A Multiresolution Homogenization of Modal Analysis with Application to Layered Media

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Abstract

We apply multiresolution techniques to study the effective properties of boundary value problems. Conditions under which boundary values are preserved in the effective ("homogenized") formulation are developed and discussed. Relations between the eigenfunctions and eigenvalues of the generic formulation and those of the effective formulation are explored. Applications to the construction of effective Green function in a complex lamination are discussed. The analytic results are demonstrated via numerical computations.

Keywords: Multiresolution, wavelets, homogenization, smoothing, boundary value problems, effective properties.

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1 Introduction

The model problem motivating the reported study is the propagation of a wavefield through a waveguide that possesses *fine-scale layering*–spatial variations in the waveguide properties, along the depth (cross-range) direction, with length scales ℓ much smaller than its total depth L. We denote L as the macro-scale, and length-scales much smaller than L, say $L/10$ and below, as the micro-scales. Thus, the fine-scale layering consists of micro-scale variations, *and* possible variations on other length-scales.

The waveguide modes and propagation constants are determined by the eigenfunctions $u_n(x)$ and eigenvalues λ_n of a corresponding boundary value problem [1],

$$\mathbf{L}(q, \lambda_n) u_n \;=\; 0, \qquad x \in D \tag{1.1}$$

with appropriate boundary conditions. Here $\mathbf{L}(q, \lambda)$ is a linear operator of the Sturm-Liouville type, with λ a parameter. $q(x)$ represents a set of functions (equation coefficients) describing the system heterogeneity ($\epsilon(x), \mu(x)$ in electromagnetics, mass density and bending stiffness in elasticity, etc..) Our objective is to study the effect of the fine scale layering (expressed via the micro-scale component of $q(x)$) on the waveguide modes and propagation constants. In particular, we shall use multiresolution theory and wavelets to derive a new, self-consistent boundary value formulation governing $u_n^s(x)$–the macro-scale features of the modes in (1.1) (here and henceforth a superscript s stands for *"smooth"* and denotes a macro-scale component.)

To accomplish such a formulation, several questions are addressed. The first concerns the **existence of an effective measure in the context of boundary value problems**. That is, the existence of $q^{(\mathrm{eff})}(x)$ described on the macro-scale L, such that with an appropriate definition of boundary conditions the large scale component of the mode solutions of (1.1) satisfies exactly the same formulation, with q replaced by its effective measure,

$$\mathbf{L}(q^{(\mathrm{eff})}(x), \lambda_n^*) u_n^* \;=\; 0, \qquad u_n^*(x) = u_n^s(x) \;\text{ for } n \le N, \qquad x \in D \tag{1.2}$$

at least for some finite N, and in a prescribed asymptotic sense. We shall derive the effective measures of q and discuss the *footprint* of its micro-scale component on the macro-scale field for two canonical structures in Sec. 3.

However, the modes and associated eigenvalues depend on the boundary conditions (BC). Thus, the next fundamental question. **What are the appropriate BC for the *homogenized* Sturm-Liouville problem (1.2) and how this relates to the BC for the complete problem (1.1)?** The MRD-based homogenization provides an answer to this question. It is shown that when homogeneous Neumann or Dirichlet BC are applied to the complete formulation (1.1), they also apply to the homogenized one (1.2). However, when impedance BC are applied to the complete problem (1.1), a rule exists for determining an *effective impedance condition* that applies for the homogenized one (1.2).

An issue of interpretation arises when considering the fact that any BVP of the type (1.1) possesses an infinite set of eigenfunctions u_n and eigenvalues λ_n. The length-scale on which $u_n(x)$ varies, vanishes as $n \to \infty$. In this regard, the homogenized problem is no exception (it is true even for a constant-coefficients Sturm-Liouville equation.) Thus, only a finite number of eigenfunctions of the homogenized problem (1.2) can have a significant macro-scale component (say, u_n^* for $n \leq N$). It is only this subset of eigenfunctions, for which homogenization in general, and the identification $u_n^* = u_n^s$ in particular, can make any sense at all. Therefore, we ask: **In what sense can u_n^* approximate u_n^s ?** From the quantitative viewpoint, we show that $u_n^*(x) \to u_n(x)$ as the micro-scale $\ell \to 0$, and the rate of this convergence decreases with increasing n. This issue leads to the next, closely related question. **What is the relationship between an eigenvalue of the homogenized Sturm-Liouville problem (1.2), λ_n^*, and the corresponding eigenvalue of the complete problem (1.1), λ_n?** We demonstrate a *spectral equivalence*: $\lambda_n^* \to \lambda_n$ as the micro-scale becomes sufficiently small. As with the modes, the rate of this convergence (spectral equivalence "quality") decreases with increasing n.

The idea of homogenization is an old one. The modern literature began with the studies in [10],[11]. Formal mathematical theory for homogenization of periodic structures, presented as a weak asymptotic limit (as $\ell \to 0$) can be found in [12]. These classical works, however, do not address directly the issues articulated above, and formally require the existence of a large gap between the micro- and macro-scales. A different approach to effective properties theory is based on *statistical smoothing* of a randomly varying fine-scale structure in an otherwise deterministic formulation. The studies of Beran and McCoy [13]– [14] are representative. Homogenization and statistical smoothing were identified as following from a common *projection* methodology by Fishman and McCoy [15].

We shall use multiresolution decomposition, in conjunction with the methodology used in [2]-[7], to address the issues articulated above and to develop a formulation governing large scale locally averaged versions of the modes, free from the classical limitations. We note that multiresolution analysis and wavelets have been used recently by Brewster and Beylkin in [16] for numerical homogenization. The basic ideas are similar to those developed in [2, 3]. The work in [16] is devoted mainly for a sophisticated "decimation" process in which efficient numerical algorithm for estimating the large scale response component is developed. It does not address directly the questions articulated above. Using the multiresolution approach, the works in [4, 6] reconstruct the classical result of homogenization of the equation $(Qu')' = f$, in the context of acoustic scattering. While the classical formulations apply only as a weak asymptotic limit [12] and require the existence of a gap in the heterogeneity scales, the multiresolution formulation can in principle be applied to any multi-scale heterogeneity and the existence of a scale gap is required only for the derivation of closed form analytic expressions for the effective properties. Using the multiresolution approach, The classical result was re-derived later by Gilbert [17]. In [18], Beylkin and Coult used the algorithm developed in [16] to investigate the numerical homogenization of boundary value problems. Their work however, does not provide analytic expression for the effective properties, and does not address the issue of the appropriate boundary conditions for the homogenized problem.

The structure of the paper is as follows. In Sec. 2 we discuss the basics of multiresolution approach to homogenization and derive the basic formulation. In Sec. 3 we introduce the boundary value problem and rewrite it as an integral equation formulation, suitable for a direct application of our homogenization procedure. In this section we also derive the effective measure and the associated spectral equivalence theorem. In Sec. 4 we discuss an application of the BVP homogenization theory of the previous sections, to propagation in complex laminates. Numerical examples are provided in section 5, and concluding discussion in section 6.

2 Multiresolution and Smoothing

We shall use the theory of multiresolution decomposition (MRD) and wavelets [19] to develop a formulation governing the large scale response. Basically, the procedure is identical to that discussed in [2]-[6]. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a nested sequence of linear spaces that constitutes a MRD of $L_2(R)$ ($\mathbb{Z}$ is the set of all integers). Further, let $\phi(x)$ and $\psi(x)$ be the corresponding scaling function and wavelet, respectively. The function $\phi_{jn}(x)$ is defined via $\phi(x)$ as $\phi_{jn}(x) = 2^{j/2}\phi(2^j x - n)$, and a similar definition holds for $\psi_{mn}(x)$. Then, $\{\phi_{jn}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of V_j, and $\{\psi_{jn}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of O_j, the orthogonal complement of V_j in V_{j+1}. An approximation of a function $f(x)$ at a resolution k can be written as the sum of two mutually orthogonal functions, namely a smooth (f^s) and a detail (f^d) components. We have

$$f(x) \simeq f^s(x) + f^d(x) \tag{2.1}$$

where

$$f^s(x) \;=\; \mathbf{P}_j f(x) \;=\; \sum_n s_n \phi_{jn}(x), \qquad s_n = \langle f, \phi_{jn} \rangle \tag{2.1a}$$

$$f^d(x) \;=\; \mathbf{D}_j^k f(x) \;=\; \sum_{m=j}^{k-1} \sum_n d_{mn} \psi_{mn}(x), \qquad d_{mn} = \langle f, \psi_{mn} \rangle. \tag{2.1b}$$

The asymptotic equality in (2.1) becomes exact in the limit $k \to \infty$. Here, $\langle \cdot, \cdot \rangle$ denotes the inner product of $L_2(R)$, and $j < k$ is some reference resolution. The scaling functions and wavelets satisfy the orthonormality relations $\langle \phi_{jn}, \phi_{jn'} \rangle = \delta_{nn'}$, $\langle \psi_{mn}, \psi_{m'n'} \rangle = \delta_{mm'} \delta_{nn'}$, and $\langle \phi_{jn'}, \psi_{mn} \rangle = 0 \; \forall j \leq m$. Thus, $\mathbf{P}_j$ and $\mathbf{D}_j^k$ in (2.1a)–(2.1b) are projection operators satisfying $\mathbf{D}_j^k \mathbf{P}_{j'} = \mathbf{P}_{j'} \mathbf{D}_j^k = 0 \forall j \geq j'$. ϕ and ψ are either of compact support, or fast decreasing, and centered more or less about the origin. They can be interpreted, respectively, as defining a local low-pass and a local band-pass filters. Examples are shown in Fig. 1. From the dilation translation relations articulated above, it follows that the terms ψ_{mn} in (2.1b) are situated, respectively, around the points $x_{mn} = n2^{-m}$. The functions $u^s(x)$ and $u^d(x)$ in (2.1) can be interpreted as a locally smoothed, or averaged, description of $u(x)$ on the length scale 2^{-j}, and a signal describing the finer details covering length scales ranging from $2^{-(j+1)}$ to 2^{-k}, respectively. Here and henceforth we refer to the number 2^{-m} and the index m as a length-scale and the resolution associated with it, respectively.

