A Study of the Effective Properties of Complex Scatterers Using Multiresolution Decomposition

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Abstract

A systematic study of the across scale coupling phenomenology in scattering problems is addressed using the theory of multiresolution decomposition and orthogonal wavelets. By projecting an integral equation formulation of the scattering problem onto a set of subspaces that constitutes a multiresolution decomposition of $L_2(\mathbf{R})$, one can derive two coupled formulations. The first governs the macroscale response, and the second governs the microscale response. By substituting the formal solution of the latter in the former, a new *self consistent* formulation that governs the macroscale response component is obtained. This formulation is written on a macroscale grid, where the effects of the microscale heterogeneity is expressed via an across scale coupling operator. This operator can also be interpreted as representing the *effective properties* of the microstructure. We study the properties of this operator versus the characteristics of the Green function and the microstructure for various structural acoustics problems, using general asymptotic considerations. Specific examples of scattering from a thin, linearly elastic, fluid loaded plate with surface mass density or bending stiffness variations are provided. We show that mass microscale variation has virtually no effect on the macroscale response, and stiffness microscale variations can significantly effect the macroscale response. Classes of stiffness variations that have an identical macroscale response are derived using a one dimensional local constitutive relation developed to govern the macroscale field components. The results are supported by numerical examples.

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1 Introduction

A fundamentally difficult problem in wave theory is the prediction of the response field when an excitation is applied to a structure that is described over a wide range of length scales. Coupling across scales is the difficulties physical genesis. From the mathematical point of view it stems from the essentially nonlinear dependencies of a system response on the coefficients of the governing equations. These nonlinearities exist even if the governing formulation is linear, as is the case in many problems of interest. To set some of the issues we consider time-harmonic, line load applied to an infinite, linearly elastic plate that contains a spatially limited region in which the mass density, or local bending stiffness, vary. The plate bounds a fluid half space. The system configuration is schematized in Fig. 1a.

When a time harmonic excitation is applied to such a system, it is convenient to use the wavelength λ as a discriminator of the various length scales pertaining to the problem. Although not precisely defined yet, this wavelength would usually be that associated with the response field of a corresponding "background", or homogeneous, problem. We shall refer to the scales in the order of λ and above as the *macroscale*, and to the range of scales much smaller than λ (say, $\lambda/50$) as the *microscale*. A complexity frequently encountered in acoustic scattering is the one characterized by a microscale heterogeneity that extends over physical domains measured on the macroscale. This heterogeneity—non-stationary when observed over the macroscale dimensions—is referred to as a **"complex scatterer."** It is schematized in Fig. 1b. We wish to estimate the acoustic field radiated far from a plate with a mass or bending stiffness heterogeneity of the articulated complex scatterer type, and to investigate its dependence on the microscale component of the heterogeneity.

Traditional formulations developed to govern the response of systems, obtained by directly applying basic physical laws, are written in physical space. Thus, the resulting frameworks govern the complete response of the body and require a complete description of the system heterogeneity (by "complete" here is meant the incorporation of the entire range of length scales.) When a complex scatterer is considered, the inherent completeness of these formulations is a source of great computational difficulties for the following reasons.

1. A direct numerical approach for the solution of such scattering problems requires a numerical grid sufficiently large to cover the entire heterogeneity, yet sufficiently fine to capture the microscale component of the heterogeneity - the numerical dimension of which may become too large to handle.

2. Any change made in either the micro- or the macro-structure formally introduces a completely

new problem—one defined again on a microscale grid of macroscale outer dimensions. That is, there is no reduction in the required computations, occasioned by a lack of change of either the micro- or the macro-structure.

A step in the direction of resolving some of these difficulties is the recognition that "fine details" of the body *response* associated with heterogeneity possessing the complexity articulated above, reside essentially in the evanescent spectrum domain, thus generating a vanishingly small contribution to the radiation far-field. A formulation of the scattering problem associated with complex scatterer, that is *a priori* tuned to adequately describe the response component measured on the λ scale (but not smaller) is therefore suggestive. This formulation is termed here as the *formulation smooth*. The response governed is termed here as the *response smooth*, or *macroscale response*. Now, two fundamental questions can be posed as regards the *footprint* of a spatially limited, small-scale plate heterogeneity in the macroscale response (far field). The first question is: **What types of heterogeneity have a footprint?** for those heterogeneity for which a footprint exists, one might expect that only a limited information of the small scale heterogeneity is required to estimate its footprint in the acoustic far field. The second question is: **What is this information?**

Let us articulate these research issues in a fashion tailored to the computational difficulties 1 and 2 described above. It is recognized that, in general, a numerical scheme discretization based on a macroscale grid would not yield predictions of the macroscale response, of satisfactory accuracy, if no further precautions designed to capture the effects of the microscale heterogeneity on the macroscale response are exercised. It is further recognized that the coupling between the microscale heterogeneity and the macroscale response is essentially a nonlinear process. Thus, the latter cannot be characterized as a response governed by the same scattering formulation with the heterogeneity simply made smooth. The self consistent accommodation of these effects within a macroscale tuned grid, via a precisely defined across scale coupling operator, is the first role addressed by the formulation smooth—a role addressed toward relaxing difficulty 1 above.

The difficulties are not resolved, however, if the precise definition or derivation of the across scales coupling operator itself requires detailed description of the structure heterogeneity, or massive computations. Unfortunately, this is usually the case. Thus, the degree to which that operator can be made invariant under certain classes of variations of the macro- or micro-structures of a complex scatterer, and the precise characterization of these classes is the second, and fundamental, role addressed by the formulation smooth—a role addressed toward relaxing (partially, at least) difficulty 2 above.

Since the articulated operator describes the effects of the microstructure on the macroscale

response, it is interpreted as an effective properties description of the complex scatterer, or effective material.

The purpose of the present study is two-fold. As a first objective, we report the **self consistent derivation** of the *formulation smooth*. That is, a formulation of the scattering problem, that is written on a *macroscale grid*, and governs the macroscale component of the response field for a complex scatterer. The term "self consistent" here meant to imply that the effect of the microscale heterogeneity on the macroscale response is expressed via a precisely defined effective properties operator. This formulation is to be applied to non stationary microscale heterogeneity, as manifested by the complex scatterer definition. The motivation for this objective is relaxing difficulty 1 above.

The second objective is **Effective properties study**. That is, an investigation as to the degree to which the effective properties operator or some of its components are invariant under classes of variations of the macro or micro structures of a complex scatterer, and the precise characterization of these classes. The motivation is relaxing difficulty 2 above.

In the context of the second objective, two fundamental results are reported; a microscale mass variation has virtually no footprint in the macroscale response (far field), whereas a microscale stiffness variation can significantly effect the macroscale response. This conclusion is traced to the behavior at the origin of the Green function in the integral equation formulation governing the appropriate measure of response. For mass variation the Green function is smooth; for stiffness variation it is singular. Nevertheless, we will show that in the one dimensional case, one can find classes of different stiffness microstructures that have the same footprint in the macroscale response.

The structure of the paper is as follows. In Sec. 2 we briefly review the tools of multiresolution theory needed for subsequent derivations, and develop the formulation smooth. In Sec. 3 we study the properties of the new formulation, essentially via a study of the effective properties operator. It is shown that this operator is predominantly determined by the nature of the singularity of the associated integral operator kernel. Two cases are discussed - the smooth kernel and the "mildly singular" kernel. These sections are essentially a summary of results obtained in [1]–[4]. Section 4 implements the new formulation to structural acoustics problem of a thin, linearly elastic, fluid loaded plate possessing mass or stiffness heterogeneity. This section demonstrates that a microscale variation of mass has no footprint in the large-scale response, whereas a microscale variation of stiffness can significantly effect the large-scale response. These results are first derived analytically and then demonstrated by numerical simulations. In section 5 we derive an expression that determines the class of small-scale stiffness variations with identical footprints in the large-scale plate response. The results are demonstrated numerically by generating small-scale stiffness

variations that are easily seen to be different, and showing they lead to the same large-scale plate response. Concluding remarks are presented in section 6.

