Almost periodic particle chains exhibit peculiar propagation properties that are not observed in perfectly periodic ones. Furthermore, since they inherently support nonnegligible long-range interactions and radiation through the surrounding free space, nearest-neighbor approximations cannot be invoked. Hence the governing operator is fundamentally different from that used in traditional analysis of almost periodic structures, e.g., Harper’s model and almost Mathieu difference equations. We present a mathematical framework for the analysis of almost periodic particle chains, and study their electrodynamic properties. We show that they support guided modes that exhibit a complex interaction mechanism with the light cone. These modes possess a two-dimensional fractal-like structure in the frequency–wave number space, such that a modal phase velocity cannot be uniquely defined. However, a well-defined group velocity is revealed due to the fractal’s inner structure.

I. INTRODUCTION

Linear periodic chains of plasmonic nanoparticles were studied in a number of publications [1–21]. The interest stems from both theoretical and practical points of view. Particle chains were proposed as guiding structures and junctions in [3–10], as surface waves couplers [11], as polarization-sensitive waveguides [12], and as nonreciprocal one-way waveguides and isolators [13,14]. The chain Green’s function computation was considered in [15], and the modal features of these periodic chains were studied deeply and thoroughly using a general approach and spectral analysis in [9,10] and were also considered in [16]. Green’s function theories revealing all the wave constituents that can be excited in these structures, including new features such as the continuous spectrum wave not exposed before, were developed and discussed in detail in [17–20]. Scattering due to structural disorder and its effect on the chain modes were studied in [15] and in [21].

Almost periodic one-dimensional (1D) structures were also studied, mainly in the context of electron dynamics in periodic magnetized crystals, or in almost periodic crystals [22–29]. In these works the system dynamics is dominated by short-range interactions that naturally lead to nearest-neighbor approximations and tight-binding formulation. The resulting discrete Hamiltonian is of the general form $H \psi_n = \psi_{n+1} + \psi_{n-1} + \lambda \cos(\alpha n) \psi_n$ with irrational $\alpha/\pi$, termed Harper’s model (the names almost Mathieu or almost periodic Hamiltonian are also used). This operator is known to possess a fractal (Cantor set) spectrum. The dependence of the latter and the associated eigenfunctions, or modes, on the parameters $\alpha, \lambda$ were studied extensively. The existence of the critical value of the modulation contrast $\lambda = 2$ has been observed both theoretically and experimentally. For $\lambda < 2$ the corresponding eigenfunctions are extended; i.e., the structure supports guided propagating modes. Beyond the critical value ($\lambda > 2$) the eigenfunctions become localized and no extended modes are supported.

In carefully designed settings, these previous studies may apply also to optical systems. The works in [30–32] considered a 1D array of closely spaced parallel optical waveguides, arranged as an almost periodic lattice. The optical mode trapped in one waveguide may couple only to its two neighboring waveguides and cannot radiate to the free space. Therefore this system exhibits optical dynamics compliant with Harper’s model. Other works on two-dimensional quasicrystals with optical band gaps, localized modes, and directive leaky waves from slablke domains were reported, e.g., in [33–35].