2.1 The formulation governing large scale response

A boundary value problem written as a second order Sturm-Liouville differential equation requires an *explicit* specification of the associated boundary conditions at two distinct points. The generic BC are written for the complete solution $u(x)$ and not for its macro-scale component $u^s(x)$. Thus, the incorporation of explicit, local boundary conditions in the homogenized formulation requires a passage from values of $u(x)$ at isolated points to values of $u^s(x)$ at isolated points. By virtue of Eq. (2.1a), this passage cannot be uniquely described.

An alternative approach is to write the boundary value problem in terms of a second kind Fredholm integral equation. It is well known that in the latter *the boundary conditions are incorporated in the integral kernel* $k(x, y)$, and are not specified explicitly "at a point". Thus, the BC to be applied to the macro-scale field are completely determined by the changes $k(x, y)$ undergoes during the homogenization process; if its structure is preserved, the effective BC should be the same as the generic ones; if the structure is changed, the BC should be modified accordingly. Furthermore, the effective properties of a Fredholm integral equation formulation can be directly analyzed and studied by the multiresolution and smoothing procedure developed in [2]-[6]. Thus, we assume for the moment that the relevant Sturm-Liouville BVP is cast in the form of the integral equation [8],

$$v(x) \;=\; f(x) + \mathbf{L}v \tag{2.2}$$

where $\mathbf{L}$ is the integral operator

$$\mathbf{L}v \;=\; \int_0^1 k(x,y)q(y)v(y)\,dy. \tag{2.2a}$$

Here $f(x)$ is a known forcing term that can be interpreted as the system response in the absence of the heterogeneity, $q(x)$ is the heterogeneity function, and $k(x,y)$ is a known kernel. For convenience in later use, the operator $\mathbf{L}$ can be written as a background operator $\mathbf{L}_b$ operating on the multiplication of $q(x)$ and $v(x)$,

$$\mathbf{L}v \;=\; \mathbf{L}_b(qv), \qquad \mathbf{L}_b f \;=\; \int_0^1 k(x,y)f(y)\,dy. \tag{2.2b}$$

The passage from the relevant BVP to this integral equation is provided in the next section.

The procedure proposed in [2]–[6] can now be applied to derive a formulation written directly on the large scale response $v^s(x)$, defined as

$$v^s(x) \;=\; \mathbf{P}_j v(x), \tag{2.3}$$

where j corresponds to the reference or smoothing scale 2^{-j}. By applying $\mathbf{P}_j$ and $\mathbf{D}_j^k$ to both sides of Eq. (2.2), expressing $v(x)$ as the sum of $v^s(x)$ and $v^d(x)$ as in Eq. (2.1), and then casting the resulting equations in a matrix form we get,

$$\begin{pmatrix} \mathbf{I}-\mathbf{\Phi} & -\mathbf{C} \\ -\bar{\mathbf{C}} & \mathbf{I}-\mathbf{\Psi} \end{pmatrix} \begin{pmatrix} \vec{s} \\ \vec{d} \end{pmatrix} \;=\; \begin{pmatrix} \vec{s_0} \\ \vec{d_0} \end{pmatrix} \tag{2.4}$$

where the coefficients of $v^s(x)$—which apply to the scaling functions summation—are collected in the vector $\vec{s}$, and the coefficients of $v^d(x)$—which apply to the wavelet summation—are collected in $\vec{d}$. Thus, $\vec{s}$ and $\vec{d}$ are unknown vectors representing the large scale response and the remaining fine details of the response, respectively. The same interpretation applies for the known vectors $\vec{s_0}$ and $\vec{d_0}$. $\mathbf{I}$ is the identity matrix and $\mathbf{\Phi}, \mathbf{C}, \bar{\mathbf{C}}$ and $\mathbf{\Psi}$ are matrix operators with elements

$$\Phi_{n',n} \;=\; \langle \mathbf{L}\phi_{jn}, \phi_{jn'}\rangle \;=\; \langle q\phi_{jn}, \mathbf{L}_b^*\phi_{jn'}\rangle \tag{2.5a}$$

$$C_{n',mn} \;=\; \langle \mathbf{L}\psi_{mn}, \phi_{jn'}\rangle \;=\; \langle q\psi_{mn}, \mathbf{L}_b^*\phi_{jn'}\rangle \tag{2.5b}$$

$$\bar{C}_{m'n',n} \;=\; \langle \mathbf{L}\phi_{jn}, \psi_{m'n'}\rangle \;=\; \langle q\phi_{jn}, \mathbf{L}_b^*\psi_{m'n'}\rangle \tag{2.5c}$$

$$\Psi_{m'n',mn} \;=\; \langle \mathbf{L}\psi_{mn}, \psi_{m'n'}\rangle \;=\; \langle q\psi_{mn}, \mathbf{L}_b^*\psi_{m'n'}\rangle. \tag{2.5d}$$

Here $\mathbf{L}_b^*$ is the adjoint of $\mathbf{L}_b$. From the lower half of Eq. (2.4) the response detail component $\vec{d}$ can be expressed in terms of the response large scale component $\vec{s}$. When the result is substituted into the upper half of Eq. (2.4) we get a formulation governing $\vec{s}$

$$\left[\mathbf{I} - \mathbf{\Phi} - \mathbf{C}\left(\mathbf{I} - \mathbf{\Psi}\right)^{-1}\bar{\mathbf{C}}\right]\vec{s} = \vec{s_0}. \tag{2.6}$$

We have assumed here that $v_0(x)$ is described on large scales only—thus $\left\|f^d(x)\right\| \ll \|f^s(x)\|$ and the small scale forcing vector $\vec{d_0}$ has been neglected. The formulation in Eq. (2.6) governs the large scale response. It has been shown that the operator $\mathbf{\Phi}$ is nothing but a matrix representation of the operator $\mathbf{L}$ with $q(x)$ replaced by its smooth component $q^s(x)$ ([2]-[6]). The latter is given by a relation similar to Eq. (2.1a): $q^s(x) = \mathbf{P}_j q(x)$. The elements of $\mathbf{\Phi}$ can be computed by Eq. (2.5a) with $q(x)$ replaced by $q^s(x)$. The information of the small scale heterogeneity $q^d(x)$ appears only in $\mathbf{C}\left(\mathbf{I} - \mathbf{\Psi}\right)^{-1}\bar{\mathbf{C}}$; recall the interpretation of ϕ_{jn} and ψ_{mn} as low pass and band pass filters. Thus, the operator $\mathbf{C}\left(\mathbf{I} - \mathbf{\Psi}\right)^{-1}\bar{\mathbf{C}}$ is the *effective material operator* (EMO). It describes how the small scale heterogeneity $q^d(x)$ affects u^s. A general measure of the potential degree of this effect is the EMO norm. If $\left\|\mathbf{C}\left(\mathbf{I} - \mathbf{\Psi}\right)^{-1}\bar{\mathbf{C}}\right\| \ll \|\mathbf{\Phi}\|$ then q^d has practically no effect on u^s. If these norms are comparable then q^d can have a significant effect on u^s An estimate of the EMO norm can be derived by using bounds of wavelet coefficients near irregularities [8]. It is recognized that the integral kernel $k(x,y)$ is closely related to the background problem Green function $G^b(x,y)$. The latter is irregular at $x = y$. Similarly, one may expect that the heterogeneity function $q(x)$ possesses an irregular behavior of some order. These properties are used in the following theorem.

Theorem 2.1 *Let $k(x,y)$ be the integral operator kernel. Let $k(\cdot,y) \in C^p$, $q(\cdot) \in C^{p'}$, and $\phi(x), \psi(x) \in C^N$ with $N > \max(p,p')$. Then*

$$\|\mathbf{\Psi}\|^2 \leq cN_0^2 \frac{2^{-2j(p'+p+2)}}{\left[1 - 2^{-2(p+1)}\right]\left[1 - 2^{-2(p'+1)}\right]} \tag{2.7a}$$

$$\|\bar{\mathbf{C}}\|^2 \leq cN_0^2 \frac{2^{-2j(p+1)}}{1 - 2^{-2(p+1)}} \tag{2.7b}$$

$$\|\mathbf{C}\|^2 \leq cN_0^2 \frac{2^{-2j(p'+1)}}{1 - 2^{-2(p'+1)}} \tag{2.7c}$$

where $c = O(1)$, N_0 is the number of resolution 0 grid points covering the domain of interest.

