Effective Resonance Representation of Propagators in Complex Ducts—A Multiresolution and Homogenization Approach

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***Abstract*—We use the multiresolution homogenization theory to study the effective resonances representation of transient electromagnetic propagation in complex random multilayer ducts. The theory permits explicit choice of the smoothing (homogenization) scale, and can be applied to a wide range of micro-scale properties. The analytical study is based on a Wroskian equivalence result, which establishes the relation between the "true" Wronskian $\mathcal{W}$ and that of the homogenized problem $\mathcal{W}^{(\mathrm{eff})}$. Since the roots of $\mathcal{W}$ in the complex ω plane constitute the duct resonance, the Wronskian equivalence theorem is used as a basic apparatus for the effective resonance study. With this, the time-domain spectral properties of the multiresolution homogenization formulation are studied analytically and demonstrated numerically. Effective representations of reflection from complex random multilayer ducts are considered.**

***Index Terms*—Multiresolution techniques, propagators, wavelets.**

I. Introduction

COMPLEXITY in the context of propagation and scattering problems can be perceived as the analytical and computational difficulties encountered when a wave-field interacts with a heterogeneity that contains a wide range of length-scales. Consider, for a moment, a time-harmonic field in a homogeneous medium; it is characterized by a single length scale—the wavelength λ. We use λ as a discriminator of length scales; length scales on the order of λ and above are termed as *macro scales*, and length scales much smaller then λ are termed as *micro scales*. We are concerned with a *complex heterogeneity*, defined as micro (and macro) scale variation of the medium properties (ϵ, μ), occupying domains of macro-scale dimensions. When a wave interacts with such a medium, the field within and near the complex heterogeneity "inherits" the medium complexity—it contains a wide range of length scales, from micro to macro. However, in a variety of applications, the field *observables* are determined only by the *macro-scale component*, while the micro-scale component is practically irrelevant.

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The role of homogenization theory is to develop *effective* formulations that govern only the macro-scale component of the field. These formulations provide new heterogeneity measures of the medium that 1) comprise macro scales only ("effective heterogeneity") and 2) describe the coupling between the micro-scale heterogeneity and the macro-scale field. Traditional homogenization theories apply only to periodic structures and require a large gap between the micro scale and the macro scale. The "bulk properties" are described on infinitely large length-scale (that is, the effective properties are constants). A comprehensive overview of these theories is provided by several textbooks [1]–[3].

Recently, a new homogenization technique, based on the theory of multiresolution and orthogonal wavelets, was developed [4]–[11]. Unlike traditional homogenization theories, the new multiresolution homogenization theory can handle nonperiodic micro structures, and formally does not require the existence of a large gap between micro scale and macro scale. The new theory permits a prior choice of the length scale on which the medium is to be homogenized ("homogenization scale"). Hence, in general the effective properties are not constants. This last property is of fundamental importance. By choosing the homogenization scale to be less than the wavelength, one can achieve a simple effective medium and still keep large-scale variation measures that play an important role in the propagation-related phase accumulation. More specifically, the studies in [10], [12], and [13] apply the new homogenization formulation for a derivation of an effective (homogenized) modal theory for propagation in complex ducts. Several issues were discussed.

1) The relation between the boundary conditions (BCs) of the original ("complete") formulation, and those of the homogenized one. Under certain conditions, the former and the latter are identical.
2) The relation between the eigenvalues and eigenfunctions of the complete formulation λ_n, u_n and those of the homogenized one λ_n^*,u_n^* (effective eigenvalues and modes). When appropriate BCs are employed, *spectral equivalence* is established: the lowest order λ_n^* are asymptotically identical to the corresponding λ_n, and u_n^* are identical to the macro-scale component of u_n. These results play a role in establishing an *effective modal analysis*, as they pertain to preservation of phase accumulation and modes shape of the effective field construction.
3) The relation between the *Wronskian* of the complete formulation and that of the effective one. The importance is twofold. a) It provides an alternative approach to the

TABLE I
COMPLETE AND EFFECTIVE BOUNDARY CONDITIONS. HERE $v = (d/dx)u$. α IS AN ARBITRARY CONSTANT AND p IS DEFINED AFTER (2.2)

	Complete	*Effective*
Dirichlet	$u = 0$	$u^s = 0$
Neumann	$v = 0$	$v^s = 0$
Natural Impedance	$\alpha u + v/p = 0$	$\alpha u^s + v^s/p^s = 0$