2 The Formulation Smooth

The theory of multiresolution decomposition (MRD) and orthogonal wavelets [5]–[7] provides a natural framework for the study of inter-scale coupling processes. There is a rapidly growing literature on the application of wavelet transformations to operators. Examples are the works in [8]–[12]. However, the main concern in [8]–[12] is purely computational in its nature: to reduce the number of operations required for the inversion of certain classes of operators by applying wavelet transformations and reducing a densely populated matrix to a sparse one. This approach, however, will not reduce the number of operations for the complex scatterer case, as its performances rely on the smoothness of the system heterogeneity. We note that multiresolution analysis and wavelets have been used recently in [13] for numerical homogenization, i.e. for generating an equation with slowly varying coefficients whose solution has the same large scale behavior as that of the original equation. Relation of this type has been pointed out and detailed also in [1, 2]. The work in [13] is devoted mainly for a sophisticated "decimation" process in which efficient numerical algorithm for estimating the large scale response component is developed. It does not address directly any of the questions articulated in the Introduction to this work. The problem presented in this paper requires a different viewpoint, and subsequently, a different analysis. Below we summarize briefly the MRD theory in a manner tailored to the class of problems discussed here.

2.1 Multiresolution Decomposition

Let $\{V_j\}$ be a nested sequence of linear spaces that constitutes a multiresolution decomposition (MRD) of $L_2(R)$. Let $\phi(x)$ and $\psi(x)$ be the associated scaling function and wavelet, respectively, and define $\phi_{mn}(x)$ as the translated and dilated version of $\phi(x)$ via $\phi_{mn}(x) = 2^{m/2}\phi(2^m x - n)$, with m, n integers. A similar definition holds for ψ_{mn}. The sets $\{\phi_{jn}\}_n$ and $\{\psi_{jn}\}_n$ are orthonormal bases of V_j and of the orthogonal complement of V_j in V_{j+1}, respectively. This complement is denoted O_j. Thus, $V_j = \text{span}_n\{\phi_{jn}\}$ and $V_{j+1} = V_j \cup \text{span}_n\{\psi_{jn}\}$. An approximation of a function $f(x)$ at a resolution k can be written as the sum of two mutually orthogonal functions, namely a smooth (f^s) and a detail (f^d) components. We have

$$f(x) = f^s(x) + f^d(x) \tag{2.1}$$

where

$$f^s(x) \;=\; \mathbf{P}_j f(x) \quad = \quad \sum_n F^s_n \phi_{jn}(x), \qquad F^s_n = \langle f, \phi_{jn} \rangle \tag{2.1a}$$

$$f^d(x) \;=\; \mathbf{D}^k_j f(x) \quad = \quad \sum_{m=j}^{k-1} \sum_n F^d_{mn} \psi_{mn}(x), \qquad F^d_{mn} = \langle f, \psi_{mn} \rangle \tag{2.1b}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product of $L_2(R)$, and $j < k$ is some reference resolution - a judicious choice of which depends on the physics of the problem. The scaling functions and wavelets satisfy the orthonormality relations $\langle \phi_{jn}, \phi_{jn'} \rangle = \delta_{nn'}$, $\langle \psi_{mn}, \psi_{m'n'} \rangle = \delta_{mm'} \delta_{nn'}$, and $\langle \phi_{jn'}, \psi_{mn} \rangle = 0 \; \forall j \leq m$. Thus, $\mathbf{P}_j$ and $\mathbf{D}^k_j$ in (2.1a)–(2.1b) are projection operators satisfying,

$$\mathbf{D}^k_j \mathbf{P}_{j'} \;=\; \mathbf{P}_{j'} \mathbf{D}^k_j \;=\; 0 \qquad \forall j \geq j'. \tag{2.2}$$

The scaling functions and the wavelets are spatially confined functions (i.e. ϕ and ψ are either of compact support, or fast decreasing), centered more or less about the origin. They can be interpreted, respectively, as defining a local low-pass and a local band-pass filters. Examples are shown in Fig. 2. Let us define the effective support of ψ and ϕ as the intervals outside of which their respective values are practically zero, and denote it by Λ_ψ and Λ_ϕ. From the dilation translation relations articulated above, it follows that the terms ψ_{mn} in (2.1b) are situated, respectively, around the points

$$x_{mn} \;=\; n 2^{-m}, \tag{2.3}$$

each having an effective support $2^{-m} \Lambda_\psi$. A similar description applies to the terms ϕ_{jn}. The set of points in (2.3) is called the wavelet grid. The functions $f^s(x)$ and $f^d(x)$ in (2.1) can be interpreted as a locally smoothed, or averaged, description of $f(x)$ on the length scale 2^{-j}, and a signal describing the finer details covering length scales ranging from $2^{-(j+1)}$ to 2^{-k}, respectively. Here and henceforth we refer to the number 2^{-m} and the index m as a length-scale and the resolution associated with it, respectively. A wavelet has M vanishing moments,

$$\int x^m \psi(x)\, dx \;=\; 0\,, \qquad m = 0,\, 1, \ldots, M-1. \tag{2.4}$$

For the Haar wavelet $M = 1$, for the cubic-spline Battle-Lemarie wavelet $M = 4$ [5, 6]. This parameter can be related to the regularity of the multiresolution system and to the support of the associated wavelets and scaling functions (see [7] for details and examples).

2.1.1 Asymptotic expressions for the inner products

When the function $f(x)$ in the neighborhood of $x = x_{mn}$ varies slowly compared to the lengthscale 2^{-m} and possesses M first derivatives (see (2.4) for M), it is possible to derive approximate expressions for the inner products $\langle f, \phi_{mn} \rangle$ and $\langle f, \psi_{mn} \rangle$. More specifically, it can be shown that for the cubic spline Battle-Lemarie system shown in Fig. 2 (see [1]),

$$\langle f, \phi_{mn} \rangle = 2^{-m/2} f(x_{mn}) - \alpha 2^{-m(2M+1/2)} O\left(f_{2M}\right) \tag{2.5a}$$

$$\langle f, \psi_{mn} \rangle = \beta 2^{-m(M+1/2)} f_M(x_{mn} + 2^{-m-1}) + \beta 2^{-m(M+5/2)} O\left(f_{M+2}\right) \tag{2.5b}$$

where $f_M(x)$ is defined as

$$f_M(x) \;=\; \left(i \frac{d}{dx}\right)^M f(x), \tag{2.5c}$$

α and β are constants given by $\alpha = \frac{B_M}{2\,(2M)!}$ and $\beta = 2^{-M} \left[\frac{(2^{2M}-1)\,B_M}{(2M)!}\right]^{1/2}$. B_M is the M-th Bernoulli number, and x_{mn} is a point in the wavelet grid defined by Eq. (2.3). Note that for the cubic spline Battle-Lemarie system one has $M = 4$, $B_M = 1/30$, so $\alpha, \beta \ll 1$. Thus, the remaining terms in the right side of (2.5a)–(2.5b) are small. Furthermore, if $f_M = O(1)$ we get,

$$|\langle f, \psi_{mn} \rangle| \ll 1, \qquad \text{for } m \gg 1 \tag{2.6}$$

which actually states that the inner product with the wavelet is vanishingly small if the function f is smooth. Similar results can be derived for other multiresolution systems.

2.1.2 Local irregularity and wavelets

The manner in which the irregularity of a function reflects on its wavelet coefficients plays a role in the multiresolution study of scattering effects [14]. It is also a central issue in the study of effective properties of complex scatterers, as the effective material operator strongly depends on the nature of the Green's function singularity [1, 2]. Wavelet coefficients in the presence of an irregularity has been studied in the general framework of Hölder regularity condition and Hölder spaces [7, 10]. For the purposes of the present work, however, it is beneficial to sacrifice generality and to address the issue via an example tailored to the particular needs of the forthcoming analysis. Let $g(x)$ be a

function with r-th derivative discontinuity at $x = x'$ ($r = 0$ corresponds to a discontinuity in $g(x)$ itself). Assume further that $g(x)$ is bounded and can be expressed as,

$$g(x) = f(x) + h(x) \tag{2.7}$$

where $f(x)$ is a smooth function $f(x) \in C^M$, and $h(x)$ contains the irregularity of $g(x)$ in the form,

$$h(x) = \begin{cases} a^-(x - x')^r & \text{for } x < x' \\ a^+(x - x')^r & \text{for } x > x'. \end{cases} \tag{2.8}$$

For boundedness it is also assumed that $|x - x'| \leq O(1)$. The r-th order discontinuity magnitude across $x = x'$ is given by $(a^+ - a^-)r!$. The wavelet coefficients of $g(x)$ can now be expressed as $\langle g, \psi_{mn} \rangle = \langle f, \psi_{mn} \rangle + \langle h, \psi_{mn} \rangle$. Thus, if 2^{-m} is small compare to the typical length scale of $f(x)$ the first term can be approximated by Eq. (2.5b). It can be shown that if the wavelet irregularity order is higher than that of $h(x)$, the second term is bounded by [7, 10],

$$|\langle h, \psi_{mn} \rangle| \leq C\, 2^{-m(r+1/2)}, \qquad C = O(1) \tag{2.9}$$

where the constant C depends on the specific multiresolution system. This bound is valid for all m.

