single-particle and many-particle measurements, respectively. In this section we calculate $T_F^{(1)}$ and $T_F^{(N)}$ for our model and demonstrate that they generally differ one from each other and from both $T_G$ and $T_B$. As in [26] we concentrate here on space- and time-independent fluctuation-dissipation relations; their temporal dependence has recently been investigated in [36] and the spatial dependence in [37].

By averaging Eq. (15) and taking the steady state solution we see that

$$\langle x_i \rangle = \frac{\langle X \rangle}{N} = -\frac{\langle e_i \rangle}{F},$$

with $\langle e_i \rangle = T_B f(1 - h)/(\alpha_1 - f h)$ [this reduces to Eq. (4) in the $h \to 0$ limit]. Therefore,

$$r = \frac{R}{N} = \frac{\langle e_i \rangle}{F}.$$  

For the steady state averaged second moments we obtain,

$$\langle \Delta x^2 \rangle = \frac{\langle e_i^2 \rangle}{2 F^2} = b(\alpha, f, h, N) \cdot r \cdot T_G,$$

$$\langle \Delta X^2 \rangle = N \langle \Delta x^2 \rangle - \frac{\langle X \rangle^2 \langle E \rangle}{F} = b(\alpha, f, h, N) \cdot R \cdot T_G,$$

where $E \equiv \sum_{i=1}^{N} e_i$, and $b$ and $B$ are dimensionless functions of the dimensionless model parameters and of the system size. We are interested in the thermodynamic limit ($N \gg 1$), for which $b$ and $B$ reduce to the expressions given in [38].

$r$, $R$, $\langle \Delta x^2 \rangle$ and $\langle \Delta X \rangle$ all diverge as $F \to 0$ (where fluctuation-dissipation relations are normally measured), however their ratios define effective temperatures

$$T_F^{(1)} \equiv \frac{\langle \Delta x^2 \rangle}{r} = b(\alpha, f, h) T_G;$$

FIG. 6: The interactions between the system $\{e_i\}$ and the probe $\{x_i\}$, introduced in addition to those given in Fig. 1.
which are finite and independent of $F$. These fluctuation temperatures generally differ one from the other, are larger than the granular temperature $T_G \equiv \langle \epsilon \rangle$ and smaller than the bath temperature $T_B$. Only in the equilibrium limits ($\alpha = 1$ and $f = 1$) do all effective temperatures coincide with $T_B$.

$T_F^{(N)}$ is generally larger than $T_F^{(1)}$, but in the limit studied in [39] of weak coupling between the system and the probe $(h \to 0)$ the two coincide, and the expression for them simplifies to that given in [39]. Nonetheless, they differ from the granular temperature, as shown in Fig. 7. The difference between the two fluctuation temperatures is most prominent in the maximal dissipation limit ($\alpha = 0$), where they reduce to the expressions given in [40]. The ratio between them for this case is shown in Fig. 8.

The single-particle fluctuation temperature $T_F^{(1)}$ directly probes the second moment of the single-particle energy distribution and thus gives the effective temperature $T_F^{(0)}$ defined in Sec. IV [compare Eqs. (16-19) to Eq. (9)]. The many-particle fluctuation temperature $T_F^{(N)}$, on the other hand, is defined by a measurement on the entire system, thus reflects correlations between particle energies and cannot be related directly to the effective temperatures defined from the single-particle energy distribution.

In dissipative systems with strong coupling to the driving mechanism ($f \approx 1$) and large restitution coefficient ($\alpha \approx 1$) the energy distribution is close to exponential, and correlations are weak, hence the values of $T_F^{(N)}$, $T_F^{(1)}$ and $T_G$ are similar (but not identical). As has been predicted by kinetic theory [9], we expect the fluctuation temperatures to be larger than the granular temperature (see Fig. 7) in granular gases as well, where the energy distribution is non-exponential. In the cases studied numerically [7, 8] the energy distributions were only slightly non-exponential, resulting in small differences between the effective temperatures, which may explain their seeming coincidence.

**B. Entropic Temperature**

In analogy to equilibrium statistical mechanics, a further definition of an effective temperature in non-equilibrium systems may be constructed by differentiating the system’s entropy $S$ with respect to its average total energy $\langle E \rangle$, yielding the entropic temperature $T_S \equiv (\partial S / \partial \langle E \rangle)^{-1}$ [11, 13, 17, 18]. It is intriguing to inquire as to how this quantity relates to other effective temperatures of the system and whether it has a fundamental thermodynamic-like significance, stemming from a maximization of entropy upon contact between systems.