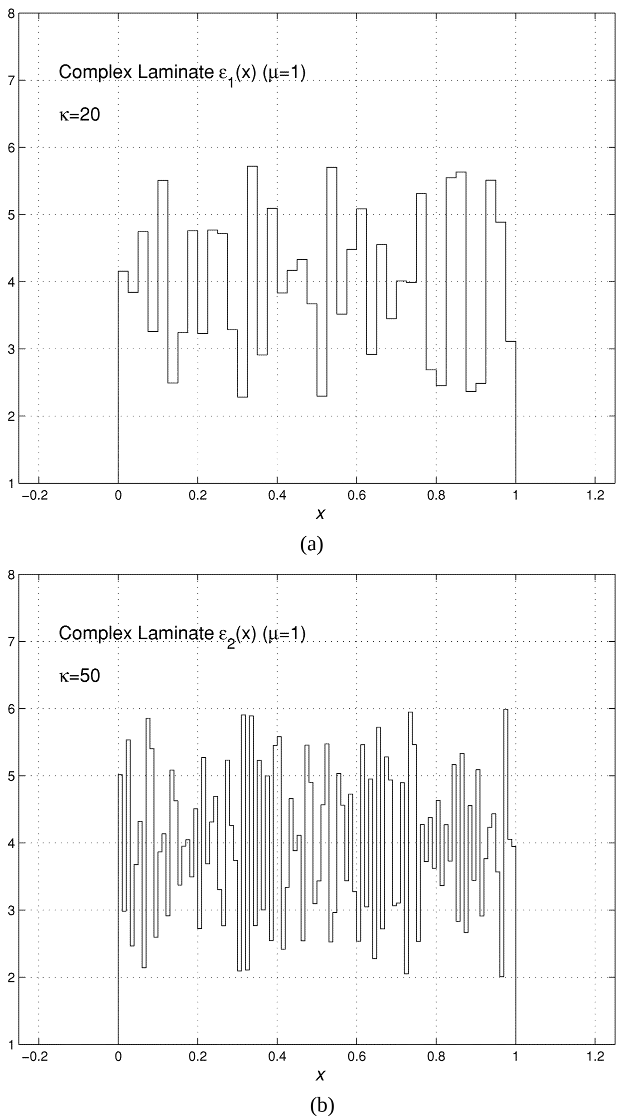

Fig. 1. Complex ducts with random multilayer microstructures. (a) $\epsilon_1(x)$, with a microscale of $1/\kappa = 1/20$. (b) $\epsilon_2(x)$ with a microscale of 1/50.

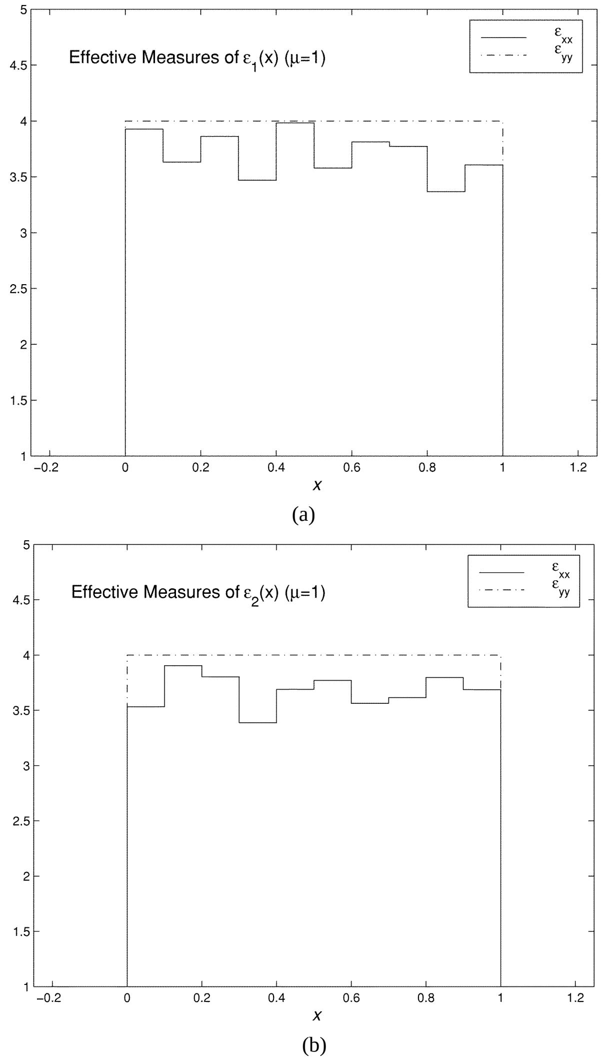

Fig. 2. Effective properties of the random multilayers for HS$= 0.1$ and Haar multiresolution system. (a) $\epsilon_1^{(\mathrm{eff})}(x)$. (b) $\epsilon_2^{(\mathrm{eff})}(x)$.

spectral equivalence issue above. b) More important, it establishes a relation between the ω_n—the poles in the complex frequency plane associated with the complete problem—and ω_n^*—the poles associated with the homogenized one. When spectral equivalence holds, we have $\omega_n^* = \omega_n$.

We note that multiresolution analysis and wavelets have been used recently by Brewster and Beylkin in [15] for numerical homogenization. The basic ideas are similar to those developed in [4] and [5]. The work in [15] is devoted mainly for a sophisticated "decimation" process in which efficient numerical algorithm for estimating the large-scale response component is developed. It does not address directly the questions articulated above. Using the multiresolution approach, the work in [6] and [8] reconstructs the classical result of homogenization of the equation $(Qu')' = f$, in the context of acoustic scattering. Using the multiresolution approach, the classical result was derived again later by Gilbert [16], together with a correction term. In [17], Beylkin and Coult used the algorithm developed in [15] to investigate the numerical homogenization of boundary value problems. Their work, however, does not address the issues articulated above.