A proof is provided in [8]. We have assumed here that the multiresolution system is "more regular" then $k(x,y)$ or $q(x)$. If this assumption does not apply, the norms bounds are even smaller [8].

2.2 Effective measures in formulations with mildly singular kernels

The results in theorem 2.1 suggest that the EMO norm is dominated by the kernel singularity [2]-[6]. For highly singular kernels the EMO affect the large scale response significantly. For continuous kernels, even for mildly singular ones, its effect is negligible and we find that $\vec{s}$ is governed by

$$(\mathbf{I} - \mathbf{\Phi})\,\vec{s} \;=\; \vec{s}_0. \tag{2.8}$$

Thus, by reversing the steps leading from the integral equation formulation (2.2)–(2.2b) to Eq. (2.6) we find that for such kernels the large scale response v^s is governed by exactly the same formulation, with $q(x)$ replaced by its smooth component $q^s(x)$—recall that $q^s(x)$ is the only measure of $q(x)$ that affect $\mathbf{\Phi}$. The formulation for $v^s(x)$ then takes on the form

$$v^s(x) \;=\; f(x) + \mathbf{L}_b(q^s v^s) \tag{2.9}$$

This means that the effective measure of $q(x)$ is simply $q^s(x)$. Since in these cases $\|\mathbf{\Psi}\| \ll 1$ and $\|\bar{\mathbf{C}}\| \ll 1$ we get from the lower half of (2.4) that $\left\|\vec{d}\right\| \ll \|\vec{s}\|$, or

$$\left\|v^d\right\| \;\ll\; \|v^s\|\,. \tag{2.9a}$$

3 Effective Properties and the Basic Boundary Value Problem

3.1 Basic equations and problem definition

We start with the standard second order boundary value problem,

$$\frac{d}{dx}\left(Q(x)\frac{d}{dx}u(x)\right) + \lambda u(x) \;=\; s(x), \qquad 0 \le x \le 1 \tag{3.1}$$

subject to the homogeneous boundary conditions,

$$\alpha_{1,2}u(x) + \beta_{1,2}\frac{d}{dx}u(x) \;=\; 0 \quad \text{ at } x = 0, 1 \tag{3.1a}$$

where the indices $1, 2$ correspond to $x = 0, 1$ respectively. The parameters $\alpha_i, \beta_i\, (i = 1, 2)$ are determined by the type of the physical boundary. Equation (3.1a) can be viewed as *local impedance boundary condition.* $s(x)$ is an arbitrary source term. We assume throughout that $Q(x)$ is strictly positive and rewrite it as,

$$Q(x) = 1 + q(x), \tag{3.1b}$$

thus $q(x)$ represents the system heterogeneity–the fine scale layering. As stated in the introduction, our goal is to develop a new boundary value problem, with (possibly) "simpler" heterogeneity function $q^{(\text{eff})}(x)$ (i.e. a heterogeneity that varies on scales of the order of 1), whose solutions (eigenfunctions) are identical with the large scale component of those of Eq. (3.1). The new heterogeneity function is termed *the effective measure* of $q(x)$.

After these results are established, we shall generalize the homogenization procedure to the more realistic cases where the problem consists of two heterogeneity functions, $Q(x)$ and $M(x)$,

$$\frac{d}{dx}\left(Q(x)\frac{d}{dx}u(x)\right) + \Omega^2 u(x) + \lambda M(x)u(x) \;=\; s(x), \qquad 0 \le x \le 1 \tag{3.2}$$

subject to the same boundary conditions as in Eq. (3.1a). Here Ω is a parameter, commonly representing frequency. This generalization provides the effective properties of Q and M in the same meaning articulated before. It is based on previously derived results [2]-[6], as well as on the results pertaining to the effective properties of $Q(x)$ when $M = 1$.

3.2 Integral equation formulation

A first step toward a Fredholm integral equation formulation would be to define the background problem–a problem presented by Eqs. (3.1)–(3.1a) with $Q(x) = 1$ ($q(x) = 0$.) The associated Green function $G^b(x, y)$ satisfies

$$\left(\frac{d^2}{dx^2} + \lambda\right) G^b(x, y) \;=\; -\delta(x - y), \tag{3.3}$$

subjected to the same boundary conditions as in Eq. (3.1a). There is a well established procedure for computing $G^b(x, y)$ analytically [1]. Let $\eta_1(x)$ and $\eta_2(x)$ be two linearly independent solutions of the *unforced* background problem (Eq. (3.3)) satisfying the boundary conditions of the original problem at $x = 0$ and $x = 1$ respectively, and the Wronskian normalization condition $\eta_1\eta_2' - \eta_1'\eta_2 = 1$. Then $G^b(x, y)$ is given by,

$$G^b(x, y) \;=\; \begin{cases} -\eta_1(x)\eta_2(y) & x \le y \\ -\eta_1(y)\eta_2(x) & x > y. \end{cases} \tag{3.4}$$

Integral equation formulation can be obtained by rewriting Eq. (3.1) with the heterogeneity term $(qu')'$ in the right hand side, and presenting the solution as a convolution of G^b with the extended source term $s(x) - (qu')'$. The result is

$$u(x) \; = \; u_0(x) + \int_0^1 G^b(x,y) \, [q(y)u'(y)]' \; dy \tag{3.5}$$

where $u_0(x)$ is the background system response to $s(x)$,

$$u_0(x) \; = \; - \int_0^1 G^b(x,y)s(y) \, dy. \tag{3.5a}$$

Equation (3.5) is integro-differential. Using the procedure developed in [8], it can be transformed to an integral equation of the second kind. The result is the desired formulation Eqs. (2.2)–(2.2b) with

$$v(x) = u'(x) \tag{3.6a}$$

$$\begin{aligned} f(x) \; = \; v_0(x) \quad & + \quad \frac{q(0)}{\Delta} \left[a_{12} v_0(1) - a_{22} v_0(0) \right] G_x^b(x,0) \\ & + \quad \frac{q(1)}{\Delta} \left[a_{11} v_0(1) - a_{21} v_0(0) \right] G_x^b(x,1) \end{aligned} \tag{3.6b}$$

$$\begin{aligned} k(x,y) \; &= \; -G_{xy}^b(x,y) \quad - \quad \frac{q(0)}{\Delta} \left[a_{12} \overline{G}(1,y) - a_{22} \overline{G}(0,y) \right] G_x^b(x,0) \\ & \qquad\qquad\qquad\;\; - \quad \frac{q(1)}{\Delta} \left[a_{11} \overline{G}(1,y) - a_{21} \overline{G}(0,y) \right] G_x^b(x,1) \\ &= \; -G_{xy}^b(x,y) \quad - \quad g_0(x,y) \; - \; g_1(x,y). \end{aligned} \tag{3.6c}$$

where $v_0(x) \; = \; - \int_0^1 G_x^b(x,y)s(y) \, dy$, and

$$G_x^b(x,y) \equiv \frac{\partial}{\partial x} G^b(x,y) \; = \; \left\{ \begin{array}{ll} -\eta_1'(x)\eta_2(y) & \quad x < y \\ -\eta_1(y)\eta_2'(x) & \quad x > y \end{array} \right. , \qquad \lim_{\epsilon \to 0} G_x^b(x,y) \Big|_{x=y-\epsilon}^{x=y+\epsilon} \; = \; -1. \tag{3.7a}$$

and

$$G_{xy}^b(x,y) \equiv \frac{\partial^2}{\partial x \, \partial y} G^b(x,y) \; = \; \delta(x-y) + \overline{G}(x,y) \tag{3.7b}$$

where $\overline{G}(x,y)$ is given by

$$\overline{G}(x,y) \;=\; \begin{cases} -\eta_1'(x)\eta_2'(y) & x \leq y \\ -\eta_1'(y)\eta_2'(x) & x > y. \end{cases} \tag{3.7c}$$

Note that $\overline{G}(x,y)$ is *continuous* at $x=y$. $\Delta = a_{11}a_{22} - a_{12}a_{21}$, and the a_{ij} are known numbers,

$$a_{11} \;=\; Q(0) + G_x^b(0^+,0)\,q(0) \tag{3.8a}$$

$$a_{12} \;=\; -G_x^b(0,1)\,q(1) \tag{3.8b}$$

$$a_{21} \;=\; G_x^b(1,0)\,q(0) \tag{3.8c}$$

$$a_{22} \;=\; Q(1) - G_x^b(1^-,1)\,q(1). \tag{3.8d}$$

As long as the determinant Δ does not vanish one can derive the Fredholm integral equation formulation. It is easy to verify that for Neumann or Dirichlet BC, and/or for cases in which the heterogeneity $q(x)$ vanishes at the boundaries, the expressions for $f(x)$, $k(x,y)$ simplify to $v_0(x)$, $-G_{xy}^b(x,y)$ respectively. We emphasize that except for the $\delta(x-y)$ singularity in $G_{xy}^b(x,y)$, the expression for $k(x,y)$ is continuously differentiable to any order, and vary on the length-scale of unity. Also, the forcing term $f(x)$ is described on the macro-scale only.