In this section we calculate $T_S$ for the continuous energy model defined in Sec. II, as well as for a discrete energy version (described below). We first identify simple scaling arguments leading to the coincidence of $T_S$ with the granular temperature $T_G \equiv \langle \epsilon \rangle$. This scaling holds for the continuous energy model with $\alpha > 0$. We then demonstrate the breakdown of this scaling both in the singular limit of the continuous energy model at $\alpha = 0$, which exhibits a condensation at $\epsilon = 0$, and by introducing a discrete energy version of the model, where the structure of the particles’ energetic levels contains an additional energy scale $\epsilon$.
1. Scaling

For systems coupled to a heat bath of temperature $T_B$ with no internal energy scale characterizing the system’s structure (such as an interaction energy or energy spacing between possible states) the only energy scale in the system is $T_B$. When $T_B$ is varied all energies in the system change linearly with its change. This is the case for the model defined in Sec. II. Different effective temperatures, or energy characteristics of the system, may differ, however they all scale linearly with $T_B$. The ratios between effective temperatures are dimensionless numbers depending on the dimensionless model parameters.

Since the $N$-particle energy distribution has dimension of inverse energy to the $N$’th power, as long as it is non-singular it scales as

$$P(e_1, \ldots, e_N) = \varphi\left(\frac{e_1}{T_B}, \ldots, \frac{e_N}{T_B}; \alpha, f\right) \frac{1}{T_B^N},$$

(20)

with $\varphi$ a dimensionless function of the dimensionless energies $\{e_i/T_B\}$ and the dimensionless model parameters ($\alpha$ and $f$ in our case).

We define the system’s entropy as

$$S \equiv -\int P(e_1, \ldots, e_N) \ln P(e_1, \ldots, e_N) de_1 \cdots de_N. \quad (21)$$

After changing the integration variables from $\{e_i\}$ to $\{e_i/T_B\}$, and using the normalization of $P(e_1, \ldots, e_N)$, we see that

$$S = N \ln T_B + \text{const.}, \quad (22)$$

with the additive constant depending only on the dimensionless parameters $\alpha$ and $f$, but not on $T_B$. Since the energy of the system $\langle E \rangle$ scales linearly with $T_B$ [see Eq. (4)], we may write

$$S = N \ln \langle E \rangle + \text{const.}, \quad (23)$$

and conclude that

$$T_S \equiv \left(\frac{\partial S}{\partial \langle E \rangle}\right)^{-1} = \frac{\langle E \rangle}{N} \equiv T_G. \quad (24)$$

The functional form of the dimensionless distribution $\varphi(e_1/T_B, \ldots, e_N/T_B)$, which depends on the model parameters, manifests itself only in the additive term in the entropy, and does not affect the relation $T_S = T_G$.

In order to observe richer behavior we turn to models with additional energy scales. We shall demonstrate this using a definition of $T_S$ which is simpler to calculate, and which for the single energy scale case exactly coincides with the calculation given above. Instead of defining the entire system’s entropy $S$ from the entire system’s energy distribution [Eq. (21)], we consider a subsystem comprised of a single particle and measure its entropic temperature. From the single-particle energy distribution $p(\epsilon)$, we define the single-particle entropy as

$$s \equiv -\int p(\epsilon) \ln p(\epsilon) d\epsilon, \quad (25)$$

and differentiate it with respect to the particle’s average energy: $T_S \equiv (\partial s/\partial \langle \epsilon \rangle)^{-1}$. Since all particles have the same single-particle energy distribution, $T_S$ is clearly equal for all particles in the system. The aforementioned scaling argument holds for the single-particle distribution, thus for systems with a single energy scale the single-particle definition gives $T_S = T_G$ as well.

2. Condensation in the Maximal Dissipation Model

For the maximally dissipative limit ($\alpha = 0$) of our model, every particle undergoing an interaction with another particle in the system is left with zero energy after the interaction, and the system exhibits a condensation at $\epsilon = 0$. Since in the steady state for $1 - f$ of the particles the last interaction was such an energy draining interaction and not an interaction with the bath, a fraction $1 - f$ of the particles have zero energy, and the single-particle energy distribution has the general form

$$p(\epsilon) = (1 - f)\delta(\epsilon) + f \varphi\left(\frac{\epsilon}{T_B}, f\right). \quad (26)$$

Due to the normalization of $p(\epsilon)$, this singular distribution yields an entropy of the form

$$s = - (1 - f) \ln (1 - f) + f \ln T_B + \text{const.}. \quad (27)$$

The average energy per particle $\langle \epsilon \rangle$ scales linearly with $T_B$. Therefore,

$$s = f \ln \langle \epsilon \rangle + \text{const.}, \quad (28)$$

and consequently $T_S = T_G/f$, which is larger than $T_G$.

