Reply to "Comment on 'Green's function theory for infinite and semi-infinite particle chains'"

Y. Hadad and Ben Z. Steinberg*

School of Electrical Engineering, Tel Aviv University, Ramat-Aviv, Tel-Aviv 69978 Israel

(Received 29 May 2012; published 11 July 2012)

Here, we reply to the comment made by Markel and Sarychev regarding our paper on the particle chains Green's function. In particular, we argue that the distinction between discrete and continuous spectra is unique, and the latter represents a novel wave in particle chains.

DOI: 10.1103/PhysRevB.86.037402 PACS number(s): 78.67.Bf, 42.82.Et, 71.45.Gm, 42.25.Bs

For the sake of clarity and in order to avoid convoluted arguments and text citations, below, we address the points raised in the Markel and Sarychev Comment (MSC) in a more general fashion. Clearly, we did not mean to imply that there are errors in the paper by MS but to expose a novel wave phenomenon that has not been discussed so far. The dispute can be distilled to three main points:

(i) Misinterpretation of the continuous spectrum wave as the "extraordinary" wave in MS's paper[1] (a statement made by us in our paper[2]).

(ii) Distinction between discrete and continuous spectra—is it unique, and what is the connection to the physics of the problem? In particular, is the branch-cut contribution a novel wave phenomenon?

(iii) The role of the insight gained by the Z transform (ZT) approach and its connection to the physics of the problem.

Although it seems that MSC was motivated mainly by point (i), point (ii) is the heart of the matter. If clarified, the discussions regarding points (i) and (iii) become much simpler. Therefore, we address point (ii) first.

We begin with a brief mathematical discussion and then connect it to the physics. MSC does not distinguish between the mere Fourier spectrum associated with an expansion and the spectrum of an operator A. The former is nothing but an expansion of *any* function [$G(z)$ included] by a basis of the Hilbert space L_2. This expansion is indeed nonunique in the sense that there are infinitely many bases to any given Hilbert space be it continuous (e.g., conventional Fourier) or discrete (e.g., Hermite-Gaussian, wavelets, etc.)—the Fourier basis is just a convenient special example. In contrast, the latter—the spectrum of an operator A—is defined as the set of values of λ for which the inverse of $(A - \lambda I)$ does not exist or exists but is unbounded (under the operator norm). If the operator is compact, then the spectrum consists of only a countable (discrete) set of eigenvalues. By Hilbert-Schmidt theorem, G can be expressed as a discrete weighted sum of the corresponding eigenfunctions. If the operator is noncompact, the spectrum consists of discrete points *as well as a continuum*. Then, the discrete eigenfunctions summation must be augmented by a contribution of a continuous summation in order to correctly get G. In any case, *the spectrum of an operator is unique*, and the distinction between the discrete points and the continuum is unique as well. It is a property of the operator itself and, in principle, it has nothing to do with the specific L_2 basis by which one chooses to describe the problem. Needless to say, our infinite matrix that governs the response of an infinite or semi-infinite chain [see, e.g., Eq. (1) in Ref. 1 and Eqs. (1)–(5) in our paper[2]] is not a compact operator. Hence, it contains a uniquely defined discrete as well as a continuous spectrum.

Clearly, the expansions in Eqs. (1) and (2) in MSC are merely a Fourier (or ZT) expansion of the function G in terms of a one specific basis of L_2. Nothing more. What, then, is the connection between these mere L_2 expansions of G and the spectrum of the operator? The values of q (or Z) that nullify the corresponding chain matrix determinant provided, *by definition*, the eigenvalues and eigenfunctions. Interestingly enough, these are exactly the poles of the integrand in MSC's Eq. (1) and the poles of the inverse ZT integrand in our analysis. If one can "close the integration contour" and can apply the residue theorem, then the contribution of this discrete spectrum is readily obtained. This is achieved by the ZT. However, this is not enough: The operator is noncompact; a *uniquely defined* continuous spectrum exists and must contribute as well. The ZT approach conveys this contribution on a silver tray: It is the contribution of all other singularities apart from the poles—the branch cut. This is unique by the very fact that the continuous spectrum of an operator is uniquely defined. Let us stress again that a Fourier basis is just an L_2 basis, and it is not necessarily related to the physics of the problem at hand. To contrast, the spectrum of an operator as defined rigorously above, is an intimate and unique property of the governing operator, and therefore, it is *inherently related to the physics of the problem*.