2.2 A multiresolution study of the scattering problem

We shall refer throughout to the Fredholm integral equation of the second kind,

$$u(x) = u_0(x) + \mathbf{L}u \qquad a \leq x \leq b \tag{2.10}$$

where $\mathbf{L}$ is the integral operator defined as

$$\mathbf{L}u = \int_a^b K(x, x')\, u(x')\, dx' \tag{2.10a}$$

and where $u_0(x)$ is a forcing term, $u(x)$ is the response to be determined and $K(x, x')$ is a known kernel. This equation occurs widely in diverse areas of mathematical physics and engineering. The formulation is said to be *complete* in the sense that it contains the entire range of lengthscales associated with the problem. In cases where the scatterer is flat and the system heterogeneity is characterized by the material properties (e.g. local constitutive relations) the scattering formulation can be written as (2.10)–(2.10a) with the following simplifications [1],

$$K(x,x') = G(x-x')h(x'). \tag{2.11a}$$

Here $G(x)$ is the Green's function of an appropriately defined background problem, usually that modeled by a linear differential equation with constant coefficients, and $h(x)$ represents a material heterogeneity with variations occurring on both the microscale and macroscale - the complex scatterer. In these cases the operation $\mathbf{L}u$ can be represented as the background operator $\mathbf{L}_b$ associated with $G(x)$, operating on the function hu,

$$\mathbf{L}u = \mathbf{L}_b(hu), \tag{2.11b}$$

where,

$$\mathbf{L}_b f = \int G(x-x')\, f(x')\, dx' \tag{2.11c}$$

and, by definition of the complex scatterer, the integration limits in (2.10a) are effected by h.

With the MRD theory, the integral equation formulation (2.10) can be decomposed into a pair of coupled formulations governing a length-scale resolved response. We express the response field $u(x)$ as a sum of two components, namely the response smooth $u^s(x)$ (macroscale response) and the response detail $u^d(x)$ (microscale response). If 2^{-k} is a lower bound on the lengthscales pertaining to the problem, then $u(x) \simeq \mathbf{P}_k u(x)$, and one can invoke (2.1),

$$u(x) = u^s(x) + u^d(x) \tag{2.12}$$

where,

$$u^s(x) = \mathbf{P}_j u(x) = \sum_{n=n_1(j)}^{n_2(j)} s_n \phi_{jn}(x) \tag{2.12a}$$

$$u^d(x) = \mathbf{D}_j^k u(x) = \sum_{m=j}^{k-1} \sum_{n=n_1(m)}^{n_2(m)} d_{mn} \psi_{mn}(x). \tag{2.12b}$$

The former and the latter represents, respectively, the response field smoothed on a reference lengthscale 2^{-j}, and the remaining fine details. s_n and d_{mn} are yet to be determined coefficients, representing the smooth and detail parts of the response. By Eq. (2.2) u^s and u^d are mutually

orthogonal. Substituting (2.12) into (2.10) and further operating on the resulting equation with $\mathbf{P}_j$ and $\mathbf{D}_j^k$, we obtain,

$$u^s(x) = u_0^s(x) + \mathbf{P}_j\mathbf{L}\mathbf{P}_j u + \mathbf{P}_j\mathbf{L}\mathbf{D}_j^k u \tag{2.13a}$$

$$u^d(x) = u_0^d(x) + \mathbf{D}_j^k\mathbf{L}\mathbf{P}_j u + \mathbf{D}_j^k\mathbf{L}\mathbf{D}_j^k u \tag{2.13b}$$

where $u_0^s(x)$ and $u_0^d(x)$ are the mutually orthogonal smooth and detail components of the integral equation forcing term, derived from $u_0(x)$ via a relation similar to (2.12a) and (2.12b). Equations (2.13a) and (2.13b) can be cast into a matrix form, written for the sets of unknown coefficients $\{s_n\}$ and $\{d_{mn}\}$. The result is the set

$$\begin{pmatrix} \mathbf{I}-\mathbf{\Phi} & -\mathbf{C} \\ -\bar{\mathbf{C}} & \mathbf{I}-\mathbf{\Psi} \end{pmatrix} \begin{pmatrix} \vec{s} \\ \vec{d} \end{pmatrix} = \begin{pmatrix} \vec{e} \\ \vec{E} \end{pmatrix} \tag{2.14}$$

where $\mathbf{I}$ is an identity matrix, $\vec{s}$ and $\vec{d}$ are the column vectors with the elements $\{s_n\}$ and $\{d_{mn}\}$, respectively, representing the smooth and detail (macroscale and microscale) response components. $\vec{e}$ and $\vec{E}$ are the excitation smooth and detail column vectors, whose elements are the respective (known) coefficients of $u_0^s(x)$ and $u_0^d(x)$, ordered similar with $\vec{s}$ and $\vec{d}$. $\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{C}$, and $\bar{\mathbf{C}}$ are matrix operators whose elements are given, respectively by,

$$\begin{aligned}
\Phi_{n',n} &= \langle \mathbf{P}_j\mathbf{L}\mathbf{P}_j\phi_{jn}, \phi_{jn'} \rangle &&= \langle \mathbf{L}\phi_{jn}, \phi_{jn'} \rangle \\
\Psi_{m'n',mn} &= \langle \mathbf{D}_j^k\mathbf{L}\mathbf{D}_j^k\psi_{mn}, \psi_{m'n'} \rangle &&= \langle \mathbf{L}\psi_{mn}, \psi_{m'n'} \rangle \\
C_{n',mn} &= \langle \mathbf{P}_j\mathbf{L}\mathbf{D}_j^k\psi_{mn}, \phi_{jn'} \rangle &&= \langle \mathbf{L}\psi_{mn}, \phi_{jn'} \rangle \\
\bar{C}_{m'n',n} &= \langle \mathbf{D}_j^k\mathbf{L}\mathbf{P}_j\phi_{jn}, \psi_{m'n'} \rangle &&= \langle \mathbf{L}\phi_{jn}, \psi_{m'n'} \rangle .
\end{aligned} \tag{2.15}$$

Equation (2.14) provides the starting point for a multiresolution study of the scattering problem (see [1, 14]), and for a self consistent development of the formulation smooth. From the lower half of (2.14), the response detail $\vec{d}$ can be expressed in terms of the response smooth $\vec{s}$ and the excitation detail $\vec{E}$. When this is substituted into the upper half of Eq. (2.14) we get a formulation governing $\vec{s}$—the formulation smooth,

$$\left[\mathbf{I}-\mathbf{\Phi}-\mathbf{C}\left(\mathbf{I}-\mathbf{\Psi}\right)^{-1}\bar{\mathbf{C}}\right]\vec{s} = \vec{e} + \mathbf{C}\left(\mathbf{I}-\mathbf{\Psi}\right)^{-1}\vec{E}. \tag{2.16}$$

It has been shown in [1] that $\mathbf{\Phi}$ can be interpreted as a "smoothed version" of the operator $\mathbf{L}$; a representation of the latter by a straightforward discretization on a reference grid separation 2^{-j}. Thus $\mathbf{C}\,(\mathbf{I}-\mathbf{\Psi})^{-1}\,\bar{\mathbf{C}}$ has been interpreted as an effective properties operator; an operator describing the coupling across scales due to the presence of a microstructure. The dependence of this operator on the heterogeneity $h(x)$ is nonlinear.