3.3 The effective heterogeneity

In the problem treated in the present work it is clear that the kernel is highly singular. It is described by a smooth function *and* a Dirac δ distribution. The immediate consequence is that micro-scale heterogeneity can have a significant effect on the large scale response. Thus, here $q^{(\mathrm{eff})}(x) \neq q^s(x)$. However, one can still derive a simple analytic expression for the effective measure of $q(x)$ by following the steps derived first in [4]. Note that in all cases discussed here, the only singular component of $k(x,y)$ is $\delta(x-y)$. The rest is continuous. By performing the integration over $\delta(x-y)$ in closed form and moving the result to the left hand side we obtain

$$v(x)Q(x) \;=\; f(x) \;+\; \mathbf{\Gamma}_b(qv) \tag{3.9}$$

where $Q(x) \;=\; 1+q(x)$, $f(x)$ is given in Eq. (3.6b), and $\mathbf{\Gamma}_b$ is an integral operator of the same structure as $\mathbf{L}_b$ in Eq. (2.2b) with the kernel $\Gamma(x,y)$,

$$\Gamma(x,y) \;=\; \begin{cases} -\overline{G}(x,y) & \text{Neumann or Dirichlet b.c} \\ -\overline{G}(x,y) - g_0(x,y) - g_1(x,y) & \text{Impedance b.c.} \end{cases} \tag{3.9a}$$

Let us define the new variable $T(x)$ via the local constitutive relation

$$T(x) = v(x)Q(x). \tag{3.10}$$

From Eq. (3.9) the formulation governing $T(x)$ is

$$T(x) = f(x) + \mathbf{\Gamma}_b \left(\frac{q}{Q} T \right) \tag{3.11}$$

which is now a second kind Fredholm integral equation formulation for $T(x)$, with a continuous kernel $\Gamma(x,y)$. The effective measure of the heterogeneity relevant for $T(x)$ is nothing but $\mathbf{P}_j\,(q/Q) = (q/Q)^s$ (see discussion in subsection 2.2). In other words, the formulation governing $T^s(x)$ is

$$T^s(x) = f(x) + \mathbf{\Gamma}_b \left[\left(\frac{q}{Q} \right)^s T^s \right] \tag{3.12a}$$

and

$$\left\| T^d \right\| \ll \| T^s \| . \tag{3.12b}$$

By rewriting Eq. (3.10) for $v^s(x)$ in terms of $T(x)$, and using Eq. (3.12b) we find,

$$v^s(x) = T^s(x) \left(\frac{1}{Q(x)} \right)^s \qquad \Longrightarrow T^s(x) = \left[\left(\frac{1}{Q(x)} \right)^s \right]^{-1} v^s(x). \tag{3.13}$$

Substituting the last expression back into Eq. (3.12a) we find a formulation governing $v^s(x)$ only,

$$v^s(x) Q^{(\mathrm{eff})}(x) = f(x) + \mathbf{\Gamma}_b \left(q^{(\mathrm{eff})} v^s \right) \tag{3.14}$$

where

$$Q^{(\mathrm{eff})}(x) = \left[\left(\frac{1}{Q(x)} \right)^s \right]^{-1}, \qquad q^{(\mathrm{eff})}(x) = Q^{(\mathrm{eff})}(x) - 1. \tag{3.15}$$

This last formulation–governing the large scale response–is identical with the generic formulation for the complete response (3.9), except that the heterogeneity $q(x)$ in the integral operator and in the lhs has been replaced by its *effective measure* $q^{(\mathrm{eff})}(x)$ the expression of which is given in Eq. (3.15). However, the forcing term $f(x)$ in the new effective formulation (3.14) as well as the kernel itself still contain expressions depending on the *numbers* $q(0), q(1)$ and not on $q^{(\mathrm{eff})}(0), q^{(\mathrm{eff})}(1)$. The importance of these observations cannot be overemphasized. While the structure of the equation is preserved, the specific prescriptions for both the forcing term $f(x)$ and the kernel function $k(x, y)$, are not. Thus, in general, the effective formulation in its entirety cannot be obtained from the complete one by a mere replacement $q \to q^{(\text{eff})}$. On the other hand, there are three specific scenarios for which the dependencies of $f(x)$ and $k(x, y)$ on $q(0), q(1)$ cancel. These are

(i) Neumann boundary conditions.

(ii) Dirichlet boundary conditions.

(iii) Natural impedance boundary conditions: Eq. (3.1a) with $\beta_1 = Q(0)$, $\beta_2 = Q(1)$.

Under these BC the prescriptions are preserved, and the effective formulation is given by the replacement $q \to q^{(\text{eff})}$). That is, for these BC one can reverse the steps leading from Eqs. (3.1)–(3.1a) to Eq. (3.9) and obtain the effective boundary value problem,

$$\frac{d}{dx}\left(Q^{(\text{eff})}(x)\frac{d}{dx}\,u^s(x)\right) + \lambda u^s(x) \;=\; s(x), \qquad 0 \le x \le 1 \tag{3.16}$$

subject to the homogeneous boundary conditions,

(i) Effective Neumann BC: $v^s = u^{s\prime} = 0$, if the generic BC were of the Neumann type.

(ii) Effective Dirichlet BC: $u^s = 0$, if the generic BC were of the Dirichlet type.

(iii) Effective natural impedance BC: $\alpha_{1,2}u^s(x) + Q^{(\text{eff})}(x)v^s(x) = 0$ at $x = 0, 1$, if the generic BC were of the natural impedance type.

Finally, note that from Eqs. (3.10) and (3.13) we can identify a rule for smoothing the product Qv

$$(Qv)^s \;=\; Q^{(\text{eff})}(x)\,v^s(x), \tag{3.17}$$

which will be found useful later.

3.4 The eigenfunctions and eigenvalues

The analysis leading to the effective formulation in Eq. (3.16) was based on the assumption that λ is not a critical value. That is, we have implicitly assumed, for example, that Eq. (2.4) possesses a unique solution. This assumption fails when λ is an eigenvalue of the problem. Here we wish to study how the eigenvalues and eigenfunctions of the complete formulation in Eq. (3.1) relate to those of the effective formulation in Eq. (3.16). We start with the following simple theorem.

Theorem 3.1 *Let $\{f_n^\epsilon\}$ be an orthonormal basis in a Hilbert space H, that depends on the parameter ϵ. Let $\{g_n\}$ be an orthonormal sequence in H. Orthonormality in both cases can be with respect to a weight function ρ. If $\forall\, m \neq n$ $\langle f_n^\epsilon, g_m\rangle \to 0$ as $\epsilon \to 0$, then $f_n^\epsilon \to g_n$ as $\epsilon \to 0\ \forall n$.*

Proof Expand g_m by the set $\{f_n^\epsilon\}$,

$$g_m = \sum_i a_i^m(\epsilon) f_i^\epsilon, \qquad a_i^m(\epsilon) = \langle g_m, f_i^\epsilon\rangle. \tag{3.18}$$

Moving the m-th term to the left and using orthogonality of the f_n^ϵ, we get

$$\|g_m - a_m^m(\epsilon) f_m^\epsilon\|^2 = \sum_{n\neq m} |\langle g_m, f_n^\epsilon\rangle|^2 \tag{3.19}$$

Since $\{f_n^\epsilon\}$ is a basis for any ϵ, the sequence (in n) $\{a_n^m(\epsilon)\}$ is square summable. Also, by hypothesis, for any given pair $(m \neq n)$ we have $\langle g_m, f_n^\epsilon\rangle \to 0$ as $\epsilon \to 0$. Therefor the sum in the right hand side of Eq. (3.19) above must vanish with ϵ. In other words

$$\|g_m - a_m^m(\epsilon) f_m^\epsilon\| \to 0 \quad \text{as } \epsilon \to 0 \tag{3.20}$$

which proves the theorem. □

Let $u_n(x)$, λ_n be the n-th eigenfunction and eigenvalue associated with the complete formulation in Eq. (3.1). Let $u_n^*(x)$, λ_n^* be those associated with the effective formulation in Eq. (3.16). The pairs u_n, λ_n and u_m^*, λ_m^* satisfy the equations

$$(Q\, v_n)' + \lambda_n u_n = 0, \tag{3.21a}$$

$$(Q^{(\text{eff})}\, v_m^*)' + \lambda_m^* u_m^* = 0, \tag{3.21b}$$

where a prime denotes a derivative with respect to the argument, $v = u'$. u_n and u_m^* satisfy Neumann, Dirichlet, or natural impedance boundary conditions, and their effective counterparts, respectively (see Eqs. (3.1a) and (3.16)). Since both equations are real and self adjoint, the eigenfunctions and eigenvalues are real in both cases. We shall assume throughout that

$$\lambda_n \neq \lambda_m^* \qquad \forall m \neq n. \tag{3.21c}$$

Since $Q^{(\text{eff})}$ is a slowly varying large scale heterogeneity the first $N = O(2^j)$ eigenfunctions of Eq. (3.21b) satisfy

$$u_n^* \in V_j \qquad \forall n \leq N = 2^j, \tag{3.22a}$$

$$v_n^* \in V_j \qquad \forall n \leq N = 2^j. \tag{3.22b}$$

We shall restrict our discussion to these first N eigenfunctions. Note also that for these first N modes the relation in Eq. (3.12b) is valid (because the results of Sec. 2.2 and Eq. (2.9a) in particular, still hold in the context of the formulation for T in Eq. (3.12a)), so the smoothing rule (3.17) still holds

$$\mathbf{P}_j\left(Qv_n\right) \equiv \left(Qv_n\right)^s = Q^{(\mathrm{eff})}v_n^s. \tag{3.23}$$

Furthermore, if $\bar{V}_j$ is a linear subspace of V_j consisting of all functions in V_j that satisfy the same boundary conditions as u_n^*, then the set $\{u_n^*\}_{n=1,\ldots N}$ is an orthonormal basis of $\bar{V}_j$.