3. Discrete Energy Model

Another way to break the scaling arguments leading to $T_S = T_G$ is by introducing an additional energy scale to the model. We now consider every particle as a quantum harmonic oscillator with possible energies $\epsilon_i = 0, \epsilon, 2\epsilon, \ldots$. The spacing $\epsilon$ between states constitutes the energy scale which invalidates the scaling arguments presented above. (Even in thermodynamic equilibrium, $T_S \equiv T$ is the equilibrium temperature, while the average energy $\langle \epsilon \rangle = \epsilon/(\exp(\epsilon/T) - 1)$ differs from the temperature, and $\langle \epsilon \rangle \gg T$ only in the continuum limit $\langle \epsilon \rangle \gg \epsilon$.)

The dynamics of the discrete model are as follows: The bath is constructed from similar quantum harmonic oscillators in equilibrium at temperature $T_B$, thus with energies distributed as $p_B(n\epsilon) \sim \exp(-n\epsilon/T_B)$. In the interaction of particle $i$ from the system with a particle
of energy $e_B$ from the bath their total energy $e_i + e_B$ is conservatively redistributed between them, by randomly choosing with equal probability a new energy $e_i' \in \{0, \epsilon, 2\epsilon, \ldots, e_i + e_B\}$. In a dissipative interaction between particles $i$ and $j$ of the system, each energy “quantum” $\epsilon$ of the total energy $e_T = e_i + e_j$ has a probability $\alpha$ to interact in the same way as $e_i$ and $e_j$ and a probability $1 - \alpha$ to be dissipated out of the system. The remaining energy $e_T'$ is then randomly redistributed between the two particles with equal probability for every outcome $e_i' \in \{0, \epsilon, 2\epsilon, \ldots, e_T\}$, and $e_j' = e_T' - e_i'$ [41].

We numerically solved this discrete energy model by Monte-Carlo simulation and obtained the average energy as well as the energy distribution $p_n \equiv \rho(n)$, from which the entropy $s = -\sum_{n=0}^{\infty} p_n \ln p_n$ was calculated. We kept the restitution coefficient $\alpha$, the coupling strength $f$, and the energy spacing $\epsilon$ fixed and scanned the bath temperature $T_B$ in order to obtain the dependence of entropy on the average energy $\langle e \rangle$ for given $\alpha$, $f$ and $\epsilon$. We compare this functional behavior to the corresponding equilibrium behavior (i.e., conservative interactions), where the entropy and average energy of a quantum harmonic oscillator are related by

$$s^{eq}(\langle e \rangle) = \ln \left( \frac{\langle e \rangle}{\epsilon} + 1 \right) + \frac{\langle e \rangle}{\epsilon} \ln \left( \frac{\epsilon}{\langle e \rangle} + 1 \right).$$

We subtracted $s^{eq}(\langle e \rangle)$ from the numerically obtained $s(\langle e \rangle)$ to yield the deviation from equilibrium behavior of dissipative systems displayed in Fig. 9.

For $\langle e \rangle \gg \epsilon$ the discrete energy model coincides with the continuous energy one: For $\alpha > 0$ the entropy of the dissipative system in this region can be seen to merely be smaller by an additive constant from the entropy of an equilibrium system with the same energy, as expected from the scaling arguments for a system with a single energy scale [Eq. (29)]. For $\alpha = 0$, on the other hand, the entropy deviation grows with energy, and $T_S$ differs from $T_G$ even in this high energy limit.

We numerically differentiated $s(\langle e \rangle)$ with respect to $\langle e \rangle$ to obtain $T_S$, and in Fig. 10 compare the functional dependence of $T_S$ on $\langle e \rangle$ to the corresponding equilibrium behavior. That is, for every value of $\langle e \rangle$ we normalize $T_S$ by the temperature $T_B^{eq}$ required to give this average energy, were the system in equilibrium. Since the entropy of a dissipative system is smaller than that of an equilibrium system with the same average energy, $T_S(\langle e \rangle)$ is generally larger than $T_B^{eq}(\langle e \rangle)$. For $\alpha > 0$, $T_S$ behaves as in equilibrium in the two extremes of very high and very low energy, and exhibits non-equilibrium behavior only for intermediate energies ($\langle e \rangle \approx \epsilon$). In the high energy limit ($\langle e \rangle \gg \epsilon$), the discrete energy model is equivalent to the continuous energy one, thus $T_S$ behaves as in equilibrium for $\alpha > 0$, and reaches a value larger by a factor $1/f$ from the equilibrium value for the singular limit $\alpha = 0$.