In this paper, we use the Wronskian equivalence reported in [13] and [14], for a *spectral plane-wave* study of the effective resonance representation of reflection of a transient plane wave from a complex duct. Specifically, we examine, for the first time, how well the effective poles/resonances reconstruct the true (full-scale) resonances under various heterogeneity parameters, and the effects of the approximate effective representation on the transient signal reflected from a complex multiscale laminate. The basic *spectral effective building blocks* studied here can then be used for a representation of two- and three-dimensional

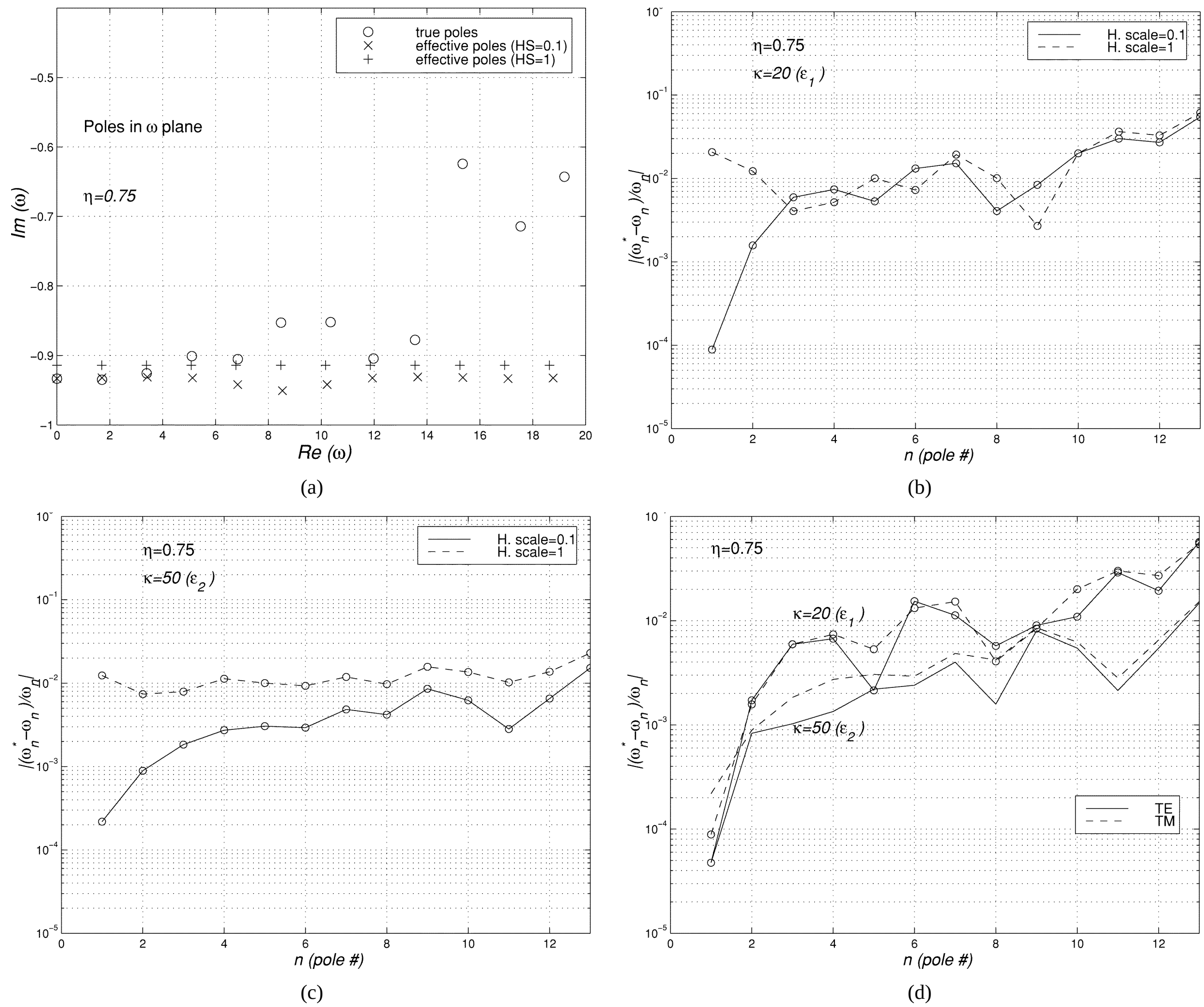

Fig. 3. The "true" (complete) poles ω_n and effective poles ω_n^*. (a) In the complex ω plane, for $\epsilon_1(x)$ and its effective measures and TM excitation. (b) The relative differences between ω_n and ω_n^* for $\epsilon_1(x)$ and two values of HS. (c) The same as (b) but for $\epsilon_2(x)$. (d) The relative difference between ω_n and ω_n^* for both polarization and heterogeneities, for HS$= 0.1$.

fields via spectral summation. This is done in a subsequent study [19].