We perform now an inner product of Eq. (3.21a) and Eq. (3.21b) with u_m^* and u_n, respectively, and subtract the second from the first. The result is

$$\langle u_m^*, (Qv_n)' \rangle - \langle u_n, (Q^{(\mathrm{eff})}v_m^*)' \rangle + \langle u_n, u_m^* \rangle \left(\lambda_n - \lambda_m^*\right) = 0. \tag{3.24}$$

where we have used the realness of the quantities involved. Integration by parts gives

$$u_m^* Q v_n|_0^1 - u_n Q^{(\mathrm{eff})} v_m^*|_0^1 + \langle v_n, Q^{(\mathrm{eff})}v_m^* \rangle - \langle v_m^*, Qv_n \rangle + \langle u_n^s, u_m^* \rangle \left(\lambda_n - \lambda_m^*\right) = 0 \tag{3.25}$$

and we have used the identity $\langle u_n, u_m^* \rangle = \langle u_n^s, u_m^* \rangle$. From Eqs. (3.22a)–(3.22b) it follows that $v_m^* = \mathbf{P}_j v_m^*$. Thus

$$\begin{aligned}
\langle v_m^*, Qv_n \rangle &= \langle \mathbf{P}_j v_m^*, Qv_n \rangle \\
&= \langle v_m^*, \mathbf{P}_j(Qv_n) \rangle \qquad (\mathbf{P}_j \text{ is self adjoint}) \\
&= \langle v_m^*, Q^{(\mathrm{eff})}v_n^s \rangle \qquad (\text{by Eq. (3.23).})
\end{aligned} \tag{3.26}$$

Similarly, since $Q^{(\mathrm{eff})}v_m^*$ is large scale function, $\langle v_n, Q^{(\mathrm{eff})}v_m^* \rangle = \langle v_n^s, Q^{(\mathrm{eff})}v_m^* \rangle$. Thus, by realness of the Q's and v's the two inner products in Eq. (3.25) give $\langle v_n^d, Q^{(\mathrm{eff})}v_m^* \rangle$ (note that $v_n - v_n^s = v_n^d$), and we are left with

$$\langle u_n^s, u_m^* \rangle \left(\lambda_n - \lambda_m^*\right) = \left[u_n Q^{(\mathrm{eff})} v_m^* - u_m^* Q v_n\right]_0^1 - \langle v_n^d, Q^{(\mathrm{eff})}v_m^* \rangle. \tag{3.27}$$

Homogeneous Neumann, Dirichlet, or natural impedance boundary conditions In this case the first term on the right hand side of Eq. (3.27) vanishes. Now, let ℓ be the micro-scale, and L_m^* be the length-scale associated with the m-th effective mode u_m^*, $L_m^* \approx 1/m$. We have $\langle v_n^d, Q^{(\mathrm{eff})}v_m^* \rangle \to 0$ as $\ell/L_m^* \to 0$. Then, by Eq. (3.27) and Theorem 3.1 we get

$$u_n^* \to u_n^s \qquad \text{as } \ell/L_n^* \to 0 \tag{3.28a}$$

and with this last result, Eq. (3.27) gives (put $m = n$)

$$\lambda_n^* \to \lambda_n \qquad \text{as } \ell/L_n^* \to 0 \tag{3.28b}$$

Note that as the effective mode order n increases, its typical length-scale L_n^* decreases. Thus the "error" $|\lambda_n^* - \lambda_n|$ decreases with decreasing ℓ, and increases with increasing the mode order. The same observations hold for u_n^*. This convergence was derived and demonstrated in [8]

3.5 Extension to the case of Eq. (3.2)

If both Q and M are present in the Sturm-Liouville equation, a single Fredholm integral equation of the second kind cannot be obtained. Instead, we suggest here a two-step approach, based on combining the procedure outlined in Sec. 3.3, with that corresponding to a mildly singular kernel. This approach was first suggested and outlined in [21]. We start by considering the problem

$$\left[\frac{d^2}{dx^2} + \Omega^2 + \lambda M(x)\right] u(x) = s(x). \tag{3.29}$$

The multiresolution based homogenization of this problem has been considered in a number of papers [2], [6]. The associated integral equation formulation possesses a continuous kernel. Thus, the effective property of $M(x)$ in Eq. (3.29) is nothing but its smooth component, and the detail component of the solution $u(x)$ or of its first derivative $u'(x)$ is much smaller than the corresponding smooth component,

$$M^{(\text{eff})}(x) = M^s(x) \tag{3.30a}$$

and

$$\left\| \left(\frac{d^n}{dx^n} u\right)^s \right\| >> \left\| \left(\frac{d^n}{dx^n} u\right)^d \right\|, n = 0, 1 \tag{3.30b}$$

The homogenization of Eq. (3.2) is based now on replacing the background problem and the associated Green function G^b of Eq. (3.3) by a new background problem,

$$\left[\frac{d^2}{dx^2} + \Omega^2 + \lambda M(x)\right] G_M^b(x, y) = -\delta(x - y). \tag{3.31}$$

With the results stated in Eqs. (3.30a)–(3.30b), away from the origin ($|x-y| \geq 2^{-j}$) G^b_M satisfies identical inequalities. Toward the homogenization of Eq. (3.2), we now apply to it exactly the same steps applied to the reduced equation Eq. (3.1), with G^b replaced by G^b_M. We use the same local constitutive relation defined in Eq. (3.10). Thus, the parallel of Eq. (3.11) is

$$T(x) \;=\; f(x) \;+\; \mathbf{\Gamma}_{b\,M}\left(\frac{q}{Q}T\,;\,M\right) \tag{3.32}$$

where the implicit dependence of $\mathbf{\Gamma}_{b\,M}$ on $M(x)$ comes from the dependence of G^b_M on it. Since also here the kernel of $\mathbf{\Gamma}_{b\,M}$ is continuous, we get $\|EMO\| << 1$. This results means that T^s is governed, to a very good approximation, by equation of the type $(\mathbf{I}-\mathbf{\Phi}_M)\,\vec{s} = s_0$, where the elements of $\mathbf{\Phi}_M$ are given by,

$$\begin{aligned}
(\Phi_M)_{nn'} \quad &= \quad \left\langle \mathbf{\Gamma}_{b\,M}\left(\frac{q}{Q}\phi_{jn}\,;\,M\right), \phi_{jn'}\right\rangle \\
&= \quad \left\langle \frac{q}{Q}\phi_{jn}, \left\langle -\overline{G}(x,\cdot;M), \phi_{jn'}\right\rangle\right\rangle \\
&\approx \quad \left\langle \frac{q}{Q}\phi_{jn}, \left\langle -\overline{G}(x,\cdot;M^s), \phi_{jn'}\right\rangle\right\rangle \qquad \text{(with Eqs. (3.30a)–(3.30b))} \\
&\approx \quad \left\langle \left(\frac{q}{Q}\right)^s \phi_{jn}, \left\langle -\overline{G}(x,\cdot;M^s), \phi_{jn'}\right\rangle\right\rangle
\end{aligned} \tag{3.33}$$

It follows that the relevant heterogeneity measures are only $(\frac{q}{Q})^s$ and M^s. Reversing the steps we end up with

$$T^s(x) \;=\; f(x) \;+\; \mathbf{\Gamma}_{b\,M^s}\left[\left(\frac{q}{Q}\right)^s T\,;\,M^s\right] \tag{3.34}$$

which is the generalization of Eq. (3.12a). Following the steps down to Eq. (3.16) yield the new homogenized equation

$$\frac{d}{dx}\left(Q^{(\mathrm{eff})}(x)\frac{d}{dx}u^s(x)\right) \;+\; \Omega^2 u^s(x) \;+\; \lambda M^{(\mathrm{eff})}u^s(x) \;=\; s(x), \qquad 0 \leq x \leq 1 \tag{3.35}$$

with the same boundary conditions as for Eq. (3.16), and

$$Q^{(\mathrm{eff})}(x) = \left[\left(\frac{1}{Q(x)}\right)^s\right]^{-1}, \qquad M^{(\mathrm{eff})}(x) = M^s(x). \tag{3.36}$$