For very low energies the system behaves as a two level system, irrespective of whether it is in equilibrium or not. For the system to have such a low average energy almost all particles must be in the ground state ($\langle e \rangle = 0$), and since the occupation of states rapidly decays with energy, only the first excited state ($\langle e \rangle = \epsilon$) is relevant, while states of higher energy have a negligible occupation. The energy distribution in a two level system is characterized by a single number (the ratio of occupation of the two states) and is hence not rich enough to exhibit any features of a non-exponential energy distribution. For intermediate energies the entropic temperature exhibits significant deviations from equilibrium behavior even for $\alpha > 0$, as clearly seen in Fig. 10.
C. Configurational Temperature

The hierarchy of effective temperatures given by \( T_R^{(n)} \) [Eq. (9a)] may be related to the hierarchy of the so-called hyperconfigurational temperatures. A recent extension of the virial theorem, states that for a system in thermodynamic equilibrium at temperature \( T \) with dynamics stemming from a Hamiltonian \( \mathcal{H} \{ p_i, q_i \} \) (1 \( \leq i \leq N \)) the following relation holds [23]:

\[
\langle \nabla \mathcal{H} \cdot \mathbf{B} \rangle \left( \nabla \cdot \mathbf{B} \right) = T,
\]

(30)

where \( \nabla \equiv (\partial/\partial p_1, ..., \partial/\partial p_N, \partial/\partial q_1, ..., \partial/\partial q_N) \) represents differentiation with respect to all phase space coordinates, and \( \mathbf{B} \) is an arbitrary vector field in phase space satisfying \( 0 < |\nabla \mathcal{H} \cdot \mathbf{B}| < \infty \), \( 0 < |\nabla \cdot \mathbf{B}| < \infty \), with \( \nabla \mathcal{H} \cdot \mathbf{B} \) growing slower than \( e^N \).

If the Hamiltonian is of the form \( \mathcal{H} = \sum_{i=1}^{N} p_i^2 + V \{ q_i \} \), that is, comprised of a kinetic term depending only on momenta and a potential term depending only on coordinates, it is useful to take \( \mathbf{B} = \nabla V \). This yields a relation between the temperature and ensemble averages of solely the particle locations, without the need of measuring momenta. This may be used to define a configurational effective temperature in non-equilibrium systems,

\[
T_C \equiv \frac{\langle |\nabla V|^2 \rangle}{\langle \nabla^2 V \rangle}.
\]

(31)

The hierarchy of hyperconfigurational temperatures \( T_C^{(n)} \) [24] generalizes this by taking \( \mathbf{B} = (0, ..., 0, (\partial V/\partial q_1)^n, ..., (\partial V/\partial q_N)^n) \), so that

\[
T_C^{(n)} = \frac{\sum_{i=1}^{N} \left( \frac{\partial V}{\partial q_i} \right)^n}{n \sum_{i=1}^{N} \left( \frac{\partial V}{\partial q_i} \right)^n \frac{\partial^2 V}{\partial q_i^2}}.
\]

(32)

In order to interpret \( T_C^{(n)} \) for our case we note that the model’s dynamics in the non-dissipative case (\( \alpha = 1 \)) manifest a uniform single-particle density of states [35]. That is, in the equilibrium limit our system is equivalent to a collection of weakly interacting harmonic oscillators, with \( c_i = p_i^2 / 2 + q_i^2 / 2 \), where \( \{ q_i \} \) and \( \{ p_i \} \) are some hidden coordinates and momenta. Each \( q_i \) and \( p_i \) change periodically with time as for an isolated harmonic oscillator with energy \( c_i \). Occasionally (that is, at a frequency much smaller than the oscillator’s frequency) this energy is changed due to an interaction with some other particle or with the bath. It is natural to extend this description to dissipative (\( \alpha < 1 \)) cases as well, thus we consider the case where \( V \{ q_i \} = \sum_{i=1}^{N} q_i^2 / 2 \).

For a harmonic oscillator of given energy \( c \), the temporal average of \( q_i^2 \) over many periods of oscillation is \( e^n (2n - 1)! / n! \). The particle’s energy changes with time due to interactions, thus we average over its steady state distribution, and obtain \( \langle q_i^2 \rangle = \langle c \rangle (2n - 1)! / n! \). Upon substitution in Eq. (32) we see that the hyperconfigurational temperatures probe ratios between succeeding moments of the energy distribution, and may thus be related to \( q_R^{(n)} \) [defined in Eq. (9)]

\[
T_C^{(2n-1)} = \frac{\langle q_i^{2n} \rangle}{(2n - 1) \langle q_i^{2n-2} \rangle} = \frac{\langle c \rangle^{(n)}}{\langle c \rangle^{(n-1)}} = T_R^{(n)}.
\]

(33)

\( T_C^{(2n)} \) is undefined since averages of odd moments of \( q \) vanish.

This demonstrates for one particular modeling the connection between the configurational temperature \( T_C \) and the single particle energy distribution. We suggest that similar relations hold in other systems, and that the hyperconfigurational temperatures may be used to characterize the non-equilibrium nature of energy distributions in general.