II. Homogenized Formulation

A. Wavelets, Scaling Functions, and Spatial Smoothing

Let $\phi(x)$ and $\psi(x)$ be the scaling function and wavelet associated with a multiresolution decomposition of $L_2(R)$. The function $\phi_{jn}(x)$ is defined via $\phi(x)$ as $\phi_{jn}(x) = 2^{j/2}\phi(2^j x - n)$, and a similar definition holds for $\psi_{mn}(x)$. The spectrum of $\phi_{jn}(x)$ is centered in the low-frequency regime $|\xi| \leq 2^j$, while that of $\psi_{jn}(x)$ is centered in the bandpass regime $2^j \leq |\xi| \leq 2^{j+1}$. An approximation of a field $u(x)$ at a resolution k can be written as the sum of two mutually orthogonal fields, namely, smooth (u^s, macro scale) and detail (u^d, micro scale) components. We have $u(x) = u^s(x) + u^d(x)$, where

$$u^s(x) = \mathbf{P}_j u(x) = \sum_n s_n \phi_{jn}(x), \quad s_n = \langle u, \phi_{jn}\rangle \tag{2.1a}$$

$$u^d(x) = \mathbf{D}_j^k u(x) = \sum_{m=j}^{k-1} \sum_n d_{mn}\psi_{mn}(x), \quad d_{mn} = \langle u, \psi_{mn}\rangle. \tag{2.1b}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product of $L_2(R)$ and $j < k$ is the reference smoothing resolution. Recalling the spectral properties of ϕ and ψ, $u^s(x)$ can be interpreted as the spatial average of $u(x)$, with the averages' being taken over intervals of the size 2^{-j}. $u^d(x)$ contains the remaining fine details; hence the terms macro- and micro-scale components. The resolution level j should be chosen such that u^s faithfully describes field components possessing spatial length scales on the order of a wavelength λ and larger. Thus, for a normalized wavelength $\lambda = 1$, we choose $j = 3$.

B. Homogenization of the Wave Equation in Layered Media

Electromagnetic wave propagation in an isotropic plane stratified medium is governed by

$$\nabla \cdot (\mathcal{Q}\nabla u) + \omega^2 g u = s(x, y), \quad \mathcal{Q} = p^{-1}\mathbf{I} \tag{2.2}$$

with boundary conditions at $x = 0, a$. ω is the frequency, $p(x)$ and $g(x)$ represent the medium heterogeneity, and $\mathbf{I}$ is the identity matrix. For TE or TM wave, u is the z-directed electric or magnetic field and $p(x) \equiv \mu(x)$, $g(x) \equiv \epsilon(x)$ or $p(x) \equiv \epsilon(x)$, $g(x) \equiv \mu(x)$, respectively. p and g vary rapidly in x. Equation