The extension of the eigenvalues/eigenfunctions results of Sec. 3.4 to the present case is not straightforward. We have, instead of Eqs. (3.21a)–(3.21b)

$$(Q\, v_n)' \quad + \quad \Omega^2 u_n \,+\, \lambda_n M u_n \;=\; 0, \tag{3.37a}$$

$$(Q^{(\mathrm{eff})}\, v_m^*)' \quad + \quad \Omega^2 u_m^* \,+\, \lambda_m^* M^s u_m^* \;=\; 0. \tag{3.37b}$$

Note that now the set $\{u_n\}$ is orthogonal with respect to the weight function M, and the set $\{u_n^*\}$ is orthogonal with respect to M^s. Following the steps of Sec. 3.4 from Eq. (3.21c) to Eq. (3.27) we find, after additional straightforward algebraic steps, for Neumann, Dirichlet, or natural impedance BC (for $\ell/L_m^* \to 0$)

$$\lambda_m^* = \lambda_m \left(1 + \frac{\langle u_m^*, M^d u_m^d \rangle}{\langle u_m^*, M^s u_m^s \rangle} \right) \tag{3.38a}$$

$$\langle u_m^*, M^s u_n^s \rangle (\lambda_n - \lambda_m^*) \;=\; -\lambda_n \langle u_m^*, M^d u_n^d \rangle. \tag{3.38b}$$

If $M^d = 0$ the results of Sec.3.4 are reconstructed. In the case of $M^d \neq 0$, the results hold only as an approximation. In certain cases this approximation is very good as demonstrated in [9, 21]. The quality of this approximation depends on how small is the value of $\langle u_m^*, M^d u_m^d \rangle$. The latter vanishes when the detail components of u_n and M are completely "uncorrelated", or when either detail component is negligibly small. As shown in [9], this is the case when $\ell/L_m^* \to 0$.

4 The Effective Green Function of a Complex Waveguide

The Green function associated with an electromagnetic or acoustic wave propagation in a complex, planarly layered medium can be modeled by a wave equation of the form

$$\left[\frac{\partial}{\partial x} P(x) \frac{\partial}{\partial x} \,+\, P(x) \frac{\partial^2}{\partial z^2} \,+\, \Omega^2 \right] G(x, z\,; x', z') \;=\; -\delta(x - x')\delta(z - z') \qquad 0 \le x \le 1, \tag{4.1}$$

subject to the same boundary conditions as given in Eq. (3.1a). An outgoing wave radiation condition is assumed at $|z| \to \infty$. Here x and z are the cross range and range directions, respectively.

By performing a Fourier transformation of Eq. (4.1) in the z direction we find an equation of the same structure as Eq. (3.1)

$$\left[\frac{d}{dx}P(x)\frac{d}{dx} + \left(\Omega^2 - \zeta^2 P(x)\right)\right] \tilde{G}(x,\zeta\,;x',z') \;=\; -\delta(x-x')e^{-i\zeta z'}, \qquad 0 \le x \le 1. \tag{4.2}$$

Here G and $\tilde{G}$ are related via the Fourier transform pair

$$\tilde{G}(x,\zeta\,;x',z') \;=\; \int_{-\infty}^{\infty} G(x,z\,;x',z')\, e^{-i\zeta z}\, dz \tag{4.3a}$$

$$G(x,z\,;x',z') \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(x,\zeta\,;x',z')\, e^{i\zeta z}\, d\zeta. \tag{4.3b}$$

The one dimensional Green function $\tilde{G}$ can be expressed by the modal summation [1]

$$\tilde{G}(x,\zeta\,;x',z') \;=\; \sum_n \frac{u_n(x)u_n(x')}{\zeta^2 + \lambda_n} e^{-i\zeta z'} \tag{4.4}$$

where $u_n(x)$ and λ_n are the eigenfunctions and eigenvalues in Eq. (3.37a) with $Q = M = P$–the transverse modal formulation associated with Eq. (4.1). When Eq. (4.3b) is used in conjunction with the Residue theorem we get the modal representation of $G(x,z\,;x',z')$

$$G(x,z\,;x',z') \;=\; \frac{1}{2}\sum_n \frac{u_n(x)u_n(x')}{\lambda_n^{1/2}} e^{-\lambda_n^{1/2}|z-z'|} \tag{4.5}$$

where $\mathrm{Re}\lambda_n^{1/2} \ge 0, \quad \mathrm{Im}\lambda_n^{1/2} \le 0$. Equation (4.5) conveys a modal representation for the *complete* Green function of the *complete* problem. Since $P(x)$ is assumed to vary on the micro-scale the question of a reduced, or smoothed, Green function arises. To address the issue systematically one must first derive an *effective* boundary value problem that can be identified as the transverse modal formulation associated with Eq. (4.1). Under the conditions for which $\lambda_n = \lambda_n^*$, we identify Eq. (3.35) or Eq. (3.37b) as the appropriate effective boundary value problem and repeat the steps in Eqs. (4.4)–(4.5). The result is the *effective* Green function

$$G^{(\mathrm{eff})}(x,z\,;x',z') \;=\; \frac{1}{2}\sum_{n=1}^{N} \frac{u_n^*(x)u_n^*(x')}{\lambda_n^{*\,1/2}} e^{-\lambda_n^{*\,1/2}|z-z'|} \tag{4.6}$$

Here summation must be restricted to the first few modes ($n \le N = O(2^j)$). Note that $G^{(\mathrm{eff})}$ can be expressed also as

$$G^{(\mathrm{eff})}(x,z\,;x',z') \;=\; \mathbf{P}_j\mathbf{P}'_jG(x,z\,;x',z') \tag{4.7}$$

where the projections $\mathbf{P}_j$ and $\mathbf{P}'_j$ (see Eq. (2.1a)) act on the x and x' dependence, respectively.

5 Numerical Example

In this section we demonstrate the results presented in Secs. 3.4-3.5. We choose two functions as two test cases for synthesizing the complex heterogeneity $P(x)$

$$\text{case 1:}\quad P(x) \;=\; Q_1(x) \;=\; M_1(x) \;=\; 1+a\cos(100\pi x) \tag{5.1a}$$

$$\text{case 2:}\quad P(x) \;=\; Q_2(x) \;=\; M_2(x) \;=\; 1+a\sin(\pi x)\sin(100\pi x). \tag{5.1b}$$

The characteristic length-scale of the inner structure is, roughly, $1/100$ in both cases. The factor a determines the micro-structure magnitude–the latter vanishes when $a=0$. $q(x)$ vanishes at the boundaries in case 2, but not in case 1. Since the smoothing operator $\mathbf{P}_j$ practically performs local spatial integration over intervals of length larger then the micro-scale (say $1/16$), a good approximation of $Q^{(\mathrm{eff})}(x)$ can be derived analytically using standard tables of integrals. The effective measures are given by

$$\text{case 1:}\quad Q^{(\mathrm{eff})}(x) \;=\; Q_1^{(\mathrm{eff})}(x) \;=\; \sqrt{1-a^2} \tag{5.2a}$$

$$\text{case 2:}\quad Q^{(\mathrm{eff})}(x) \;=\; Q_2^{(\mathrm{eff})}(x) \;=\; \sqrt{1-a^2\sin^2(\pi x)}. \tag{5.2b}$$

The functions $Q_{1,2}(x), M_{1,2}(x)$ are shown in Fig.2 together with the respective effective measures $Q_{1,2}^{(\mathrm{eff})}(x)$, for $a=0.5$. In all cases, $M_{1,2}^{(\mathrm{eff})}(x)=M_{1,2}^{s}(x)=1$.

Let us demonstrate the relations between u_n,λ_n and u_n^*,λ_n^* numerically. For the first few modes, the length-scales ratio ℓ/L_n^* is sufficiently small to establish an approximate spectral equivalence, as the limits in Eqs. (3.28a)–(3.28b) and their generalization (3.38a)-(3.38b) suggest. Computations were done with the frequency parameter $\Omega=5.5\pi$. We start with case 1 and Dirichlet boundary conditions. Figure 3a compares u_n to u_n^* for $n=1$ and $n=10$. The micro-structure magnitude is $a=0.9$. As predicted in Secs. 3.4-3.5, $(u_n)^s=u_n^*$. Figure 3b compares λ_n/π^2 to λ_n^*/π^2 vs. a for the first ten modes. Note that for $a=0$ (no micro-structure) we find $\lambda_n=\lambda_n^*=-\Omega^2+n^2\pi^2$, and the lines "bend down" as the micro-structure magnitude increases. This phenomena demonstrate that micro-scale heterogeneity can have a profound effect on the eigenvalues. As the figure shows, however, this effect can be introduced into the boundary value problem by using the effective (smooth) measure $Q^{(\mathrm{eff})}, M^{(\mathrm{eff})}$ instead of the complex Q, M. Indeed, the approximate relation $\lambda_n = \lambda_n^*$ is evident for the entire range of a. Figure 4 shows the same but for case 2 with Dirichlet boundary conditions. Figures 5 and 6 show the same, for cases 1 and 2 respectively, but with Neumann boundary conditions. All the results are consistent with the approximate spectral equivalence relation predicted by Eqs. (3.28a)–(3.28b) and its generalization in Sec. 3.5, and also demonstrate that micro-structure heterogeneity can have a profound effect on the eigenvalues. Figure 7 shows the modes and eigenvalues of case 1 with the natural impedance boundary conditions. The results are in a good agreement with the anticipated spectral equivalence. Finally, we note that the numerical simulations here were performed for $P(x)$ that is essentially periodic, but similar conclusions apply to less ordered structures [9, 21].