3 Properties of the Formulation Smooth

The manner in which the microstructure effects the effective material operator $\mathbf{C}(\mathbf{I}-\mathbf{\Psi})^{-1}\bar{\mathbf{C}}$ in (2.16) have been investigated in [1, 2] for the case of a heterogeneity with a widely separated micro- and macro-scales, pertaining to a thin, elastic, fluid-loaded plate. Since these results play a pivotal role in the present theory, we shall briefly summarize them in the following subsections and develop new results, more relevant to electromagnetic scattering. We assume that the integral operator kernel can be expressed as in (2.11a). Then, with $\mathbf{L}_b^*$, the adjoint operator of $\mathbf{L}_b$ in (2.11b), the elements can be rewritten as,

$$\Phi_{n',n} = \langle h\phi_{jn}, \mathbf{L}_b^*\phi_{jn'}\rangle = \langle h\phi_{jn}, \phi_{jn'}^f\rangle \tag{3.1a}$$

$$\Psi_{m'n',mn} = \langle h\psi_{mn}, \mathbf{L}_b^*\psi_{m'n'}\rangle = \langle h\psi_{mn}, \psi_{m'n'}^f\rangle \tag{3.1b}$$

$$C_{n',mn} = \langle h\psi_{mn}, \mathbf{L}_b^*\phi_{jn'}\rangle = \langle h\psi_{mn}, \phi_{jn'}^f\rangle \tag{3.1c}$$

$$\bar{C}_{m'n',n} = \langle h\phi_{jn}, \mathbf{L}_b^*\psi_{m'n'}\rangle = \langle h\phi_{jn}, \psi_{m'n'}^f\rangle \tag{3.1d}$$

where $\phi_{jn'}^f$ and $\psi_{m'n'}^f$ are the "fields" defined by,

$$\phi_{jn}^f(y) = \mathbf{L}_b^*\phi_{jn} = \int \phi_{jn}(x)\overline{G(x-y)}\,dx \tag{3.2a}$$

and,

$$\psi_{mn}^f(y) = \mathbf{L}_b^*\psi_{mn} = \int \psi_{mn}(x)\overline{G(x-y)}\,dx \tag{3.2b}$$

where the overline denotes a complex conjugate. These expressions will be used in Sec. 3.2 to study the effective material operator.

3.1 System characterization and scales hierarchy

Two type length scales are defined by the physical system. The first applies to the variation of $G(x)$, induced by the dynamics of the system. We refer to this as the wavelength (λ) scale and to the associated resolution as m_λ. The possible irregularity of $G(x)$ at the origin may introduce additional variability over a relatively wide range of length scales. This irregularity has an essential role in the theory and it will be discussed further. The second type length scales is due to the system heterogeneity and applies to the variations of $h(x)$. These variations can range from the *macroscale* ($\geq \lambda$) to the *microscale* ($\ll \lambda$). The former and the latter are associated with the resolutions m_a and m_i, respectively, such that $2^{-m_a} \geq \lambda$ and $2^{-m_i} \ll \lambda$. Finally, the reference, or smoothing, scale should be carefully selected. Details of scale λ must be adequately described, whereas microscale details can be averaged. The following hierarchy of resolutions and scales applies,

$$m_a \leq m_\lambda < j < m_i \Longleftrightarrow 2^{-m_a} \geq 2^{-m_\lambda} \approx \lambda \gg 2^{-j} \gg 2^{-m_i}. \tag{3.3}$$

Essential to the present study is the concept of a "scatterer", as a localized region of *microscale* heterogeneity that is confined to a domain in space measured on the *macroscale*. Also essential is the existence of a "gap" in the scales for observing heterogeneity. Specifically, the system heterogeneity is not to posses any component in the range of scales $(2^{-m_a-1}, 2^{-m_i+1})$. We term this case a two-scale (macro/micro) variation. When referred to the j scale; i.e., the chosen smoothing scale; the macroscale variation is the smooth component and the microscale variation is the detail component. We write

$$h(x) \;=\; h^s(x) + h^d(x) \tag{3.4}$$

where,

$$h^s(x) \;=\; \sum_n a_n \phi_{m_a\, n}(x) \tag{3.4a}$$

is expressible as a synthesis of scaling functions and

$$h^d(x) \;=\; \sum_n b_n \psi_{m_i-1\, n}(x). \tag{3.4b}$$

as a synthesis of wavelets. The index n, in both these equations, span a region with a physical dimensions described on the macroscale. Note that $h^s(x)$ and $h^d(x)$ are mutually orthogonal,

and the associated scales are widely separated. The synthesis in (3.4a)–(3.4b) identifies the spatial averages of $h(x)$ over neighborhoods in the order of the macroscale 2^{-m_a} with h^s, and further implies that spatial averages of the detail component $h^d = h - h^s$ vanish if performed over neighborhoods larger than the microscale 2^{-m_i}. Denoting $\|\cdot\|_2$ the L_2 norm, we shall assume throughout the rest of this work that

$$\left\| h^d(x) \right\|_2 \;=\; O\left(\|h^s(x)\|_2 \right) \;=\; O(1) \tag{3.5a}$$

which is equivalent to

$$\sum_n |b_n|^2 \;=\; O\left(\sum_n |a_n|^2 \right) \;=\; O(1). \tag{3.5b}$$

In other words, the energy associated with the microstructure is of the order of that associated with the macrostructure, and both are beyond the perturbative regime.

Remark: The results obtained in this section, especially in subsection 3.2, hold also for scatterers that possess macro- and micro-structures $h^{s,d}$ more general then those specified in (3.4a)–(3.4b). For example, one can have more scales in the wavelet synthesis of h^d (3.4b), but the large gap between the micro- and macro-scales, as well as the "finite energy" conditions (3.5a), must be maintained.

3.2 The response components u^s, u^d, and the microstructure signature

An interesting general relation between the smooth and detail components of the response field can be derived from the set (2.14). The integral equation excitation term u_0 is nothing but the background system response. In most applications the latter varies on the scale of λ - the macroscale. Thus, with (2.5b)–(2.6) and (3.3) we obtain that the detail component of the excitation term in (2.14), namely $\vec{E}$, becomes vanishingly small. In this case the lower half of (2.14) yields,

$$(\mathbf{I} - \mathbf{\Psi})\, \vec{d} \;=\; \bar{\mathbf{C}}\, \vec{s} \tag{3.6}$$

which states that, when the excitation is described only on the macroscale, the relation between the smooth (macroscale) and detail (microscale) components of the response field is independent of the excitation. This relation will be useful in subsequent derivations.

Below we derive some results pertaining to the effect of the microstructure on the macroscale response—the microstructure signature—and relate it to the nature of G.

3.2.1 smooth $G(x)$

We assume the integral equation kernel function $G(x)$ is regular at the origin (that is, it varies on the scale of λ for all x). Then, an estimate of the matrix elements in Eqs. (2.15) can be obtained by invoking Eqs. (2.5a)–(2.6), (3.1a)–(3.2b), and the scale hierarchy characterization (3.3). We get (see [1]),

$$\Phi_{n',n} \simeq \overline{\phi^f_{jn'}(x_{jn})}\,\langle h, \phi_{jn}\rangle \tag{3.7a}$$

$$\Psi_{m'n',mn} \simeq \overline{\psi^f_{m'n'}(x_{mn}+2^{-m-1})}\,\langle h, \psi_{mn}\rangle \tag{3.7b}$$

$$C_{n',mn} \simeq \overline{\phi^f_{jn'}(x_{mn}+2^{-m-1})}\,\langle h, \psi_{mn}\rangle \tag{3.7c}$$

$$\bar{C}_{m'n',n} \simeq \overline{\psi^f_{m'n'}(x_{jn})}\,\langle h, \phi_{jn}\rangle, \tag{3.7d}$$

and ϕ^f, ψ^f are given by,

$$\phi^f_{jn}(y) = \mathbf{L}^*_b\phi_{jn} \simeq 2^{-j/2}\overline{G(x_{jn}-y)} \tag{3.8a}$$

$$\psi^f_{mn}(y) = \mathbf{L}^*_b\psi_{mn} \simeq \beta\, 2^{-m(M+1/2)}\overline{G_M(x_{mn}+2^{-m-1}-y)}. \tag{3.8b}$$

With the articulated conditions on G and the results in Eqs. (2.5b)–(2.6) it follows that $\left|\psi^f_{mn}(y)\right| \ll 1$. Together with (3.5a)–(3.5b) it induces the norm relations,

$$\|\mathbf{\Psi}\| \ll 1 \tag{3.9a}$$

$$\|\bar{\mathbf{C}}\| \ll 1 \tag{3.9b}$$

$$\|\mathbf{C}\| \leq O(1), \tag{3.9c}$$

where $\|\cdot\|$ denotes the matrix norm induced by the Euclidean vector norm. These estimates are derived in the appendix. It should be emphasized that they are valid only for smooth $G(x)$, and for the scatterers characterized by (3.3) and the discussion thereafter. With (3.9a)–(3.9c) one obtains for the effective material operator $\left\|\mathbf{C}(\mathbf{I}-\mathbf{\Psi})^{-1}\bar{\mathbf{C}}\right\| \ll 1$. The conclusion that, under (3.5a) and the smoothness condition on G, the macroscale response practically bares no footprint of the microscale

heterogeneity, follows directly from this last result (see [1]). Furthermore, this result establishes that a new formulation, governing the macroscale response component $u^s(x)$, is readily obtained form (2.10)–(2.11c),

$$u^s(x) \;=\; u_0(x) \;+\; \mathbf{L}_b \, h^s \, u^s \tag{3.10}$$

where $\mathbf{L}_b$ is the background operator defined in (2.11c), $u_0(x)$ is the excitation term, assumed here to vary on the macroscale only, and $h^s(x)$ is the locally averaged (smooth component) of the system heterogeneity (see (3.4)–(3.4b) and discussion thereafter). Equation (3.10) identifies $h^s(x)$ as a local effective property of the system heterogeneity $h(x)$.