V. DISCUSSION

This Paper investigates several non-equilibrium phenomena observed in a minimal stochastic model for driven dissipative dynamics. Our model is inspired by granular gases, nevertheless we believe it may be relevant to a broader class of driven dissipative systems. The model is simple enough to admit an exact solution of the single-particle distribution in terms of its moments, in the steady state as well as during the evolution from any initial condition to this state. When considering particles with energies slightly larger than the average energy in the system, the high energy tail of the single-particle distribution is seemingly overpopulated, as has been found in granular gases. However, we have calculated the very high energy tail and found that it behaves exponentially with a decay rate corresponding to the temperature of the driving bath \( T_R \). Generally, very high energy tails manifest the bath distribution since dissipative interactions within the system (where typical energies are smaller than in the bath) hardly affect this tail. It will be interesting to investigate the tail of very high energies in other dissipative system, and to test whether they agree with the bath distributions (as has been found in [16])

Due to the non-exponential energy distributions of non-equilibrium systems, the average energy, or granular temperature \( T_C \), is just one energy scale characterizing the system. Higher moments of the distribution define hierarchies of effective temperatures \( T_R^{(n)} \) and \( T_M^{(n)} \), which generally vary with the order \( n \) of the moments, and coincide with the actual temperature if the system is in equilibrium. The large \( n \) limit of these effective temperatures relates to the high energy tail of the distribution. In our model these effective temperatures converge to the bath temperature in this limit, reflecting the tail’s exponential behavior. Furthermore, we related these effective
temperatures to the hyperconfigurational temperatures $T_{CN}^{(n)}$, defined in Hamiltonian system from particle locations.

Fluctuation-dissipation measurements characterize aspects of the second moment of the energy distribution, and thus show an effective temperature generally differing from thegranular temperature (or first moment). Here we have extended the calculations presented in [26] to general coupling strength $A$ between the system and the probe, and moreover showed that single-particle and many-particle measurements yield different effective temperatures $T_{F}^{(n)}$ and $T_{F}^{(1)}$. These results, together with those dealing with time dependent measurements presented in [36], call for examining these phenomena in more realistic dissipative systems.

Finally, we considered the entropic temperature $T_{S}$ obtained by differentiating the entropy with respect to energy. We showed that generally for a non-singular system without an internal energy scale, simple scaling arguments lead to the exact coincidence of the entropic temperature with the granular temperature. When this scaling is not valid, $T_{S}$ is generally larger than the corresponding equilibrium value. It is intriguing to test this scaling and its breakdown in the more complex systems where $T_{S}$ can be measured.

We can identify an ordering of effective temperatures in our model: For the hierarchies defined from the energy moments, both $T_{R}^{(n)}$ and $T_{H}^{(n)}$ grow with $n$; The fluctuation temperatures $T_{F}$ are larger than the granular temperature $T_{G}$; The many-particle fluctuation temperature $T_{F}^{(n)}$ is larger than the single-particle one $T_{F}^{(1)}$; All these effective temperatures are smaller than the bath temperature $T_{B}$. It will be interesting to see whether such ordering occurs in other driven dissipative systems as well.

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[38] For $N \gg 1$ we obtain

$$\delta = \left\{ f(k^2 + 2h - 1) + 2(1 - 3h) + A_1/f\right\} A_1$$
\[
+ (1 - h)(3 - A_2)/[(1 - h)(2A_2 - fh)],
B = f h [(18(1 - h) - 3f(4 - h - 7h^2) + f^2(2 + 3h - 15h^2 - 6h^3)] + 2[15 - 27h
- f(5 - 6h - 21h^2)] A_1 - [18(1 - h) + f(3h + 1)(10h - 2)] A_2 - 2(5 - 9h)A_1 A_2
+ A_2^2[9 - 3h - f(3 - 10h + 21h^2)] - (3 - h)A_2
+ 6f/A_1 + f(1 - 2h)/A_1]
/[(6(1 - h)A_1 - fh)(A_1 + fh)(2A_2 - fh)].
\]

[39] For \( h \to 0 \) we obtain
\[
b = B = [(4 - f)(1 - f + 2A_1) - (4 - 3f)A_2] / (2fA_2).
\]

[40] For \( \alpha = 0 \) we obtain
\[
b = [4(2 - f) + fh(f(h + 7) - 12)] /
\{ f(1 - h)[6 - f(2 + h)] \},
B = [32 - 16f(3 + 2h) - 2f^2(12 + 27h - 11h^2)
- f^3(1 - h)(4 + 28h - 13h^2) + f^4(7 + h)(1 - h - h^2)]
/\{ f(1 - h)[6 - f(2 + h)][(2 - f)^2 - f^2h^2]\}.
\]

[41] Note that energy “quanta” are dissipated as if they were
distinguishable entities, whereas for redistributing the re-
maining energy they are treated as indistinguishable.
8 Discussion

Granular materials are an interesting example of complex classical systems, which exhibit a growing number of phenomena and features observed in experiments and in numerical simulations, but not well understood theoretically. This Thesis explores thermodynamic analogies for describing the statistical mechanics of jammed and dissipative systems, and refers to granular solids and gases as visual examples for such non-equilibrium systems.