In this work we study the propagation of optical signals in almost periodic particle chains. The chains considered here—two typical examples of which are schematized in Fig. 1—possess the following general properties. The particles are equally spaced by a distance $d$, and all possess an identical resonant frequency governed for convenience by a plasmonic-Drude model. The resonant wavelength is much larger than the particles’ typical size. At least one physical/geometrical property of the particles constitutes an almost periodic sequence; in Fig. 1(a) this property is related to the (spherical) particle’s volume, and in Fig. 1(b) it is related to the (ellipsoidal) particle’s spatial orientation (see details below). Due to these features, our structures differ from the previously studied ones by several important physical aspects. These differences pertain, first and foremost, to long-range vs short-range interactions. Since the free-space dyadic Green’s function describing the radiation from an excited particle decays algebraically with distance, long-range interactions between remote particles cannot be neglected and Harper’s model ceases to hold. Studies of periodic chains show that the long-range interactions are essential to expose the (possible) interaction of the chain with the free-space radiation and the ensuing light cone [9]. The light-cone and radiation modes are present in our structures and play an intricate role in determining the guided modes and chain dynamics—a mechanism absent in Harper’s model. Second, the internal particle resonance plays a role in the chain spectra; it eliminates the critical passage from extended modes to localized ones. Last but not least, we show that due to the fractal nature of the chain spectra, phase velocity of the chain guided modes does not exist. However, due to the fractal’s inner structure, a definite group velocity exists.
summation by replacing the spectrum \{\Lambda_n\} by its additive group \{\hat{\Lambda}_n\}, and add more matrix coefficients \hat{a}_n that may or may not take the value 0. For convenience of subsequent derivations, the summation is indeed treated as over the entire module of \{a_n^{-1}\}.

Borrowing from the theory of differential equations with almost periodic coefficients [37], the solution can be expressed as

\[ p_n = \tilde{p}_n e^{i\beta nd}, \]

where \( \tilde{p}_n \) is by itself an almost periodic sequence whose module must be contained within the module of the almost periodic coefficients \( a_n^{-1} \) in the governing formulation in Eq. (1) (i.e., within the set \( \{\hat{\Lambda}_n\} \)). This lets us write the solution as

\[ p_n = \sum_{\ell=-\infty}^{\infty} \Gamma_\ell e^{i\ell n\hat{\Lambda}_\ell} e^{i\beta nd}. \]

We seek a solution for the spectral vector sequence \( \Gamma_\ell \) and for \( \beta \). Note that the physical meaning of \( \beta \) is different from that seen in periodic systems, in the sense that it does not exclusively control the phase accumulation from one particle to its neighbor (or in the more general sense from one unit cell to its neighbor). This phase accumulation is also governed by the dominant frequency components in the expansion given in Eq. (6). By using Eqs. (4)–(6) in Eq. (1) we obtain

\[ \epsilon_0 \sum_\ell \sum_r \hat{a}_r \Gamma_\ell e^{i(m\hat{\Lambda}_\ell + \hat{\Lambda}_r)} = \sum_\ell \left[ \sum_{q,q\neq 0} A(q,d) \Gamma_\ell e^{-i\ell q - i\beta dq} \right] e^{i\beta \ell \hat{\Lambda}_\ell}, \]

where all summations above extend from \(-\infty\) to \(\infty\). This equation holds several important properties which will allow further simplification in particular cases. It is an equation between two a.p. sequences, both displayed in their corresponding formal Fourier representation. The term \( \hat{\Lambda}_r + \hat{\Lambda}_\ell \) is included within the \( \hat{\Lambda}_r \) sequence itself. The uniqueness of these expansions [36] implies that one must require equality between the coefficients of identical frequencies. This imposes the relation \( \hat{\Lambda}_r + \hat{\Lambda}_\ell = \hat{\Lambda}_\ell \). We denote by \( C_\ell \) the set of all pairs \((r,\ell)\) that satisfy the latter relation [beware: the pairs \((r,\ell)\) \(\in C_\ell \) satisfy \( \ell = \ell - r \) only in the spacial case where \( \hat{\Lambda}_r \) is linear with \( \ell \)]. Then Eq. (8) reduces to a difference equation for the unknown spectral vectors \( \Gamma_\ell \),

\[ \epsilon_0 \sum_{(r,\ell) \in C_\ell} \hat{a}_r \Gamma_\ell - D_\ell \Gamma_\ell = 0, \]

where \( D_\ell \) is a diagonal matrix, given by the summation

\[ D_\ell = \sum_{q,q\neq 0} A(q,d) e^{-iq(\hat{\Lambda}_r + \beta d)} = \text{diag}(D_{\ell x}, D_{\ell y}, D_{\ell z}), \]