The inequalities in (3.9a)–(3.9c) can be used to derive an estimate on the relation between u^s and u^d. From the multiscale resolved formulation (2.14),

$$(\mathbf{I} - \mathbf{\Psi}) \; \vec{d} \;=\; \bar{\mathbf{C}} \, \vec{s} \;+\; \vec{E}. \tag{3.11}$$

Using (3.9a), and subsequently applying the triangle and operator norm inequalities, we get,

$$\left\| \vec{d} \right\| \;\leq\; \|\bar{\mathbf{C}}\| \; \|\vec{s}\| \;+\; \left\| \vec{E} \right\| \tag{3.12}$$

where $\|u\|$ denotes the Euclidean vector norm if u is a vector, or the matrix norm induced by the Euclidean vector norm if u is a matrix. In cases where the excitation term u_0 is described on the macroscale only (i.e. $\left\| \vec{E} \right\| = 0$, so (3.6) holds) we get (with (3.9b))

$$\left\| \vec{d} \right\| \;\ll\; \|\vec{s}\| \tag{3.13a}$$

or, denoting $\|\cdot\|_2$ the L_2 norm (taken over the support of the heterogeneity),

$$\left\| u^d \right\|_2 \;\ll\; \|u^s\|_2 \, . \tag{3.13b}$$

3.2.2 singular $G(x)$

In cases where the integral equation background kernel $G(x)$ is singular at the origin, the regularity assumptions, leading to (3.9a)–(3.9c), cannot be used. Thus, the inequality $\left\| \mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1} \bar{\mathbf{C}} \right\| \ll 1$ does not hold in general. The immediate consequence is that a microstructure can have an effect on the macroscale response. The degree of this effect depends strongly on the irregularity order. If the

wavelet irregularity order is higher than that of $G(x)$, the results of Sec. 2.1.2 can be used to study the various operators and estimate their norms (see, for example [4]). Note that this condition can be satisfied by a proper choice of the MRD system. Furthermore, in such cases, if the irregularity of $G(x)$ is "mild" (that is, it is continuous and it possesses few continuous derivatives) one still gets from (2.9) $\left|\psi^f_{mn}(y)\right| \ll 1$, and the results of the previous section still apply, especially (3.10) and (3.13a)–(3.13b). That is: the microscale heterogeneity has virtually no footprint in the macroscale response.

However, if $G(x)$ is "strongly" singular at $x = 0$ (unbounded, or not continuous) the inequalities (3.9a)–(3.9c) cannot be used. The implications of this case are that the microscale heterogeneity has a significant effect on the macroscale response.

The next section demonstrates the two cases - "mildly singular" $G(x)$ and "strongly singular" $G(x)$, corresponding, respectively to mass and stiffness variation in thin, linearly elastic, fluid loaded plate.

4 Examples

4.1 Statement of the problem

We shall be concerned with the flexural velocity response $v(x)$ of a thin elastic plate lying in the $z = 0$ plane, due to a time harmonic ($e^{i\omega t}$) y-independent excitation. The problem statement is depicted in Fig. 1. A surface mass density heterogeneity and a bending stiffness heterogeneity are introduced, respectively, by the y-independent distributions $m(x)$ and $Q(x)$. The differential equation governing $v(x)$ is given by

$$\frac{d^2}{dx^2}\left\{Q(x)\frac{d^2}{dx^2}v(x)\right\} - \omega^2 m(x)v(x) \;=\; -i\omega\left[F(x) \;-\; p(x,0)\right], \tag{4.1}$$

where $F(x)$ is the y independent forcing, and $p(x,z)$ is the fluid pressure that satisfies the Helmholtz wave equation in the fluid half space $z > 0$. The fluid pressure is related to the flexural velocity via the boundary condition $v(x) = -\frac{i}{m_f\omega}\frac{\partial}{\partial z}p(x,0)$, where m_f is the volume mass density of the fluid. It is convenient to work with a dimensionless coordinate system throughout the examples section. Towards this end, we rewrite the plate surface mass density $m(x)$ and the bending stiffness $Q(x)$ in the following way,

$$m(x) \quad = \quad m_0\left[1 + \rho(x)\right] \tag{4.2a}$$

$$Q(x) \quad = \quad Q_0\,[1+q(x)] \tag{4.2b}$$

where $\rho(x)$ and $q(x)$ are normalized, dimensionless distributions expressing the local variations with respect to the background values m_0 and Q_0. The flexural wavenumber associated with m_0 and Q_0 is given by $k_{f_0} = (m_0\omega^2/Q_0)^{1/4}$. With the dimensionless length coordinate $x \mapsto k_{f_0}x$, the equation of motion (4.1) becomes,

$$\frac{d^2}{dx^2}\left\{[1+q(x)]\,\frac{d^2}{dx^2}v(x)\right\} - [1+\rho(x)]\,v(x) \;=\; -\frac{i}{\omega m_0}\,[F(x) \;-\; p(x,0)]\,. \tag{4.3}$$

The background plate Green function (i.e. the response of the homogeneous plate $q(x) = \rho(x) = 0$ to a line forcing $F(x) = \delta(x)$) is given by the inverse Fourier transform [15],

$$g(x) \;=\; \frac{1}{2\pi}\int \tilde{g}(\xi)e^{i\xi x}\,d\xi. \tag{4.4}$$

$$\tilde{g}(\xi) \quad = \quad \frac{i}{\omega m_0}\,\frac{\zeta}{(1-\xi^4)\,\zeta + i\epsilon/\hat{M}} \tag{4.4a}$$

$$\epsilon \quad = \quad \frac{m_f\,k_0}{m_0\,k_{f_0}^2} \tag{4.4b}$$

$$\hat{M} \quad = \quad \sqrt{\frac{\omega}{\omega_c}} \;=\; \frac{k_0}{k_{f_0}}. \tag{4.4c}$$

Here ϵ and $\hat{M}$ are the fluid loading parameter and the root of the ratio of ω to the coincidence frequency ω_c, and ζ is the z-component of the fluid pressure wavenumber,

$$\zeta \;=\; (\hat{M}^2 - \xi^2)^{1/2}\,, \qquad \mathrm{Im}\zeta \;\geq\; 0, \quad \mathrm{Re}\zeta \;\geq\; 0. \tag{4.4d}$$

The last two inequalities are the radiation and convergence conditions pertaining to the fluid pressure waves in the $z > 0$ half space. This response is shown in Fig. 3. It is a smooth function, described essentially on the scale of 2π. It has a third derivative discontinuity at $x = 0$ and a variation scale of $\lambda = 2\pi$ away from the origin. Consistent with (3.3) the macro, reference, and micro resolutions are defined as $m_a < 1, \quad j = 0$, and $m_i = 2$, respectively. These values correspond to the wavelength λ, a reference scale of about $\lambda/6$, and a microscale in the order of $\lambda/24$. Figures 4a and 4b show two different manifestations of microstructures, namely $h_1^d(x)$ and $h_2^d(x)$, synthesized

via (3.4b) with the articulated definitions of scales. For each manifestation, the coefficients b_n were synthesized by a random number generator. Figure 4c shows an example of a macrostructure $h^s(x)$. Evidently, $h^d_{1,2}(x)$ vary on a lengthscale much smaller than the (normalized) wavelength $\lambda = 2\pi$, and the $h^s(x)$ variation lengthscale is in the order of λ.

In the next subsections we shall examine the plate response associated with mass or stiffness complex scatterers. Specifically, we shall show that

1. Microscale mass heterogeneity with macroscale outer dimension has no footprint in the macroscale response.

2. Microscale stiffness heterogeneity with macroscale outer dimension can significantly effect the macroscale response.

3. It is possible to synthesize classes of different stiffness complex scatterers such that the macroscale response associated with each of the members of the class is the same.