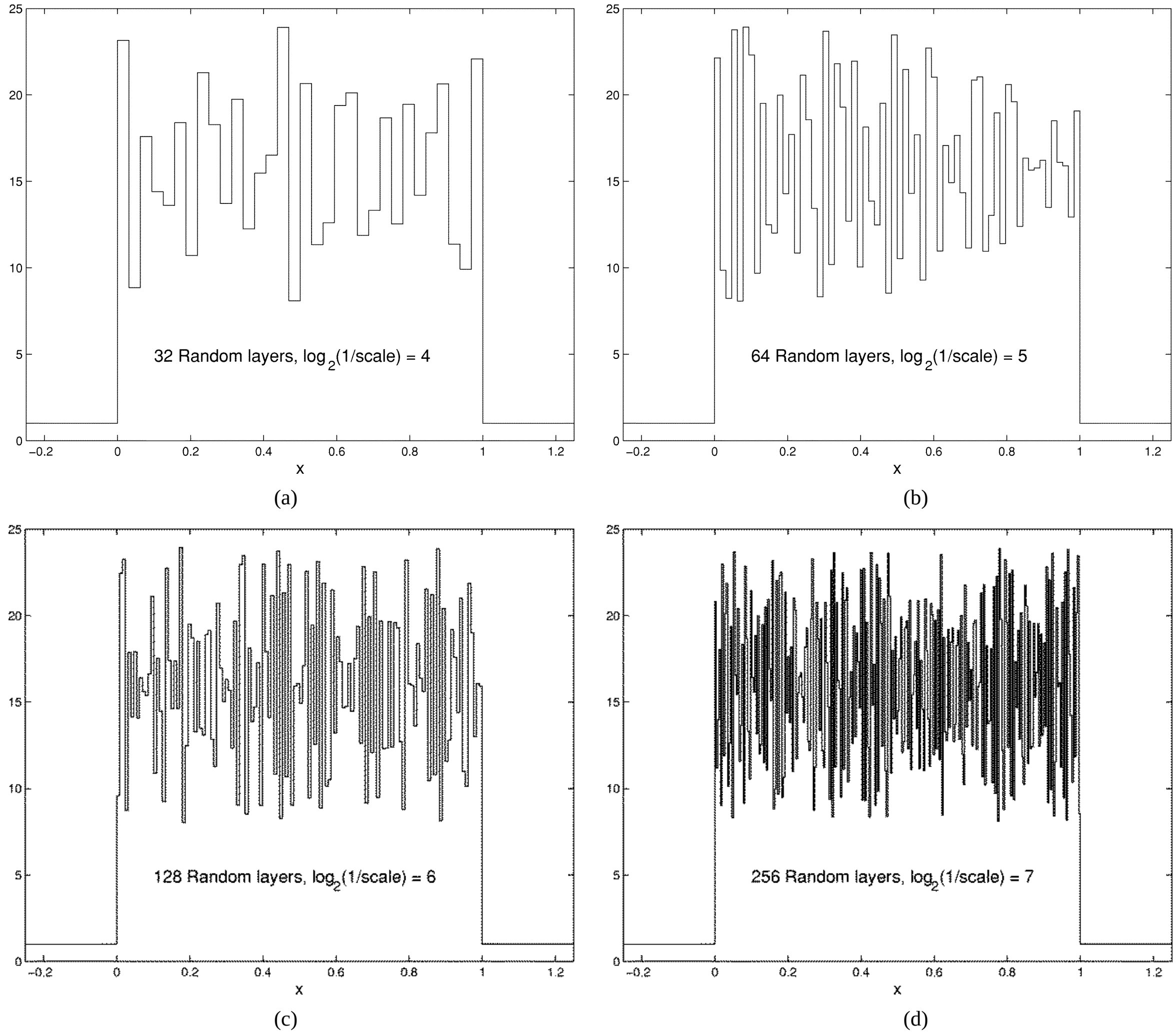

Fig. 4. Four different manifestations of complex multilayer ducts, with micro-scale of 1/32 (a), 1/64 (b), 1/128 (c), and 1/256 (d).

(2.2) can be reduced to a (1-D) Sturm–Liouville problem by a Fourier transform along the y coordinate. The result is

$$\left[\frac{d}{dx}\frac{1}{p(x)}\frac{d}{dx}+\omega^2 g(x)-\frac{\omega^2\eta^2}{p(x)}\right]\tilde{u}(x,\eta)=\tilde{s}(x,\eta) \tag{2.3}$$

subjected to the same BC as (2.2) in the x direction. Here η is the spectral variable associated with the y direction. This formulation is now amenable for a direct application of the theory in [10]–[13]. The field $u(x)$ is expressed as the sum of two mutually orthogonal fields: the smooth (u^s, macro scale) and the detail (u^d, micro scale) components. We have $u(x)=u^s(x)+u^d(x)$, as defined in (2.1a) and (2.1b). It can be shown that if the complete field is subject to Neumann, Dirichlet, or natural impedance BC, the macro-scale field u^s satisfies the BC summarized in Table I (see [10], [12], and [13]) and is governed by

$$\left[\frac{d}{dx}\frac{1}{p^s}\frac{d}{dx}+\omega^2 g^s-\omega^2\eta^2\left(\frac{1}{p}\right)^s\right]\tilde{u}^s(x,\eta)=\tilde{s}(x,\eta). \tag{2.4}$$

Here f^s denotes the macro-scale component of f, as defined by the projection operation in (2.1a). Applying an inverse Fourier transform, we obtain the effective wave equation governing $u^s(x,y)$. It is identical in form to the complete formulation (2.2), except that g and $\mathcal{Q}$ are replaced by their effective measures $g^{(\text{eff})}$, $\mathcal{Q}^{(\text{eff})}$. We have $g^{(\text{eff})}(x)=g^s(x)$, while $\mathcal{Q}^{(\text{eff})}(x)$ is a diagonal matrix with elements $q_{11}=1/p^s$, $q_{22}=(1/p)^s$, $q_{21}=q_{12}=0$. In general, $1/p^s\neq(1/p)^s$. This introduces an effective anisotropy into the macro-scale formulation.

III. Spectral Equivalence and Effective Wronskian

Let $\lambda_m=\eta_m^2$ be an eigenvalue of the complete problem (2.3) and u_m the corresponding mode. λ_m^*, u_m^* are the corresponding *effective* quantities associated with (2.4). Let ℓ be the micro scale and L_m^* be the length scale associated with the effective mode u_m^* (since the latter is an eigenfunction of an equation with coefficients that vary on the macro scale, one has approximately $L_m^*=O(1/m)$). Then [10], [12], [13]

$$u_n^*\to u_n^s \quad \text{as } \frac{\ell}{L_n^*}\to 0 \tag{3.1a}$$

$$\lambda_n^*\to\lambda_n \quad \text{as } \frac{\ell}{L_n^*}\to 0. \tag{3.1b}$$

Hence a "spectral equivalence": if the micro scale is small compare to the length scale of the effective mode u_n^*, the effective eigenvalue λ_n^* approximates the true eigenvalue λ_n and the effective mode u_n^* approximates the nth mode macro-scale component u_n^s.