\( D_{\ell x} = D_{\ell y} \). Eq. (8b) \( D_\ell \) can be expressed in terms of the polylogarithm functions \( \text{Li}_r \), for which efficient summation formulas exist (see [9,38]...
and Appendix in [14]),
\[ D = \frac{k^3}{4\pi} \sum_{s=1}^{3} \alpha_s \hat{u}_s(kd, \beta d + \hat{A}_r) A_s, \]  
(8c)
where \( (u_1, u_2, u_3) = (1, -i, 1) \), \( A_3 = A_2 \), and
\[ f_{s}(x, y) = \chi^{-1}[L_s(e^{ix+iy}) + L_s(e^{ix-iy})], \]  
(8d)
and where \( L_s(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n} \) is the \( s \)-th order polylogarithm function.

### III. Analysis and Examples

The spectral domain formulation in Eqs. (8a)–(8d) governs the chain dynamics. Generally, it is not a tight-binding model for the sequence \( \alpha \). For simplicity, in this work we assume an isotropic Drude model for \( \alpha \). The scalar case: Spherical particles (40)–(43) are obtained from Eqs. (8b)–(8d), with \( D_{s} = D_{s(r)} \) for transverse (longitudinal) excitation.

Since \( \Delta \theta/\pi \) is irrational, the sequence \( Q_{\ell} \) never repeats itself. However, we emphasize that despite the structural similarity between Eq. (12) and Harper’s equation, the former is not a result of the tight-binding approximation. Furthermore, while Harper’s model governs the lattice response itself, Eq. (12) is written on the response’s spectral decomposition. The effects of long-range interactions in our lattices are encapsulated within the structure of \( D_{s} \) or \( Q_{\ell} \). In addition, the fact that the equation involves the \( \ell \)-th spectral component plus its two neighbors \( \pm \ell \) only is due to the simplicity of the particle’s polarizability; only three terms are involved in the spectral decomposition in Eq. (11a). Generally, the number of spectral neighbors involved in this equation is strictly determined by the number of spectral terms in the expansion of the a.p. sequence in Eq. (4). Finally, we note that in the traditional Harper’s model the index-dependent coefficient is real and is of a simple cosine form, while the present formulation is generally complex and with a more complicated dependence on \( \ell \).

Equations of this type, with a general periodic or a.p. complex \( Q_{\ell} \), were studied in [40], where sufficient conditions for the existence of Bloch solutions for that equation were developed. However, recall again that Eq. (12) is written for the spectral decomposition of the chain modes; see Eq. (6). Hence, a Bloch-wave solution of Eq. (12) implies a localized solution for the chain response, and vice versa; a localized non-Bloch solution of Eq. (12) implies a Bloch-wave solution (i.e., a propagating mode) for the chain response. Thus, borrowing from [40], we find that a necessary condition for the latter is
\[ q_{\ell} \equiv \inf_{\ell} |Q_{\ell}| \leq 2 |M|. \]  
(13)

The result above defines the domain in the \( \beta, \omega \) space in which Bloch-wave solutions of the original a.p. difference equation, Eq. (1), exist. It is used in our numerical examples below.

It is interesting to examine the chain dynamics exactly at the particle resonance \( \omega = \omega_r \) for lossless material (radiation losses are still kept). In this case \( \alpha_{r, n}^{(1)} = 0 \), hence \( \alpha_{r, n}^{(2)} = \alpha_{n} \) becomes scalars. The Fourier series for the sequence \( \alpha_{r, n}^{(2)} \)—i.e., the first series in Eq. (4)—contains only 3 nonzero coefficients so \( \alpha_{r, n}^{(1)} \) may be written as
\[ \alpha_{r, n}^{(1)} = \frac{\delta}{2} \alpha_{n}^{(1)}, \]  
(11a)
with coefficients
\[ a_{r, n}^{(1)} = a_{1} = \frac{\delta}{2} \alpha_{n}^{(1)}, \]  
(11b)
where \( \alpha \equiv \alpha_{r, n}^{(1)} \) and \( \alpha \equiv \alpha_{n} \) correspond to a particles whose volume modulation contrast shrinks to zero. We note that the series in Eqs. (11a)–(11b) has a finite spectrum \( \{\varLambda_{r}\} \); it is the set \( \{\varDelta \theta, 0, \varDelta \theta\} \). Hence, the module of \( \alpha_{n} \) is
\[ \left| \varLambda_{r} \right| = |\varDelta \theta|_{|r| = \infty}. \]  
(11c)