6 Conclusions

The effective properties of a complex boundary value problem were studied. We have developed a new formulation with slowly varying coefficients $Q^{(\mathrm{eff})}(x)$, $M^{(\mathrm{eff})}(x)$, whose solution is identical to the large scale component of the solution to the generic boundary value problem. A key issue in such a study is the appropriate representation of the boundary conditions;since the latter are written as values of the *complete* solution at two isolated points, they cannot be directly applied to the large scale formulation, as the latter is written on the large scale solution component only. We have addressed this difficulty by rewriting the boundary value problem as a second kind Fredholm integral equation formulation. The latter is known to incorporate boundary conditions inherently and in a "natural way". We have studied the relation between the eigenfunctions and eigenvalues of the generic problem (u_n, λ_n), and these of the effective problem (u_n^*, λ_n^*). For cases in which the fine-scale layering is expressed only via the coefficient $Q(x)$, it has been shown that for homogeneous Neumann or Dirichlet conditions, or for the natural impedance boundary conditions $u_n^* \to u_n^s$ and $\lambda_n^* \to \lambda_n$ as the micro-scale $\ell \to 0$, while for a general impedance conditions these relations do not apply. The more general form of Sturm-Liouville problem given in Eq. (3.2) has been considered too. The effective measures of Q and M were derived. It is shown, however, that the spectral equivalence $u_n^* \to u_n^s$ and $\lambda_n^* \to \lambda_n$ as $\ell \to 0$, is approximately true if $\langle u_m^*, M^d u_m^d \rangle$ is small. This is a subject for further research [9].

References

[1] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves,* Prentice Hall, 1973.

[2] B. Z. Steinberg and J. J. McCoy, "Towards local effective parameter theories using multiresolution decomposition," *J. Acoust. Soc. Am.*, Vol. 96, pp. 1130–1143, 1994.

[3] B. Z. Steinberg and J. J. McCoy, "Effective measures for nonstationary microscale stiffness variation using multiresolution decomposition," *J. Acoust. Soc. Am.*, Vol. 98(6), pp. 3516-3526, 1995.

[4] B. Z. Steinberg and J. J. McCoy, "A class of one dimensional stiffness microstructures resulting in identical macroscale response," *Wave Motion*, Vol. 23, pp. 237-258, 1996.

[5] B. Z. Steinberg and J. J. McCoy, "A Multiresolution Study of Effective Properties of Complex Electromagnetic Systems," *IEEE Trans. Ant. Propag.*, Vol. 26, pp. 971-981, 1998.

[6] B. Z. Steinberg and J. J. McCoy, "A Study of The Effective Properties of Mass and Stiffness Microstructures - A Multiresolution Approach," to appear in *Quarterly Journal of Applied Mathematics.*

[7] B. Z. Steinberg and J. J. McCoy, "A multiresolution analysis of scattering by a pair of local regions of complex heterogeneity", *The Journal of the acoustical society of America*, Vol. 103(3), pp. 1273-1281, 1998.

[8] B. Z. Steinberg, J. J. McCoy, and M. Mirotznik, "A Multiresolution Approach to Homogenization and Effective Modal Analysis of Complex Boundary Value Problems," *SIAM Journal on Applied Mathematics*, accepted for publication.

[9] B. Z. Steinberg, "Homogenization and Effective Properties Formulations for Propagation in Finely Structured Laminates - A Multi-Resolution Approach," submitted to *Wave Motion*, special issue on Electrodynamics of Complex Environments.

[10] I. Babuska, "Homogenization and its application. Mathematical and computational problems," in *Numerical Solutions of Partial Differential Equations—III*, B. Hubbard, Ed. New York: Academic, 1976, pp. 89-116.

[11] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures.* North Holland, Amsterdam, 1978.

[12] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *"Homogenization of Differential Operators and Integral Functionals,"* Springer-Verlag, 1994.

[13] M. J. Beran and J. J. McCoy, "Mean field variation in random media," *Quart. Appl. Math.*, Vol. 28(2), pp. 245-258, 1970.

[14] M. J. Beran and J. J. McCoy, "Mean field variation in statistical sample of homogeneous linearly elastic solids," *Int. J. Solids Structures*, Vol. 6, pp. 1035-1054, 1970.

[15] L. Fishman and J. J. McCoy, "A unified view of bulk property theories for stochastic and periodic media," *J. Appl. Math. Phys.*, (ZAMP), Vol. 32, pp. 45-61, 1981.

[16] M. E. Brewster and G. Beylkin, "A multiresolution strategy for numerical homogenization," *Appl. Comp. Har. Anal.*, Vol. 2, pp. 327–349, 1995.

[17] A. C. Gilbert, "A Comparison of Multiresolution and Classical One-Dimensional Homogenization Schemes," *Appl. Comp. Har. Anal.*, Vol. 5, pp. 1-35, 1998.

[18] G. Beylkin, and N. Coult, "A Multiresolution Strategy for Reduction of Elliptic PDEs and Eigenvalue Problems," *Appl. Comp. Har. Anal.*, Vol. 5, pp. 129-155, 1998.

[19] I. Daubechies, *Ten lectures on wavelets*, CBMS–NSF Series in Applied Mathematics, SIAM Publ., Philadelphia, 1992.

[20] S. Jaffard and Ph. Laurencot, "Orthonormal wavelets, analysis of operators, and applications to numerical analysis," in *Wavelets: A Tutorial in Theory and Applications,* C. K. Chui ed., Academic Press 1992.

[21] B. Z. Steinberg, and J. Oz, "A Multiresolution Approach for the effective properties of Complex Propagation Ducts and Effective Modes/Rays Theories," Proceedings of the 1998 URSI International Symposium on Electromagnetic Theory, Thessaloniki, Greece, May 25-28, 1998.

Figure captions

Figure 1. The cubic spline Battle-Lemarie multiresolution system. (a) The scaling function $\phi(x)$. (b) The wavelet $\psi(x)$.

Figure 2. Examples of $Q(x)$ (solid line) and its effective measure $Q^{(\text{eff})}(x)$ (dashed line) for $a = 0.5$. (a) Case 1 (equations (5.1a) and (5.2a)). (b) Case 2 (equations (5.1b) and (5.2b)).

Figure 3. Eigenfunctions and eigenvalues of the complete problem Eq. (3.21a) and the effective problem Eq. (3.21b), for case 1 with Dirichlet boundary conditions. $Q(x)$ and $Q^{(\text{eff})}(x)$ are shown in Fig.2a. (a) $u_n(x)$ (solid line) and $u_n^*(x)$ (dashed line) for $n = 1, 10$. (b) λ_n/π^2 (solid line) and λ_n^*/π^2 (dashed line) vs. a for the first 10 modes.

Figure 4. The same as Fig.3 but for case 2 with Dirichlet boundary conditions.

Figure 5. The same as Fig.3 but for case 1 with Neumann boundary conditions.

Figure 6. The same as Fig.3 but for case 2 with Neumann boundary conditions.

Figure 7. The same as Fig.3, but for case 1 with the natural impedance boundary conditions.