These results are first obtained by using theoretical considerations, and then demonstrated numerically. As seen in the next two subsections, points 1 and 2 above follow directly from the analysis of Sec.3.2. Point 3 needs more detailed analysis and is treated in a separate section.

4.2 Mass heterogeneity

Here we examine the case of mass density variation and constant stiffness. Thus, $q(x) = 0$ and the equation of motion (4.3) can be rewritten as an integral equation by expressing the solution $v(x)$ as a convolution of $g(x)$ with the extended forcing term $F(x) + i\omega m_0 \rho(x) v(x)$. The result is the Fredholm integral equation formulation (2.10)–(2.11c) with

$$u(x) = v(x) \tag{4.5a}$$

$$h(x) = \rho(x) \tag{4.5b}$$

$$G(x) = i\omega m_0 g(x) \tag{4.5c}$$

$$u_0(x) = v_0(x) = \int g(x - x')\, F(x')\, dx' \tag{4.5d}$$

where $g(x)$ is defined in (4.4)–(4.4c). It is seen from (4.5c) that the kernel $G(x)$ associated with the integral operator is smooth. The results of Sec.3.2.1 can be cited directly; (i) microscale mass variation has no footprint in the macroscale response, the latter is governed now by the formulation

(3.10), and (ii) if v_0 in (4.5d) is described on the macroscale only, then a relation similar to (3.13b) applies for $v(x)$. Note that this last condition holds for every "reasonable" $F(x)$ - a line forcing, for example.

A numerical simulation demonstrates these results. We have computed the plate responses, associated with the mass heterogeneities

$$\rho_1(x) = h_1^d(x) \tag{4.6a}$$

$$\rho_2(x) = h^s(x) \tag{4.6b}$$

$$\rho_3(x) = h^s(x) + h_1^d(x), \tag{4.6c}$$

where $h_1^d(x)$ and $h^s(x)$ are shown in Fig. 4. Thus, $\rho_1(x)$ possesses microscale variations only, and $\rho_3(x)$ possesses both a microstructure ($h_1^d(x)$) and a macrostructure ($h^s(x)$). The forcing term, $F(x)$, is given by a square pulse of a unit amplitude and a unit spatial width, situated at $x = 15$. It is shown in Fig. 5, drawn to scale together with the corresponding background solution (4.5d). Figure 6 shows the complete response that corresponds to the heterogeneity $\rho_1(x)$. A comparison with the background solution (Fig. 5) shows that $\rho_1(x)$, which consists of microscale variations only, has no footprint on the response. Figure 7 compares the macroscale responses associated with $\rho_2(x) = h^s(x)$ and $\rho_3(x) = h^s(x) + h_1^d(x)$. These mass density distributions differ only by their microscale components (0 and $h_1^d(x)$, respectively). The macroscale responses are indistinguishable within the graphic resolution. Finally, Fig. 8 depicts the various components of the plate response associated with $\rho_3(x)$ - the inequality of (3.13b) is evident.

4.3 Stiffness heterogeneity

If we set $\rho(x) = 0,\ q(x) \neq 0$, the solution $v(x)$ of the equation of motion (4.3) can be expressed as a Green function superposition with the extended source term $F(x) - i\omega m_0 \left[q(x)v''(x)\right]''$ as a forcing (a prime denotes a derivative with respect to the argument). Taking the second derivative of the resulting formulation with respect to x, we get the integro-differential equation,

$$u(x) = u_0(x) - i\omega m_0 \int g''(x-y)\ \left[q(y)\,u(y)\right]'' dy \tag{4.7}$$

where

$$u(x) = v''(x) \tag{4.7a}$$

$$u_0(x) \quad = \quad \int g''(x-y)\, F(y)\, dy. \tag{4.7b}$$

Note that $u(x)$ can be interpreted as the reciprocal of the plate local radii of curvature. Integrating (4.7) by parts twice, and assuming that $q(x)$ and all its derivatives vanish at infinity, we get the Fredholm integral equation formulation (2.10)–(2.11c) with u, u_0 defined in (4.7a)–(4.7b) and with,

$$h(x) \quad = \quad q(x) \tag{4.8a}$$

$$G(x) \quad = \quad \Gamma(x) - \delta(x) \tag{4.8b}$$

where $\delta(x)$ is the Dirac delta function, and $\Gamma(x)$ is given by an inverse Fourier transform of [2],

$$\tilde{\Gamma}(\xi) \; = \; \frac{\zeta + i\epsilon/\hat{M}}{(1-\xi^4)\,\zeta + i\epsilon/\hat{M}}. \tag{4.9}$$

The function $\Gamma(x)$ is shown in Fig. 9. It has essentially the same properties of the Green function $g(x)$; it possesses a third derivative discontinuity at $x = 0$ and a variation scale of 2π away from the origin. Evidently, from (4.8b), the kernel function $G(x)$ of the integral operator is now highly singular at the origin. Referring to Sec. 3.2.2, the immediate consequence is that a stiffness microstructure can have a significant effect on the macroscale response.

A numerical example demonstrates these results. We have computed the response of a plate with the following stiffness heterogeneities,

$$q_1(x) \quad = \quad h_1^d(x) \tag{4.10a}$$

$$q_2(x) \quad = \quad h_2^d(x) \tag{4.10b}$$

$$q_3(x) \quad = \quad h^s(x) \tag{4.10c}$$

$$q_4(x) \quad = \quad h^s(x) + h_1^d(x) \tag{4.10d}$$

where $h_{1,2}^d(x)$ and $h^s(x)$ are shown in Fig. 4. Note that q_1 and q_2 possesses microscale variations only, q_3 possesses macroscale variations, whereas q_4 possesses both, with the macroscale component identical to that of $q_3(x)$. The forcing term is the same as in Sec. 4.2, and is shown in Fig. 10 drawn to scale together with the corresponding background solution (4.7b). Figure 11 compares the

macroscale response component u^s associated with q_1 to that associated with q_2. The difference is evident, and it is also evident that both deviate significantly from the background response. Recall that $q_{1,2}$ possess microscale variations only - these results confirm our assertion that microscale stiffness variation significantly effects macroscale response. Furthermore, it is seen that in general two different microscale stiffness structures can result in significantly different large scale responses. This holds also in the presence of a macroscale variation. Figure 12 compares u^s associated with q_3 to that associated with q_4. Again, the difference is evident.

5 The Effective Constitutive Relations

By substituting (4.8b) into the integral equation formulation (2.10)–(2.10a), performing the integration over the $\delta(x-x')q(x')u(x')$ term, and moving the result to the left hand side, we get the integral equation,

$$u(x)\,[1+q(x)] \;=\; u_0(x) \;+\; \int \Gamma(x-y)q(y)u(y)\,dy. \tag{5.1}$$

In this equation, $Q(x) = 1 + q(x)$ is the local bending stiffness of the plate; the plate bending moment, $T(x)$, is related to $u(x)$ via the local constitutive relation,

$$T(x) \;=\; u(x)Q(x) \;=\; u(x)\,[1+q(x)]\,. \tag{5.2}$$

The last results suggest a new integral equation, written for the bending moment $T(x)$,

$$T(x) \;=\; T_0(x) \;+\; \mathbf{\Gamma}_b r T \tag{5.3}$$

with $T_0(x)$ as the background bending moment (corresponds to $q(x) = 0$),

$$T_0(x) \;=\; u_0(x). \tag{5.3a}$$

$\mathbf{\Gamma}_b$ is an integral operator associated with the smooth kernel $\Gamma(x)$,

$$\mathbf{\Gamma}_b f \;=\; \int \Gamma(x-y)f(y)\,dy \tag{5.3b}$$

and $r = r(x)$ is a heterogeneity measure related to the stiffness variability,

$$r(x) \;=\; q(x)/\,[1+q(x)]\,. \tag{5.3c}$$

Thus, the bending moment $T(x)$ is governed by a second-kind Fredholm integral equation, a formulation completely analogous to (2.10)–(2.11c) with a *smooth* background kernel function. The forcing term $F(x)$ is assumed to possess variations on the macroscale only. The spectrum of $\Gamma(x)$, namely $\tilde{\Gamma}(\xi)$, behaves like ξ^{-4} for $\xi > 2\pi/\lambda$ (see (4.9) and [1])—rendering negligible the detail component of the excitation term $T_0 = u_0$. Hence, the results of Sec. 3.2.1 directly apply,