By using Eqs. (11a)–(11c) in Eq. (8), we obtain the difference equation for the \( \alpha_{r, n}^{(1)} \) spectral amplitudes \( \Gamma_{r} \):
\[ M\varGamma_{r+1} + Q_{r+1} \varGamma_{r} + M\varGamma_{r-1} = 0, \quad \forall \ell \in \mathbb{Z}. \]  
(12)
the above results do not hold, but they may still provide an approximate solution for low loss material.

When \( \omega \neq \omega_0 \) or/and when material loss is present, the structure may still support Bloch modes, but their analysis and the associated dispersion are not as straightforward and transparent as the case of precise resonance. To obtain a condition for the existence of a nontrivial solution sequence \( \{ \Gamma_n \}_{n=1}^{\infty} \) to Eq. (12), we employ a procedure as in [41] (it is not limited to periodic medium) and obtain a double continued fraction relation,

\[
k_{\ell,+}^{(+)}, k_{\ell,-}^{(-)} = 1,
\]

where

\[
k_{\ell,+}^{(\pm)} = \frac{1}{q_1 - \frac{1}{q_{\ell+1} - \frac{1}{q_{\ell+2} - \ldots}}},
\]

and where \( q_1 = Q_1/M).\) Formally, the dispersion is obtained by the “solution pairs”—the pairs \((\omega, \beta)\) for which this relation is satisfied. In Appendix B we prove the following important properties:

1. All solution pairs are \( \ell \) independent.
2. If \((\omega_0, \beta_0, d)\) is a solution pair, so is the pair \((\omega_0, \beta_0 d + \Delta \theta)\).
3. If \(\{ \Gamma_n \}_n \) and \(\{ \Gamma_n \}_n \) are the coefficient sequences that correspond to \((\omega_0, \beta_0, d)\) and \((\omega_0, \beta_0 d + \Delta \theta)\), respectively, then they are related by a simple shift: \(\Gamma_n + \Delta \theta = \Gamma_{n+1}\).

Generally, for each \( \omega_0 \) there could be more than a single wave number \( \beta_0 \). We number the latter as \( \beta_0^{(m)} \). From property 2 it follows that for any solution pair \((\omega_0, \beta_0^{(m)} d)\), there exist infinitely many additional solution pairs \((\omega_0, \beta_0^{(m)} d + \Delta \theta)\) for which this relation holds.

For any \( \omega_0 \), \( \beta_0 \), and \( d \), denote the set of the infinitely many corresponding \(\{ \beta_0^{(m)} d + \Delta \theta \}_m\) as \( S^{(m)} \). For example, for \( \omega_0, \beta_0 d, d \) within the boundaries in Eq. (13), seek for \( S^{(m)} \) of \( \beta_0 d \) for which \( |\beta_0 d| < c \).

Finally, although the proper way to analytically define and study the dispersion relation is to use Eqs. (14a)–(14b), as done above, we found it very inconvenient numerically. Therefore, to get the dispersion we truncate the infinite matrix in Eq. (12) to a finite equation and solve it by searching numerically for pairs \((\omega_0, \beta_0 d)\) for which the matrix is rank deficient. Naturally, some clipping of the data occurs, meaning that we set some threshold and treat only \( \Gamma_n \)’s which surpass the threshold.