We isolated the effects of jamming and dissipation by considering the extreme limits of static dense granular packing and of dilute dynamical granular gases. For jammed systems we assumed Edwards’ thermodynamic description [6], and tested the consequences drawn from it. For dissipative systems, on the other hand, we tested various aspects of thermodynamic behavior by analyzing the emerging behavior of a system from its dynamical rules.

8.1 Jammed Systems

In Section 3 [119] we generalized Edwards’ thermodynamic analogy to include the role of friction, and studied compaction and segregation of jammed granular solids. For this, we constructed a simple mechanical model, incorporating the effect of friction on the local volume and entropy of the packing. Together with original mean-field geometrical calculations, and within the framework of Edwards’ thermodynamic assumption, this model enabled calculating the global behavior of a granular solid from its local properties. In the case of identical friction for all grains, systems comprised of smoother grains were found to compact more than those comprised of rougher ones. For mixtures of grains differing in friction a phase diagram for segregation was constructed. Moreover, the compactivity (which to date is inaccessible experimentally) has been eliminated from the results, by using the resulting volume fraction (which may easily be measured experimentally) as a scale for compaction.

This work provides simple geometrical and mechanical models for testing Edwards’ thermodynamic hypothesis, and to our knowledge, is the first to suggest eliminating the compactivity. We elucidated the role of friction in compaction and segregation of static (or tapped) granular packings. This lead to an experimental demonstration [125] of different volume fractions due to friction differences in systems undergoing the same shaking protocol. Following our theoretical observations on segregation due to friction differences, such differences have been minimized in an experimental and numerical study [126]. Segregation due to friction differences was then demonstrated in numerical simulations [127].
The ideas we presented call for experimental or numerical investigation of both the dependence of compaction on friction, and of segregation due to frictional differences between grains. Our procedure of eliminating the compactivity may be useful in expressing theoretical results obtained within Edwards’ description in terms of experimentally measurable quantities, as opposed to the yet unclear notion of compactivity.

8.2 Driven Dissipative Systems

In Section 4 [120] we introduced a simple but powerful stochastic model to describe driven dissipative systems. The model incorporates the essential features of a dissipative system, namely energy transfer and loss upon particle interactions. This is done with maximal randomness. That is, any pair of particles in the system has an equal probability to interact, and whenever two particles interact, they randomly redistribute the undissipated energy between them. Driving is similarly performed by similar conservative interactions with an equilibrium heat bath.

Our model, together with the variety of exact results calculated for it, focuses attention at the statistical mechanics of driven dissipative system. It provides a simple framework for theoretically investigating systems far from equilibrium. Below, we overview our central results presented in detail in Sections 4 [120], 5 [122], 6 [123], and 7 [124], comment on their relation to other driven dissipative systems, and suggest directions for continuing this research.

Much of our results are expected to be applicable to more realistic driven dissipative models and to experiments, and one of the central questions we pose is the generality of our results. We tested some aspects of this by comparison to numerical simulations of granular gases [120]. Furthermore, we suggested adding spatial structure to our model by placing the particles on a lattice. This variant of our model has been investigated in [128]. We are currently testing some of the results obtained within our model on models for granular gases. Furthermore, we are searching similar results in experimental studies of colloidal suspensions and bacterial baths as additional prototypes of driven dissipative systems.
8.2.1 Energy Distribution

In Section 4 [120] we calculated the single particle energy distribution and demonstrated its deviation from an equilibrium-like distribution. This may serve to test whether a system is effectively in equilibrium or not. Driven dissipative systems typically may not be mapped onto equilibrium states, and possess non-Boltzmann energy distributions.

In Section 7 [124] we showed that the high energy tail of the distribution is proportional to the high energy tail in the driving bath. This arises from the fact that in order for a particle to reach a very high energy it must receive most of its energy from the bath, and therefore the probability for such an event is proportional to the probability of having such high energies in the bath. A similar result was found in numerical simulations of a granular gas driven by an equilibrium bath [64], where the high energy tail behaved as in the driving bath. This leads us to the speculation that high energy tails in dissipative systems driven by equilibrium baths tend to behave as the high energy tail in the bath. This speculation should be tested in other systems and the conditions for its validity should be identified.

Much of the interest in driven dissipative systems is in their steady states, and, in particular, in comparing these states to thermal equilibrium. Our solution of the energy distribution is not limited to the steady state. In Section 7 [124] we considered the approach to steady state, and showed that high energy components of the distribution reach their steady values slower than the low energy components. It will be interesting to consider the approach to steady state in other driven dissipative systems as well, and to inquire as to the generality of the results obtained within our model.