$$T^s(x) \;=\; T_0(x) \;+\; \mathbf{\Gamma}_b r^s T^s \tag{5.4a}$$

which follows from (3.10), and

$$\left\| T^d \right\|_2 \;\ll\; \|T^s\|_2 \tag{5.4b}$$

which follows from (3.13b). An *effective* constitutive relation—a relation written for u^s—can now be obtained by reversing (5.2) and using (5.4b),

$$u(x) \;\simeq\; T^s(x)/Q(x) \;=\; T^s(x) \left[\left(\frac{1}{Q(x)} \right)^s \;+\; \left(\frac{1}{Q(x)} \right)^d \right] \tag{5.5}$$

where $\left(\frac{1}{Q(x)}\right)^{s,d}$ denote the smooth (s) and detail (d) components of $\frac{1}{Q(x)}$. By taking the smooth part of both sides, we obtain now (a smooth component multiplied by a detail component gives a function with essentially zero local average, i.e. a "detail function"),

$$u^s(x) \;\simeq\; T^s(x) \left(\frac{1}{Q(x)} \right)^s . \tag{5.6}$$

This is a local (i.e. algebraic) effective constitutive relation connecting u^s and T^s.

For the task of calculating the plate macroscale response $u^s(x)$, the effective constitutive relation should be combined with the integral equation formulation governing $T^s(x)$, given in (5.4a). A noteworthy feature of the latter is that only a smooth measure of the system heterogeneity, namely $r^s(x)$, is required.

5.1 An important implication

Microscale stiffness variation can have a significant effect on the macroscale response. This general conjecture—not evidently plausible at first glance—is shown correct in Sec. 4.3. Three fundamental questions immediately arise,

1. Is there exist a class of *different* stiffness microstructures such that the macroscale response associated with each of the members of the class is the same?

2. If the answer is 'yes', what is the precise definition of such class?

3. Again—if the answer is 'yes'—can one synthesize an *effective stiffness*; a *smooth* stiffness heterogeneity, for which the associated complete response is the same as the macroscale response associated with each of members of the class?

The effective constitutive relation, (5.6), combined with the integral equation (5.4a), provide the answers; 'yes' for 1, then, an algorithmic characterization of the "classes of invariance", and an effective stiffness of the class. The latter is an inherently nonlinear process. To derive these results, note that,

$$\frac{1}{Q(x)} \;=\; 1 - r(x) \tag{5.7}$$

where $r(x)$ is defined in (5.3c). Then, (5.6) takes on the form,

$$u^s(x) \;=\; T^s(x)\,[1 - r(x)]^s \;=\; T^s(x)\,[1 - r^s(x)] \tag{5.8}$$

Notably, the stiffness heterogeneity measure that effects $T^s(x)$ is $r^s(x)$ (see (5.4a)). It therefore follows from (5.8) that this is the *only* stiffness heterogeneity measure effecting $u^s(x)$. Thus, two different microscale variation of $q(x)$, that give the same smoothed $r(x)$, have the same footprint on the macroscale response. A simple synthesis procedure for generating stiffness variations of the same class, as articulated above, follows. Let $r_1(x)$ and $r_2(x)$ be two functions differing only by their detail components,

$$r_i(x) \;=\; r^s(x) \,+\, r_i^d(x), \qquad i = 1, 2 \tag{5.9}$$

that is,

$$r_1(x) \;\neq\; r_2(x), \tag{5.9a}$$

$$(r_1(x))^s \;=\; (r_2(x))^s \;=\; r^s(x). \tag{5.9b}$$

These variations can be obtained by choosing at random a function for $r_1(x)$, and adding to it an arbitrary microstructure for $r_2(x)$. Or, choosing $q_1(x)$ for the stiffness variation, computing the corresponding $r_1(x)$ via (5.3c), and performing the articulated step. The associated stiffness variations are obtained by inverting (5.3c),

$$q_i(x) \;=\; r_i(x)/\left[1 - r_i(x)\right] \qquad i = 1,2 \tag{5.10}$$

Clearly, we have $q_1(x) \neq q_2(x)$. However, from (5.9b) it follows that the associated macroscale responses would be the same. This procedure can be used also to obtain an effective stiffness heterogeneity. For the class defined by (5.9) with $i = 1,2,3,\ldots$ (infinitely many different microstructures), the effective stiffness would be that associated with $r^s(x)$, i.e. $q_{\text{eff}}(x) = r^s(x)/[1 - r^s(x)]$. Or, alternatively, the effective stiffness associated with the stiffness variation $q_1(x)$ (that possesses both smooth and detail components) is obtained by first computing the corresponding $r_1(x)$ via (5.3c), then smoothing it via, say, the projection operator of (2.1a); $r_1^s(x) = \mathbf{P}_j r_1(x)$, and finally obtaining $q_{\text{eff}}(x)$ via the inverse of (5.3c)

$$q_{\text{eff}}(x) = r_1^s(x)/[1 - r_1^s(x)]. \tag{5.11}$$

5.2 Example

We turn to demonstrate the last results numerically. The procedure described by Eqs. (5.9)–(5.10) and the discussion thereafter is used to synthesize examples of stiffness complex scatterers with identical macroscale response, and with a prescribed effective stiffness. Specifically, we used the macroscale structure $h^s(x)$ in Fig. 4c as a smooth stiffness, computed the corresponding $r(x)$, added the complex structures $0.6\,h_1^d(x)$ or $0.6\,h_2^d(x)$, and re-invert the results to get two different complex structures of stiffness $q_1^*(x)$ and $q_2^*(x)$, respectively. These are shown in Figs. 13a and 13b. Clearly these stiffness distributions are not the same; they differ in both their macro- and microscale components. However, they should have the same macroscale response, and further, it should be identical to that associated with $q_3(x) = h^s(x)$ of Sec. 4.3, the latter being the corresponding effective stiffness. Figure 14 shows the macroscale component of the corresponding response field $u^s(x)$, subject to the same loading as before, and compares the results to u^s associated with $q_3(x) = h^s(x)$. The results are indistinguishable within the graphic resolution.

6 Conclusions

Multiresolution technique has been used to develop a theory for effective properties of complex scatterers. Using scaling functions and wavelets, a scattering formulation can be resolved to two coupled formulations. The one governing the macroscale response via the operator $\mathbf{\Phi}$ with the microscale response appearing as an additional forcing term through a coupling operator $\mathbf{C}$, and

the second governing the microscale response via the operator $\mathbf{\Psi}$ with the macroscale response appearing as an additional forcing through the coupling operator $\bar{\mathbf{C}}$ - see (2.14). By substituting the formal solution of the latter into the former, a new formulation that governs the macroscale component of the response is obtained - the *formulation smooth* (2.16). This formulation is written on a *macroscale* grid, whereas the effects of the microscale heterogeneity on the macroscale response are expressed via a precisely defined across scales coupling operator. The latter can also be interpreted as an *effective material operator* (EMO) description of the heterogeneity. The EMO norm is predominantly determined by the singularity at the origin of the associated integral equation kernel. The more singular the kernel the higher the norm, and the stronger is the effect of the microscale heterogeneity on the macroscale response. Since the articulated kernel is usually nothing but the Green function associated with the corresponding "background" problem, it follows that the effects of the microscale heterogeneity h^d on the macroscale response u^s are predominantly determined by the *near field* behavior of the corresponding background system. This conclusion is intuitively appealing; the microscale structure can be viewed as a collection of point scatterers separated by distances much smaller then the wavelength. Then, their effect on the macroscale response is interpreted as the result of accumulated multiple near field interactions.

In the context of the specific example treated, i.e. the response of a thin, linearly elastic plate, we have demonstrated that a small-scale mass density variation has no footprint in the large-scale response of a linearly elastic plate, whereas a small-scale bending stiffness variation does. It means that, in general, different stiffness microstructures lead to different large scale responses. However, we showed that the small-scale stiffness variation has no footprint in the large-scale "bending moment" variation in the plate. This was used to identify a condition for a class of small-scale stiffness variations which all lead to the same large-scale plate response. This class is defined as the collection of all stiffness variation that lead to the same effective measure q_{eff} (5.11). This result also lead to a significant decrease in the computational effort required to solve the problem, if one wishes to predict only large scale responses. Given a complex stiffness heterogeneity $q(x)$, one can compute its effective measure $q_{\text{eff}}(x)$ via the algebraic procedure detailed in Sec. 5.1. Since the effective measure is *slowly varying*, solving the problem with $q_{\text{eff}}(x)$ is much simpler then solving it with $q(x)$, and the solution associated with the former is nothing but the large scale component of the solution associated with the latter.