This has no significant effect on the solution accuracy since dealing with the guided modes implies a localized nature of the coefficients as mentioned before. To summarize, we choose \( \omega_\text{a priori} \) frequencies within the boundaries in Eq. (13), seek for solutions with diminishing values of \( \Gamma_n \), and set the threshold in values well below the diminishing tail of the \( \Gamma_n \) distribution. Typical values were below 10 orders of magnitude relative to the maximal \( \Gamma_n \). As shown below, this approach can provide very accurate results when compared to an actual simulation of an excited particle chain.

To demonstrate the properties discussed above, consider a chain with \( \delta = 1/3, \delta = 0.5, \) and \( \Delta \theta = 0.4 \) radians. We applied the numerical approach described above to compute the solution pairs \((\omega_\text{a}, \beta_\text{a}^{(m)} d + \Delta \theta)\) for which \( |\beta_\text{a}^{(m)} d| < c \).

The results are shown in Fig. 2, color-coded according to the \( \beta \)’s magnitudes. We emphasize that although these pairs are solutions of the dispersion relation defined by Eqs. (14a)–(14b), the results should not be perceived as a “dispersion” in the usual sense. That is, a single point in the chart does not constitute a chain solution. Rather, all points in the charts at a given frequency are excited, each with its own excitation magnitude, in order to constitute together a wave solution. Hence, we refer to Fig. 2(a) as the excitation chart. This chart possesses a fractal-like nature in the sense that formally the band depicted in the figure is filled with solution pairs, due to the fact that the set \( S^{(m)} \) of \( \beta_0 d \) is dense in the interval \([-\pi, \pi]\).

The results shown in the excitation chart imply that it is impossible to define a single or even a finite set of phase...
velocities that will characterize the propagating modes along the chain. However, an inner structure of different but parallel “branches” is clearly observed, along which all solution pairs are ordered. Hence, it is possible to define a group velocity as the slope. As we show below, this uniquely defined velocity is consistent with the properties of wave-packet propagation along the chain. Panel (b) zooms in a selected region, showing better this inner structure.

The wave numbers $\{\beta_{\lambda}\}^\infty_{\lambda}$ may be complex. Our numerical solutions for their values verified what has been predicted in the discussion following properties 4–6: all possess the same imaginary part. Panel (c) shows this calculated imaginary part of $\beta d$. Dashed lines show the frequencies for which significant transition occurs in the magnitude of $\text{Im}[\beta d]$.

$\text{Im}[\beta d]$ predicted by Fig. 2(c) is $4.09 \times 10^{-5}$. This implies an attenuation by about 16% over 80\x. From the simulation we obtain attenuation of 15.9% so the match is very good. Figure 4(b) shows the Fourier transform of this response (blue), and compares it to the data predicted by the excitation chart (red dots). Again, excellent agreement is observed. Note the two peaks inside the light cone that provide the radiation-loss mechanism that leads to the response attenuation.

Next, we examine how the modulation frequency $\Delta \theta$ affects the range of frequencies for which propagating modes may be excited. The results are displayed in Fig. 5. This plot shows a structure which has many features that resemble Hofstadter’s butterfly [22]. The fractal nature is clearly visible. The frequency range for which guided modes exist, predicted...
by Eq. (13), is $\omega/\omega_p \in [0.5316, 0.6093]$ and is used as limits for the vertical axis.

Note that the limit $\Delta \theta = 0$ corresponds to a conventional periodic chain of identical spheres. In this case both the spectrum and module of $\alpha^{-1}_m$ collapse to a point; see Eq. (11c). Then Eq. (12) governing $\Gamma_\ell$ becomes shift invariant since $Q_\ell$ is rendered $\ell$ independent. Moreover, the shift-and-duplicate process described in points 1–6 and in Eq. (15) produce the same dispersion $\forall \ell$ as the shift itself is zero. Hence the fractal structure of the excitation chart disappears, and the conventional chain dispersion is reconstructed.