8.2.2 Time Dependent Fluctuation-Dissipation Relations

We used our model to investigate time-dependent fluctuation-dissipation relations about non-equilibrium steady states. Most investigations of fluctuation-dissipation relations were limited to the issue of the definitions and possible coincidence of different effective temperatures. In Section 6 [123] we addressed a more fundamental aspect of the equilibrium fluctuation-dissipation theorem and questioned whether two-time correlation functions and the corresponding delayed response functions have the same functional dependence on the waiting time (or measurement time). To our knowledge, we were the first to theoretically demonstrate that correlation and response functions generally have different time dependences in driven dissipative systems.
By investigating the origin of this result we identified that the same time scales enter the correlation and response functions. In simple measurements, there is a single time scale in the correlation and response, therefore these functions have the same temporal dependence. In general measurements, with multiple time scales, the correlation and response functions may depend differently on time. However, since they share the same time scales they approach the same asymptotic dependence for long waiting times. We expect this behavior to hold for a very wide class of non-equilibrium systems. These results call for further investigation of time-dependent fluctuation-dissipation relations about non-equilibrium steady states in other systems, and to the formulation of the general requirements for these results to hold in other systems.

8.2.3 Effective Temperatures

Another controversial issue we addressed is effective temperatures in driven dissipative systems. A major unanswered question related to effective temperatures is whether different definitions of effective temperatures on a single system coincide. We calculated several effective temperatures on our model, and demonstrated that they generally differ. We calculated the granular temperature defined as the average energy per degree of freedom, the fluctuation temperature scaling fluctuations in the system, the entropic temperature from differentiating entropy with respect to energy, the configurational temperature from averages of particle locations, and hierarchies of effective temperatures from moments of the single particle energy distribution.

Although all effective temperatures generally differ, we identified cases where different effective temperatures coincide. In Section 6 [123] we showed that the fluctuation temperature and granular temperature exactly coincide for Green-Kubo measurements on the inelastic Maxwell model. We explained this exact coincidence and propose to test its validity in more realistic models for granular gases as well. In Section 7 [124] we presented a scaling argument predicting the exact coincidence of the entropic temperature and granular temperature. It would be interesting to test this scaling as well as its breakdown in more realistic driven dissipative systems.
We found that in our model effective temperatures are ordered. In our model fluctuation temperatures are always larger than the granular temperature. Multi-particle fluctuation temperatures are larger than single particle ones. Effective temperatures defined from energy moments increase with the order of moments, but are always smaller than the bath temperature, as are all other effective temperatures in the system. We suggest checking whether similar ordering occurs in other driven dissipative systems as well.

8.3 Identifying Non-Equilibrium Systems

Using the identification of the correct density of states, we showed in Section 5 [122] that an alleged non-equilibrium system [121] is in fact effectively in equilibrium. We presented a procedure which may be used in order to distinguish between systems that are effectively in equilibrium and systems that are truly away from equilibrium.

In dissipative systems the dynamics may not be expressed in a form which is symmetric with respect to time reversal. Therefore, dissipative systems may not be mapped onto equilibrium states obeying detailed balance, and are hence truly away from equilibrium.
References

[108] T.C. Lubensky (private communication).
(2005).


We have studied (driven) systems in the context of granular gas mechanics. In these systems, the granular gas provides an example of systems with energy losses occurring in collisions between particles and entering the system through dissipative processes. We have also moved to a level in an attempt to describe the behavior of specific systems in this place. In any case, we have tried to understand the statistical mechanics of dissipative systems through an abstract, more and so we have tried to understand the statistical mechanics of dissipative systems through an abstract, more general framework. The model contains a minimal random model that we propose for the dynamics of these systems. In each interaction, the energy of a pair of particles in the system is lost with a probability different from that of all other researchers. This energy is then shared among the particles in a random way, the particles losing energy with an ambivalent field, and the field that we defined in our model is independent of the energy exchange. We have calculated the entropy at the level of the model for each energy exchange and found that the entropy changes with time in a way depending on the fluctuations of the system. We have also found that the entropy changes with time in a way depending on the fluctuations of the system. We have also found that the entropy changes with time in a way depending on the fluctuations of the system. We have also found that the entropy changes with time in a way depending on the fluctuations of the system. We have also found that the entropy changes with time in a way depending on the fluctuations of the system.
basal theorems and their implications on the microscopic heat conduction. Additional predictions and consequences for the field are explored in the subsequent sections.

The article discusses the implications of the microscopic heat conduction for the macroscopic heat flow in various systems, highlighting the role of microscopic processes in determining the macroscopic behavior.

The study further explores the relationship between microscopic and macroscopic heat conduction, emphasizing the importance of understanding the microscopic mechanisms that govern heat flow in complex systems.
תקציר

החברת אפרואיזים היא לוותר לשחורים וגרגירים במא gestión המכנית.