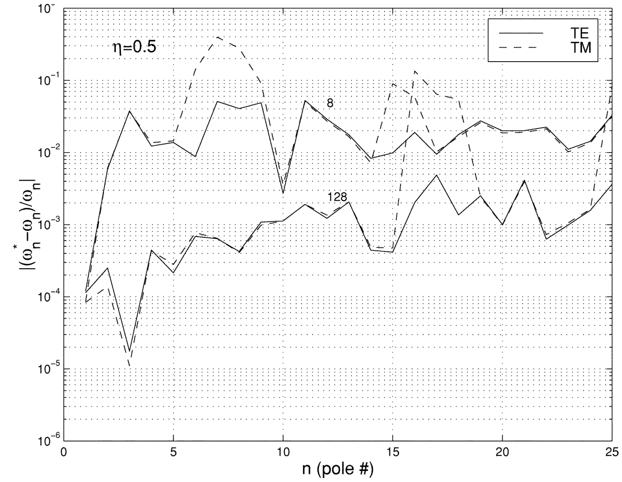

Fig. 5. The same as Fig. 3(d) but for the eight and 128 random micro layers of the second set.

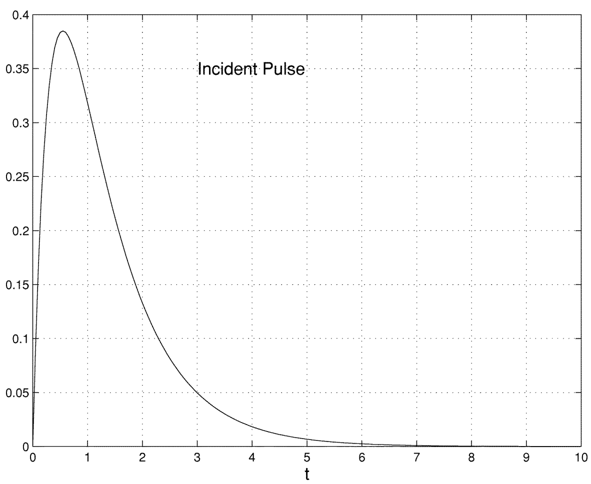

Fig. 6. The incident field.

The Wronskian of the complete problem is given by

$$\mathcal{W}[u_\ell, u_r] = u_\ell u_r' - u_\ell' u_r \tag{3.2}$$

where u_ℓ,u_r are solutions of the source free problem satisfying BC on the left and on the right boundary, respectively. It is well known that $\mathcal{W}/p$ is independent of x. This quantity, however, depends in general on the two parameters ω, η. Thus, we write

$$\frac{\mathcal{W}[u_\ell, u_r]}{p(x)} = \text{const.} = F(\omega, \eta). \tag{3.3}$$

As $\ell \to 0$, each of the two independent solutions u_ℓ, u_r satisfies (see [10], [12], and [13])

$$\left[\frac{u'(x)}{p(x)}\right]^s \simeq \frac{u'^s(x)}{p^s(x)} \tag{3.4}$$

and

$$\frac{\|u^d\|}{\|u^s\|} \to 0 \tag{3.5}$$

where both the error of (3.4) and the ratio in (3.5) are bounded from above by $(\ell/L)^\alpha$, where L is the outer duct dimension and $\alpha \geq 1$. Applying the smoothing operator (2.1a) on (3.3) and using (3.4) and (3.5), we find

$$u_\ell^s u_r'^s - u_\ell'^s u_r^s \simeq p^s(x) F(\omega, \eta) \quad \text{as } \ell \to 0. \tag{3.6}$$

However, the quantity on the left is nothing but the *effective* Wronskian $\mathcal{W}^{(\text{eff})}$; the Wronskian associated with the effective formulation (2.4). Equations (3.3) and (3.6) imply the "Wronskian equivalence"

$$\frac{\mathcal{W}^{(\text{eff})}}{p^s} \simeq \frac{\mathcal{W}}{p} = F(\omega, \eta), \quad \text{as } \ell \to 0. \tag{3.7}$$

Thus, as $\ell \to 0$, the dependence of $\mathcal{W}^{(\text{eff})}$ on ω, η approaches that of $\mathcal{W}$. It is well known that for a fixed ω, the roots of $\mathcal{W}$ and $\mathcal{W}^{(\text{eff})}$ in the complex η plane are $\sqrt{\lambda_n}$ and $\sqrt{\lambda_n^*}$. Thus, this result reestablishes the spectral equivalence (3.1b). More important, the relation between $\mathcal{W}$ and $\mathcal{W}^{(\text{eff})}$ yields a spectral equivalence result in the complex ω plane too. Let $\omega_n(\eta)$ and $\omega_n^*(\eta)$ be the roots of $\mathcal{W}$ and $\mathcal{W}^{(\text{eff})}$ in the complex ω plane, for a fixed η. Then, the last results yield

$$\omega_n^*(\eta) \to \omega_n(\eta) \quad \text{as } \ell \to 0. \tag{3.8}$$

Thus, spectral equivalence exists in both the spatial wavenumber plane and the temporal frequency plane. These results are important for establishing effective modal representations, as well as for *effective resonance frequency* analysis (effective SEM).