Finally, it is possible to excite several frequencies together, and observe the chain response to a pulse-excitation. Towards this end, we have simulated the chain response in the time domain due to a $\xi$-directed dipole excitation of the central particle at 30 equally spaced frequencies in the range $\omega = [0.57, 0.5725] \omega_p$. The frequencies are weighted by a Hamming window. This excitation creates a pulse whose temporal width is about 400 $\tau_p$ where $T_p$ is the oscillation period of $\omega_p$. Snapshots of the chain response as a function of $z$, at four equally spaced times, are shown in Fig. 6. This response shows a pulse that preserves its shape while propagating along the chain at constant velocity. This velocity is consistent with the local slopes of the inner structure revealed in Figs. 2(a)–2(b). Hence, as predicted, although a phase velocity cannot be defined, a uniquely defined group velocity does exist.

B. The vector case: Rotating ellipsoidal particles

We now turn to analyze the chain presented in Fig. 1(b). This chain is a.p. for irrational $\Delta \theta/\pi$. Many of the results reported in Sec. III A hold here, and particularly the formal properties

1–6 discussed there. However, there are some important differences. First and foremost, due to the ellipsoids rotation the longitudinal and transverse polarizations are coupled, and the general matrix formulation in Eqs. (8a)–(8d) cannot be reduced to a scalar one. Also, unlike the scalar case, here the ideal (lossless material) a.p. chain does not possess a solution identical to that of a perfectly periodic one. Last but not least, in our numerical calculations and simulations we were not able to find a case in which a significant spatial harmonic enters the light cone. Hence, we may conclude that the modal solutions of this chain are “better isolated” from the free space surrounding it, and their attenuation due to radiation loss is practically irrelevant. This observation, although based for the moment on numerical simulations, may have important practical implications.

From the almost-periodicity we again assume the solution given in Eqs. (5)–(6), which will take the full vector nature this time. Writing $\alpha^{-1}_m$ explicitly we obtain

$$\alpha^{-1}_m = T_m \alpha^{-1} T_{-m}, \quad (16)$$

where $\alpha$ is the polarizability of the reference ellipsoidal particle and $T_m$ is the rotation operator by $m \Delta \theta$ in the $(x,z)$ plane. The entries of this matrix are given in Appendix C. Note that although the angle of rotation from one particle to its neighbor is $\Delta \theta$ the spectrum of the polarizability sequence is the set $\{\Lambda_m\} = \{2r \Delta \theta\}_r=-\infty, \infty$. Hence the decomposition of the sequence $\alpha^{-1}_m$ according to Eq. (4) is

$$\alpha^{-1}_m = \sum_{r=-\infty}^\infty \tilde{\alpha}_r e^{ir 2 \Delta \theta m}, \quad (17)$$

and in our specific case all the matrices $\tilde{\alpha}_r$ are the zero matrix except $r = -1, 0, 1$, for which they are identical to $a_r$. These matrices are also listed in Appendix C. Since there are again only three terms in this expansion, the dynamics formulation in Eqs. (8a)–(8d) reduce to a form identical to Eq. (12), but of matrix nature

$$M \Gamma_{\ell+1} + Q_\ell \Gamma_\ell + M \Gamma_{\ell-1} = 0, \quad \forall \ell \in \mathbb{Z}, \quad (18)$$

where $M = \epsilon_0 a_1$ and $Q_\ell = \epsilon_0 a_0 - D_\ell$. 

\[ \text{FIG. 5. (Color online) Excitable frequencies as a function of the modulation frequency $\Delta \theta$. White areas represent frequencies for which guided modes exist.} \]

\[ \text{FIG. 6. (Color online) A pulse propagating through the a.p. chain of modulated spherical particles.} \]
For numerical example, we consider a chain with $\Delta \theta = 0.4\,\text{rad}$, and prolate ellipsoid aspect ratio of $0.9$. The corresponding excitation chart is shown in Fig. 7. As with the scalar case, it possesses a fractal-like structure in the sense that a frequency band is filled with solution pairs; a phase velocity is hard to define. However, an inner structure of parallel lines is identified, along which all solution pairs $(\omega, \beta)$ are ordered. The corresponding slope can be associated with definite group velocity (see below). We found numerically that all the corresponding wave numbers were real. This can be attributed to the fact that at the frequency range shown, the weight of $|\Gamma_1| \ell$ that resides inside the light cone is overwhelmed by those that reside outside it.