הסיטטסית. גרגירים וגרגירים מספקים מקום קלאסי lemברנקובר: התהות המכנית של
גרגיר בבד פשיהם של הנחת התא היסוד של עיקוף של גרגירים במברק, וניהל ליצוף
בחזית שעון מחוון את אחר עוקב אל ממבר את התיה.

ההכהות על מטרות גרגרים החוקה לשחורים וגרגירים有机 פאקטוריסטים בחק
לסקאלות התאוריה האטוטו. מחפה kart שמעונתר מחברת מסמס גוי של גרגיר שלוקות
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תרמודומג מיצולתulls החברת את המחונת החומר ומכך התוכן החלקים המרכזים אחד.
ולש בשあり הפאקטוריסטים של הנגרים, נהל להתח יש מאפיינים של תומרים גרגרים, అשר
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לסקאטרוס האמיצית לשחיב (𠂈mayor הצהרה) ואת הלוגיסטית כי היא גישה בחס
לאינוגי המכונה הדوحد הניתוח גרגרים מקארוקספסים. בל, תומרים גרגרים גורחים בליה
לפיור בוחר החתם של גרגרים משלコー וה בנייתו של גרגרים incapacitate על גרגרים
מועטים הצטמצמו ביציב מקוממי. אי לכל, תומרים גרגרים אנוי תוריים
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בניילול את האטוטים האטוטים שלדה. מבילב או על האטוטים לחולה הרבח ייחזקו עם
גרגרים המראים בצעפיה, מבילם שיש לדיר שיכור ממוח של מסמס ודיל של גרגרים התיאורית
מנת בדורות בין יכל לעשת את מיומן המאמאיפי על שומני 앞ילא ממעורר גרגרים דלילת
מליחות בשיוו-משקל תורמודומג או דיסיפיציה, ולא איבד האטוטים בתנאיים המ önüית
בין גרגרים מקארוקספסים. אי לכל, המנור של מיעוטי מבית המכון להכרותה שלמודומגית
ולמכהית היישוע יש לי-מייקל דין להסรอย
הסיטטסית של שיוו-משקול נייא לולוים אחרים.

במוך מכוסי מדארכות גרגרים דמיון (וגם שוני בר) לביצעי העברית השניה. בל, נהל
לעשות את פテスト: דומקינ גרגירים של אריאיזי יינוח שגרגיר בצלעה משקド אריאז
גרגירים מראים ממדיק מLOY בצלעה משקד אריאז בין גזים גרגרים ממדיק מLOY
עניק שמוו erased של היל א以後. (מדיק אריאזי מגרד בהרי חית במקומ הפרסה של גרגרים
לבל י_list של 앞י מעובד בכרוח, זה עץ הילימיות ביכורים.

בוחן זה המביליים בט פייקום בל הידיספסיציה גנוצרת שunes המוניציפות המארכות
גרגרים מראים בשיוו-משקל תורמודומג. על הטעה של תיבת את השפעות של[][גרגיר
פדידיס נאוס ומפתחים בשני מקיר קוצני: יש להארת צפיפות של גרגרים נייחים, תיא.
פייקום משמעתי והשיני של גזים גרגרים דليل, שיש דיסיפיציה נייא לולוים.
רשימת ציורים

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הדוגמאות ל�ורית ניוגון של אראיה גורית ניוגון

1. ניסיון של חומר גורית ניוגון שעובר צифור psz תכנית תכנית
2. תפרימו בחפף מסותבב
3. תפרימו בחפף במדידה מאשר מאפיין
4. תפרימו בחפף מחטאט מתחלית אנכית
5. ניסיון של גורית
6. פילוג המתריעות מניסיון של גורית
7. ניסיון של תרבות גורית
8. מיקומי החלים קיים בaternity נמיה של גורית ממקורר
9. פילוג המתריעות בaternity נמיה של גורית ממקורר
10. מודל המטריצות-דיפיציה בaternity נמיה של גורית ממקורר

1.11 מודעה פלקטואיציה-דיפיציה בaternity נמיה של גורית ממקורר
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3.3 מודל העצמה בתלות-מיני
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4.1 הספวรת הגרגרית פילוג הארגיניות-
4.2 הספวรת הגרגרית פילוג הארגיניות-
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5.10 מיקום הספวรת הגרגרית פילוג הארגיניות-
ברגני להJNIEnv את הערכתי העמוקה לדב ולון. נבעד והלא היה היא את פועל מביי הניחית.

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המחק.
אנלוגיות תרמודינמיות בחומרים גרגליים

חיבר על מחקר

לשם مليולי חלקי של הדרישות לעיבוד החומרים
דוקטור לפילוסופיה

יאיר שוקף

הונג לסטט הטכניות - מכון טכנולוגי לישראל

תמונה משישית

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