IV. Application and Numerical Example

Consider a complex duct with $\mu = 1$ and with $\epsilon(x)$ that possesses a random microstructure. Typical examples are shown in Fig. 1. The 1-D Green function of (2.3) associated with such 1-D heterogeneities is given by [18]

$$\tilde{g}(x, x'; \eta) = \frac{u_\ell(x_<) u_r(x_>)}{\frac{\mathcal{W}[u_\ell, u_r]}{p(x)}}. \tag{4.1}$$

The poles of $\tilde{g}$ in the complex ω plane are the duct resonances, formally given by the roots of the Wronskian $\mathcal{W}$. We demonstrate the implications of the Wronskian equivalence on the poles' effective representations. Effective resonances representation of the reflected transient field, due to a transient excitation, is also considered. Using the results of the previous sections, we have $\mu^{(\text{eff})} = 1$, and

$$\epsilon^{(\text{eff})} = \text{diag}\left[\epsilon_{xx}^{(\text{eff})}, \epsilon_{zz}^{(\text{eff})}\right], \quad \epsilon_{xx}^{(\text{eff})} = \frac{1}{\left(\frac{1}{\epsilon}\right)^s}, \quad \epsilon_{zz}^{(\text{eff})} = \epsilon^s \tag{4.2}$$

where the superscript s denotes local smoothing operation as defined in (2.1a).

We have used a random number generator to synthesize random multilayer ducts. Two manifestations $\epsilon_1(x)$, $\epsilon_2(x)$ are shown in Fig. 1. The number of random micro layers is 2κ, and the width of each micro layer is $1/2\kappa$—the micro scale. $\kappa = 20, 50$ for $\epsilon_{1,2}(x)$, respectively. The value of ϵ for the micro layers is chosen at random, uniformly distributed between two and six, yielding an average contrast of four with the surrounding vacuum. We have computed the effective properties of $\epsilon_{1,2}(x)$ via (4.2), using Haar multiresolution system. This was performed for two choices of the homogenization scale (HS): $2^{-j} = 1$ and 0.1 (the scale on which the smoothing