The simplicity of the results suggest there must be easily grasped underlying physical reasons. These are: First, although the experiment posed is dynamical, this designation applies for large-scale distances. The travel times for distances identified with the small-scale heterogeneity are small

when compared to the inverse of the frequency of the time-harmonic forcing. Thus, the interaction of the plate response with the small-scale heterogeneity is essentially statical. This leads to the conclusions that a small-scale mass density variation should have no effect and that the effects of a small-scale stiffness variation should be the same as that calculated for a statical forcing. The second underlying physical reason is that the line load forcing results in a prediction problem that is one-dimensional; i.e., the plate responds as a beam; and, one-dimensional beam theory is statically determinate. That is, the internal stress resultant—the internal bending moment—can be determined from the laws of equilibrium alone. One conclusion of this is that the small scale material properties of the beam will have no effect in determining the internal bending moment. Thus, the achieved conclusions are consistent with an intuitive understanding of the underlying physics.

Appendix: Bound estimates for the matrix norms - smooth $G(x)$

The results given in (3.9a)–(3.9c) are derived here. The discussion is limited to the scatterer defined by (3.4)–(3.5b), with the scales hierarchy articulated by (3.3). Thus, (3.7a)–(3.8b) apply. Further, with the smoothness of G,

$$|G_{M}(x)| \leq \lambda^{-M}. \tag{A.1}$$

It is essential to recognize the scatterer outer dimension as measured on the scale of λ. Thus, denoting N_0 the total number of resolution zero grid points spanning the scatterer extent, we have (see (3.3))

$$N_0 = 2^{-m_a} O(1). \tag{A.2}$$

A.1 Bound estimate for $\|\mathbf{\Psi}\|$

With (3.7b), (3.8b), and the scatterer characterization (3.4)–(3.4b) one has,

$$\Psi_{m'n',mn} \simeq \beta b_n 2^{-m'(M+1/2)} G_M(x_{m'n'} - x_{mn} + 2^{-m'-1} - 2^{-m-1}) \delta_{m\, m_i - 1} \tag{A.3}$$

where $\delta_{i\,j}$ is the Kronecker delta function. Using (A.1)

$$\left|\Psi_{m'n',mn}\right| \leq \beta \left|b_n\right| 2^{-m'(M+1/2)} \lambda^{-M} \delta_{m\, m_i - 1}. \tag{A.4}$$

With the Euclidean matrix norm inequality $\|\mathbf{\Psi}\|^2 \leq \sum_{m',n',m,n} \left|\Psi_{m'n',mn}\right|^2$, one has

$$\|\Psi\|^2 \leq \beta^2\lambda^{-2M}\sum_{m',n',n}|b_n|^2\,2^{-2m'(M+1/2)} \qquad \text{using (A.4)} \tag{A.5a}$$

$$= \beta^2\lambda^{-2M}\sum_{m',n'}2^{-2m'(M+1/2)}. \qquad \text{using (3.5b)} \tag{A.5b}$$

The summation over n' depends on m', as the number of grid points associated with the resolution m' scales like $N_0 2^{m'}$, where N_0 is the number of points at resolution 0. Thus, with (A.2)

$$\|\Psi\|^2 \leq \beta^2\lambda^{-2M}2^{-m_a}\sum_{m'=j}^{\infty}2^{-2Mm'} \tag{A.6a}$$

$$= \beta^2\lambda^{-2M}2^{-m_a}2^{-2Mj}\left(1-2^{-2M}\right)^{-1}. \tag{A.6b}$$

The last result can be rewritten as,

$$\|\Psi\|^2 \leq \beta^2\lambda\left(1-2^{-2M}\right)^{-1}(\lambda 2^{m_a})^{-1}\left(\lambda 2^j\right)^{-2M}. \tag{A.7}$$

However, by multiplying (3.3) with 2^j one finds $\lambda 2^j \gg 1$ and by multiplying it with 2^{m_a} one finds $\lambda 2^{m_a} \approx 1$. In addition, recall $|\beta| \ll 1$ and usually $M \geq 2$. Thus, for finite λ we obtain the result in (3.9a).

A.2 Bound estimate for $\left\|\bar{\mathbf{C}}\right\|$

With (3.7d), (3.8b), and the scatterer characterization (3.4)–(3.4b),

$$\bar{C}_{m'n',n} \simeq \beta a_n 2^{-m'(M+1/2)}G_M(x_{m'n'} + 2^{-m'-1} - x_{jn}). \tag{A.8}$$

Thus, with the norm inequality $\left\|\bar{\mathbf{C}}\right\|^2 \leq \sum_{m',n',n}\left|\bar{C}_{m'n',mn}\right|^2$, one has

$$\left\|\bar{\mathbf{C}}\right\|^2 \leq \beta^2\lambda^{-2M}\sum_{m',n',n}|a_n|^2\,2^{-2m'(M+1/2)} \qquad \text{using (A.8) and (A.1)} \tag{A.9a}$$

$$= \beta^2\lambda^{-2M}\sum_{m',n'}2^{-2m'(M+1/2)}. \qquad \text{using (3.5b)} \tag{A.9b}$$

The inequality (3.9b) follow by repeating the steps of (A.6a)–(A.7).

A.3 Bound estimate for $\|\mathbf{C}\|$

With (3.7c), (3.8a), and the scatterer characterization,

$$C_{n',mn} \simeq 2^{-j/2} b_n G(x_{jn'} - x_{mn} - 2^{-m-1}) \delta_{m\, m_i - 1} \tag{A.10}$$

Thus, by (A.10) and (A.1),

$$\|\mathbf{C}\|^2 \leq \sum_{n',m,n} |C_{n',mn}|^2 \leq 2^{-j} \sum_{n,n'} |b_n|^2 . \tag{A.11}$$

The summation over n is $O(1)$ by the scatterer characterization (3.5b). The n' index counts the grid points at resolution j. The latter is $O(2^j)$. Hence the result in (3.9c).

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Figure captions

Figure 1. A schematization of the problem. (a) The geometry. A thin linearly elastic plate separates a fluid ($z > 0$) and a vacuum ($z < 0$). The plate is excited by a time harmonic line load. (b) The heterogeneity. The plate properties (surface mass density or local bending stiffness) vary within a spatially limited region of an extent in the order of λ. The variations have two components, one is measured on a lengthscale of the order of λ and the second is measured on a lengthscale much smaller then λ.

Figure 2. The cubic spline Battle-Lemarie multiresolution system. (a) The scaling function $\phi(x)$. (b) The wavelet $\psi(x)$.

Figure 3. The homogeneous (background) plate Green function for $\epsilon = 0.129$ and $\hat{M} = 0.5$.

Figure 4. Examples of heterogeneity. (a) The micro-structure $h_1^d(x)$. (b) The micro-structure $h_2^d(x)$. (c) The macro-structure $h^s(x)$.

Figure 5. The forcing $F(x)$ and toe background system response for the mass heterogeneity problem.

Figure 6. The response associated with $\rho_1(x) = h_1^d(x)$ (no macro-structure).

Figure 7. Magnitude of the macroscale response component $u^s(x)$, associated with $\rho_2(x)$ (solid line), and $\rho_3(x)$ (dot line).

Figure 8. Magnitudes of the microscale and macroscale components of the plate response, $u^d(x)$ and $u^s(x)$, associated with the surface mass heterogeneity $1\rho_3(x) = h^s(x) + h_1^d(x)$. The inequality in (3.13b) is evident.

Figure 9. The function $\Gamma(x)$ in the kernel associated with the stiffness heterogeneity problem.

Figure 10. The forcing $F(x)$ and the background system response for the stiffness heterogeneity problem.

Figure 11. Magnitudes of u^s associated with $q_1(x)$ (solid line) and with $q_2(x)$ (dot line).

Figure 12. Magnitudes of u^s associated with $q_3(x)$ (solid line) and with $q_4(x)$ (dot line).

Figure 13. Two different complex structures of stiffness, designed via the procedure of Sec. 5.1 to have the same macroscale response. (a) $q_1^*(x)$. (b) $q_2^*(x)$.

Figure 14. The macroscale response u^s associated with q_1^*, q_2^*, and $q_3 = h^s$.