Figure 8(a) shows the chain response due to a $\hat{x}$-directed unit dipole excitation at $\omega = 0.536445 \omega_p$. No attenuation is observed over propagation distances of hundreds of wavelengths. In Fig. 8(b) we show the corresponding Fourier transform (blue) compared to the data of the excitation chart (red dots). Excellent agreement is seen. Note that there are no significant peaks inside the light cone.

Finally, Fig. 9 displays the chain response to a point dipole excitation that consists of 100 equally spaced frequencies in the band $\omega = [0.546213, 0.546324] \omega_p$, weighted by a Hamming window. A pulse that preserves its shape while propagating with a constant group velocity is observed.

IV. CONCLUSIONS

Theoretical analysis of almost periodic particle chains was presented, and a fractal-like dispersion relation, termed here as the excitation chart, was obtained. New chain modes existing in a.p. particle chains were extracted and confirmed by simulations. It is shown that while phase velocity cannot be uniquely defined, these guided modes do possess a well-defined group velocity due to the inner structure of the fractal-like excitation chart. An intricate radiation mechanism that depends on the number of significant spatial harmonics inside and outside the light cone has been observed.

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APPENDIX A: A SOLUTION TO EQUATION (12) AT RESONANCE FOR LOSSLESS MATERIAL

We examine Eqs. (12)–(13) exactly at the particle resonance $\omega = \omega_r$ for lossless material but with radiation loss. In this case $\alpha_{r-1} = 0$ hence $M = 0$, and Eq. (12) reduces to the requirement $Q \Gamma \ell = 0$ $\forall \ell$. Obviously, this can be satisfied for every specific choice of $\ell$ provided that

$$\Gamma_\ell = 0 \quad \forall \ell \neq \ell',$$

and the last equation implies

$$D_\ell|_{\omega_0} = -ik_0^2/(6\pi),$$

where $k_0 = \omega_0/c$. Now note that Eq. (A1c) is identical in form to the dispersion relation of the modes of conventional periodic particle chains [9] and thus always possesses a solution at resonance. Let $\beta_p(\omega_r)$ be that solution. Then at resonance $\beta$ of our a.p. chain must satisfy [use Eq. (11c) in Eqs. (8b)–(8c)]

$$\beta d + \ell \Delta \theta = \beta_p(\omega_r)d.$$  (A2)

Since this solution is associated with a single nonzero coefficient $\Gamma_\ell$, it constitutes a Bloch solution of the a.p. chain. Furthermore, using this fact in Eq. (6), we find that at resonance
the solution of the corresponding periodic chain always holds for its a.p. counterpart.

APPENDIX B: PROPERTIES OF THE CONTINUED-FRACTION DISPERSION EQUATION (14)

First, we note that $K^{(\pm)}$ satisfy the following property:

$$\frac{1}{K^{(\pm)}} = q_\ell - K^{(\mp)}_{\ell \pm 1}. \quad (B1)$$

Now assume that Eq. (14a) is satisfied by a pair $(\omega_0, \beta_0)$ and rewrite it as

$$\frac{1}{K^{(+)}} = K^{(-)}_{\ell - 1}. \quad (B2)$$

APPENDIX C: THE POLARIZABILITY SEQUENCE OF ROTATING ELLIPSOIDS CHAIN AND ITS FOURIER DECOMPOSITION

The entries of the $\alpha^{-1}_m$ are given by

$$\alpha_{m, (11)}' = \frac{1}{2} \alpha_{xx}' + \frac{1}{2} \alpha_{zz}' - \frac{1}{2} \alpha_{xy}' \cos 2m \Delta \theta + \frac{1}{2} \alpha_{xz}' + \frac{1}{2} \alpha_{yz}' \sin 2m \Delta \theta, \quad (C1a)$$