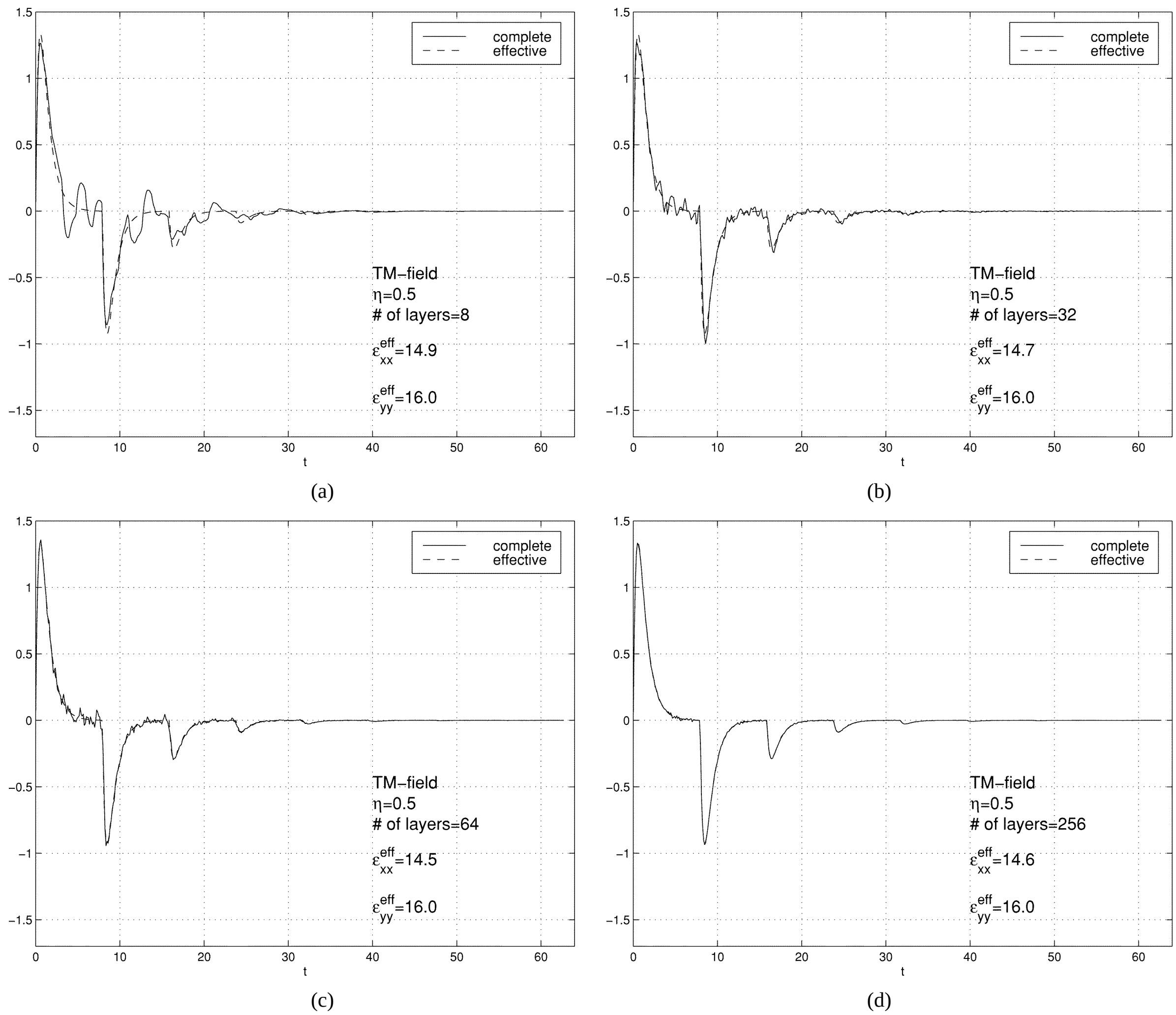

Fig. 7. The reflected field for $\eta = 0.5$ and TM polarization. (a)–(d) show the response of the 8, 32, 64, and 256 random micro-layer ducts, respectively.

is performed—see (4.2) or (2.1a). For HS$=1$, $\epsilon^{(\text{eff})}$ is constant; $\epsilon_{1,2}^{(\text{eff})} = \text{diag}[3.66, 4.06]$, $\text{diag}[3.59, 3.96]$, respectively. For HS$=0.1$, the resulting effective properties are depicted in Fig. 2. We have used standard transfer matrix method, in conjunction with a numerical root-finder, to compute the poles of (4.1) in the complex ω plane, for the complete (all scales) $\epsilon_{1,2}(x) - \omega_n$ and for the $\epsilon^{(\text{eff})}(x) - \omega_n^*$. Fig. 3(a) shows ω_n for $\epsilon_1(x)$ and ω_n^* that correspond to HS$=0.1$ and 1. TM polarization and $\eta = 0.75$ (incident plane wave, at 48.5° relative to the z-axes). To get a better quantitative measure of the approximation of ω_n by ω_n^*, Fig. 3(b) plots the relative difference $|(\omega_n^* - \omega_n)/\omega_n|$ versus n for $\epsilon_1(x)$ with HS$=0.1$ and 1. Fig. 3(c) shows the same but for $\epsilon_2(x)$. It is seen that, in general, smaller HS gives a better approximation $\omega_n^* \to \omega_n$, and this effect is more evident in the lower resonances, say, $n \leq 3$. Fig. 3(d) shows the relative difference for both polarizations and for $\epsilon_{1,2}(x)$ with the "better" choice of HS$=0.1$. It is seen that $|\omega_n^* - \omega_n|$ is an order of magnitude smaller for the smaller micro-scale $(\epsilon_2(x))$, as predicted in the previous sections.

To study the implications of the effective resonances representations on the computation of the total reflected field, we have synthesized a second set of random multilayer ducts with a unit total width, and with a number of micro layers ranging from eight (micro scale$=1/8$) to 256 (micro scale$=1/256$). The values of ϵ in the micro layers are uniformly distributed between eight and 24, yielding an average ϵ contrast of 16 with the surrounding vacuum. Four manifestations are shown in Fig. (4a)–(d). Fig. 5 shows the same as Fig. 3(d), but for the eight and 128 random layers cases, and $\eta = 0.5$ (plane-wave incidence at 30°). Again it is evident that the relative difference decreases with the micro scale, as predicted by (3.8). This set of random ducts was excited by an incident field

$$u^i(0,t) = e^{-t} - e^{-3t}, \quad t \geq 0. \tag{4.3}$$

This excitation signal is shown in Fig. 6. Note that inside the laminated ducts the pulse typical width is $O(1/2)$—narrower than the total width of the ducts but larger than the micro scale. Fig. (7a)–(d) shows the transient reflected fields for four typical manifestations of the random multilayer ducts, computed exactly (solid line) and computed using the corresponding effective resonances (dashed lines). It is seen that the effective representations reconstruct the large-scale components of the complete responses. Furthermore, it is seen that the magnitude of the micro-scale component of the response is decreasing as the micro scale of the heterogeneity decreases, as predicted by (3.5).

V. Conclusion

A multiresolution homogenization theory provides formulations governing large-scale solution components for problems of wave propagation and scattering in complex multilayer duct environments. With the relations between the true and effective Wronskians, given in (3.7) (the "Wronskian equivalence"), a *space-time spectral equivalence* is established; the homogenized formulation *effective modes, eigenvalues, and resonances* provide reliable estimates of the corresponding fundamental constituents associated with the complete formulation. These results are used here for *effective resonance frequency* analysis (effective SEM) of random multilayer ducts.

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