To get a feeling about this relation, let us put aside for a moment the hot potato of our chain problem and discuss the excitation of a conventional waveguide by a point source. For a waveguide with metallic (impenetrable) walls, the problem can be cast in terms of a compact and self-adjoint operator; an infinite countable set of modes are excited—the problem eigenfunctions. The response can then be described by the discrete sum of modes. However, for "open" structures, e.g., dielectric waveguide with penetrable walls or layered media, compactness is lost. The response is described by a sum over a countable set of modes (an incomplete set) + a continuous spectrum. The latter is manifested by the presence of branch-cut integrals in the rigorous Green's function representation. This branch cut encapsulates a plethora of different waves, each with genuinely different physics: leaky modes/"improper modes" (appearing as poles in the lower Riemann sheet but partly contributing to the field structure in some limited domains in space), and lateral waves—obtained by a branch-cut integration—that propagate along the dielectric interface. These waves shed energy away as they propagate and generally do not possess a constant magnitude. The lateral wave, for example, decays algebraically as it propagates. Most

 ©2012 American Physical Society

importantly, each of these wave types represents a *different* radiation mechanism.

The *unique* distinction among guided modes, leaky modes, and lateral waves is well established and is recognized by the scientific community. This distinction is *not* a mere mathematical formality. All of these waves were even measured in experiments. They have been explored in numerous papers,[3,4] and they convey a physical picture that is uniquely defined by the waveguide properties.

The analogy with the chain problem is clear. Simple poles in the upper Riemann sheet, belonging to the (uniquely defined) discrete spectrum, represent modes that are trapped by the chain and do not radiate out energy as they propagate (poles on the unit circle) or decay exponentially as they propagate due to radiation (poles off the unit circle). The continuous spectrum wave, obtained by the branch-cut integral in our analysis, is *a novel wave species*, whose physics is the analog of the lateral wave: It radiates energy out at an algebraic rate as it propagates. This is clearly seen by our results: The branch-cut wave attenuates as $1/[n(\ln n)^2]$ when it propagates in the chain, independent of material loss. Hence, it may become dominant in realistic chains. The fact that this result is "only asymptotic" (i.e., valid for $|n| \gg 1$) does not change the essence of things: It represents a different wave mechanism. Furthermore, it agrees excellently with the exact numerical evaluation of the branch-cut contribution even for short distances.

We turn to discuss points (i) and (iii) above.

Point (i): The *unique* distinction among the trapped wave that is represented by a simple pole (a part of the discrete spectrum), the wave that is due to a branch cut (continuous spectrum wave), and the different physics conveyed by them, remain unexposed in MS's paper.

Point (iii): In hindsight, we believe this has been clarified already in the discussion of point (i) above. The ZT leads us naturally to express the total response as a sum of different waves that convey different physics, each represented by a different kind of singularity. In particular, it rescued us from ignoring the differences between discrete and continuous spectra.

Finally, we note that we have stated up front, in the Introduction section of our paper,[2] that the ZT is obtained by a mapping of the Fourier transform variable (see column 2, first page). The point is not the transform itself but the transparency by which it conveys the physics.

*steinber@eng.tau.ac.il

[1]V. A. Markel and A. K. Sarychev, Phys. Rev. B **75**, 085426 (2007).
[2]Y. Hadad and B. Z. Steinberg, Phys. Rev. B **84**, 125402 (2011).
[3]L. B. Felsen and N. Markuvitz, *Radiation and Scattering of Waves* (Prentice Hall, Eaglewood Cliffs, NJ, 1973).
[4]G. W. Hanson and A. B. Yakovlev, *Operator Theory for Electromagnetics* (Springer-Verlag, New York, 2002).