$$\alpha_{m, (13)}' =\frac{1}{2} \alpha_{xx}' - \frac{1}{2} \alpha_{zz}' + \frac{1}{2} \alpha_{xy}' \cos 2m \Delta \theta + \frac{1}{2} \alpha_{xz}' - \frac{1}{2} \alpha_{yz}' \sin 2m \Delta \theta, \quad (C1b)$$

$$\alpha_{m, (22)}' = \alpha_{xy}' \quad (C1c)$$

$$\alpha_{m, (31)}' =\frac{1}{2} \alpha_{xx}' - \frac{1}{2} \alpha_{zz}' + \frac{1}{2} \alpha_{xy}' \cos 2m \Delta \theta + \frac{1}{2} \alpha_{xz}' + \frac{1}{2} \alpha_{yz}' \sin 2m \Delta \theta, \quad (C1d)$$

$$\alpha_{m, (33)}' =\frac{1}{2} \alpha_{xx}' + \frac{1}{2} \alpha_{zz}' - \frac{1}{2} \alpha_{xy}' \cos 2m \Delta \theta - \frac{1}{2} \alpha_{xz}' + \frac{1}{2} \alpha_{yz}' \sin 2m \Delta \theta, \quad (C1e)$$

where $\alpha_m' = \alpha_m^{-1}$ is the inverse polarizability of the $m$th particle and $\alpha' = \alpha^{-1}$ is the inverse polarizability of the reference particle. The corresponding Fourier coefficients according to Eqs. (C1a)–(C1e) are

$$a_0 = \begin{pmatrix} \frac{1}{2} \alpha_{xx}' + \frac{1}{2} \alpha_{zz}' & \frac{1}{2} \alpha_{xx}' - \frac{1}{2} \alpha_{zz}' & 0 & 0 \\ 0 & \alpha_{xy}' & 0 & 0 \\ -\frac{1}{2} \alpha_{xx}' + \frac{1}{2} \alpha_{zz}' & 0 & \frac{1}{2} \alpha_{xy}' + \frac{1}{2} \alpha_{xy}' & 0 \\ \frac{1}{2} \alpha_{xx}' - \frac{1}{2} \alpha_{zz}' & -\frac{1}{2} \alpha_{xy}' & \frac{1}{2} \alpha_{xy}' + \frac{1}{2} \alpha_{xy}' & \frac{1}{2} \alpha_{xz}' - \frac{1}{2} \alpha_{yz}' \end{pmatrix}, \quad a_1 = (a_{-1})^*, \quad (C2a)$$

$$a_{-1} = \begin{pmatrix} \frac{1}{4} \alpha_{xx}' + \frac{1}{4} \alpha_{xx}' & \frac{1}{4} \alpha_{xx}' - \frac{1}{4} \alpha_{xx}' & 0 & 0 \\ 0 & \frac{1}{4} \alpha_{xy}' & 0 & 0 \\ \frac{1}{4} \alpha_{xx}' + \frac{1}{4} \alpha_{xx}' & \frac{1}{4} \alpha_{xx}' - \frac{1}{4} \alpha_{xx}' & 0 & 0 \\ \frac{1}{4} \alpha_{xx}' - \frac{1}{4} \alpha_{xx}' & \frac{1}{4} \alpha_{xx}' + \frac{1}{4} \alpha_{xx}' & 0 & 0 \\ \frac{1}{4} \alpha_{xx}' + \frac{1}{4} \alpha_{xx}' & \frac{1}{4} \alpha_{xx}' - \frac{1}{4} \alpha_{xx}' & 0 & 0 \\ \frac{1}{4} \alpha_{xx}' - \frac{1}{4} \alpha_{xx}' & \frac{1}{4} \alpha_{xx}' + \frac{1}{4} \alpha_{xx}' & 0 & 0 \end{pmatrix}. \quad (C2b)$$