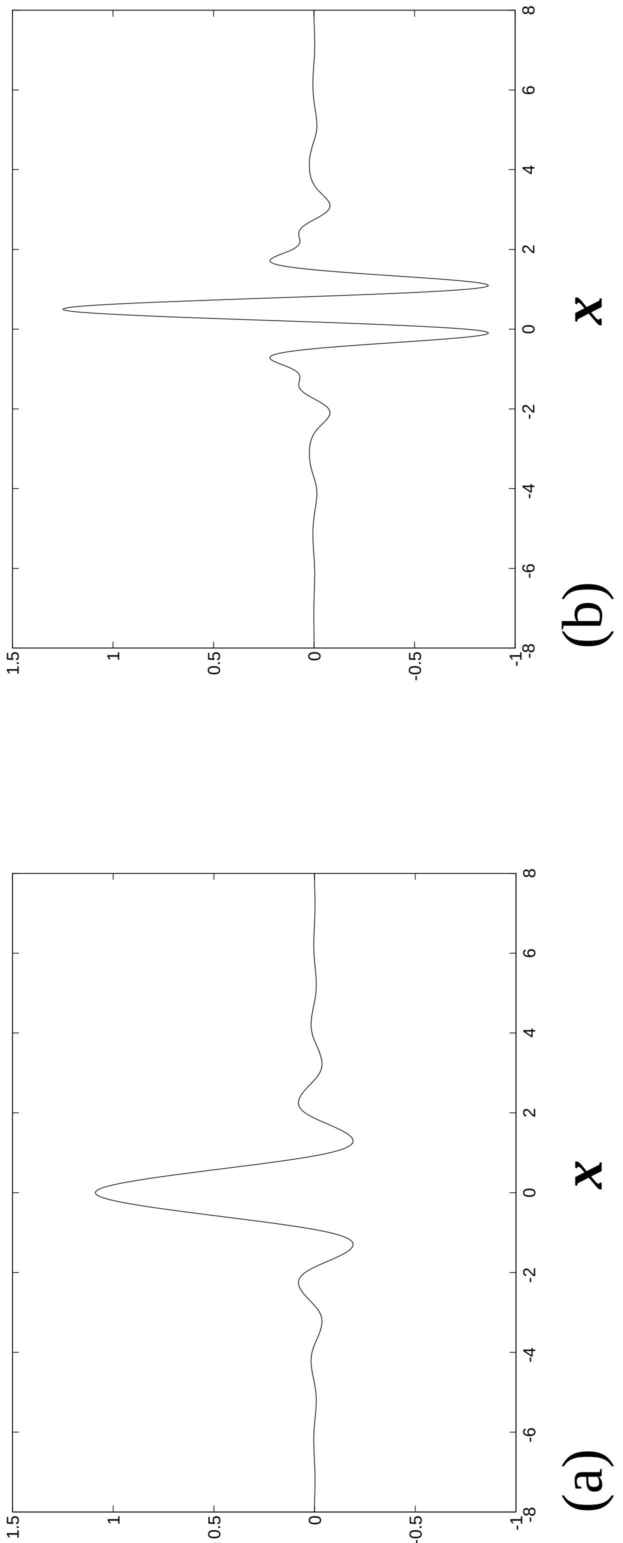


Fig. 1

Fig. 2a

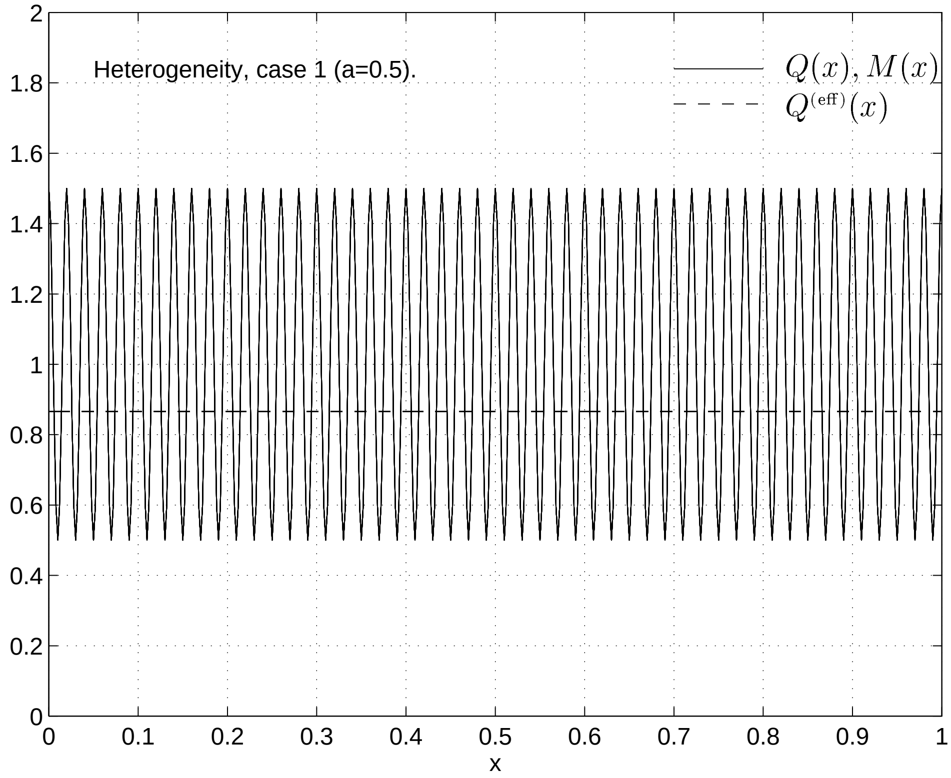

Heterogeneity, case 1 (a=0.5).
$Q(x), M(x)$
$Q^{(\mathrm{eff})}(x)$
x

Fig. 2b

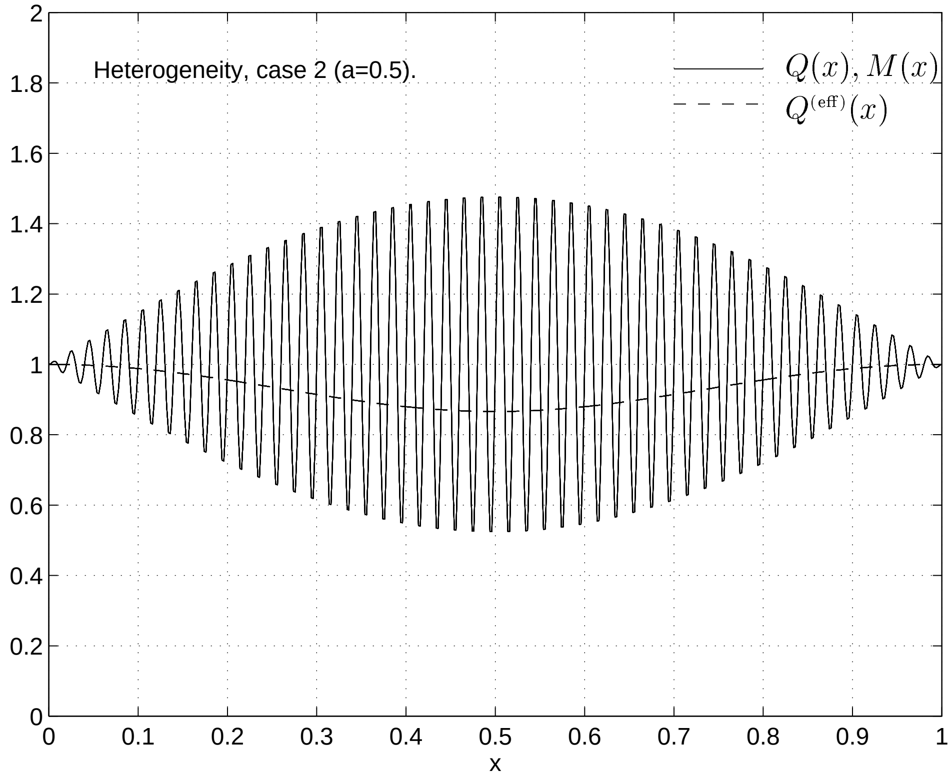

Heterogeneity, case 2 (a=0.5).
$Q(x), M(x)$
$Q^{(\mathrm{eff})}(x)$
x

Fig. 3a

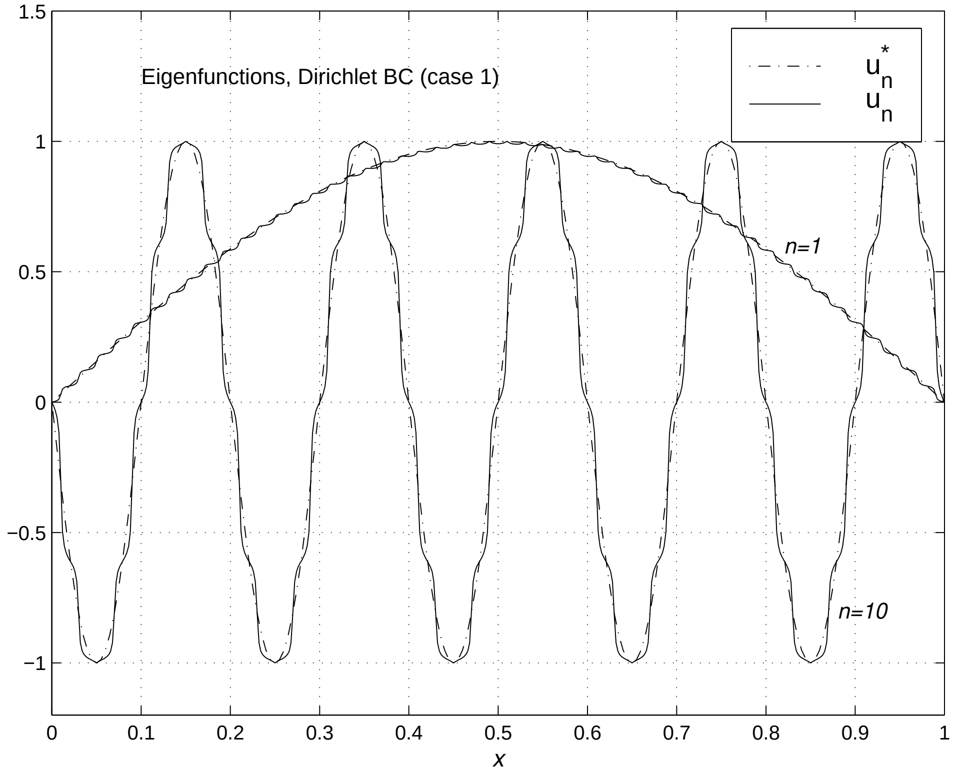

Eigenfunctions, Dirichlet BC (case 1)
u_n^*
u_n
n=1
n=10
1.5
1
0.5
0
−0.5
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Fig. 3b

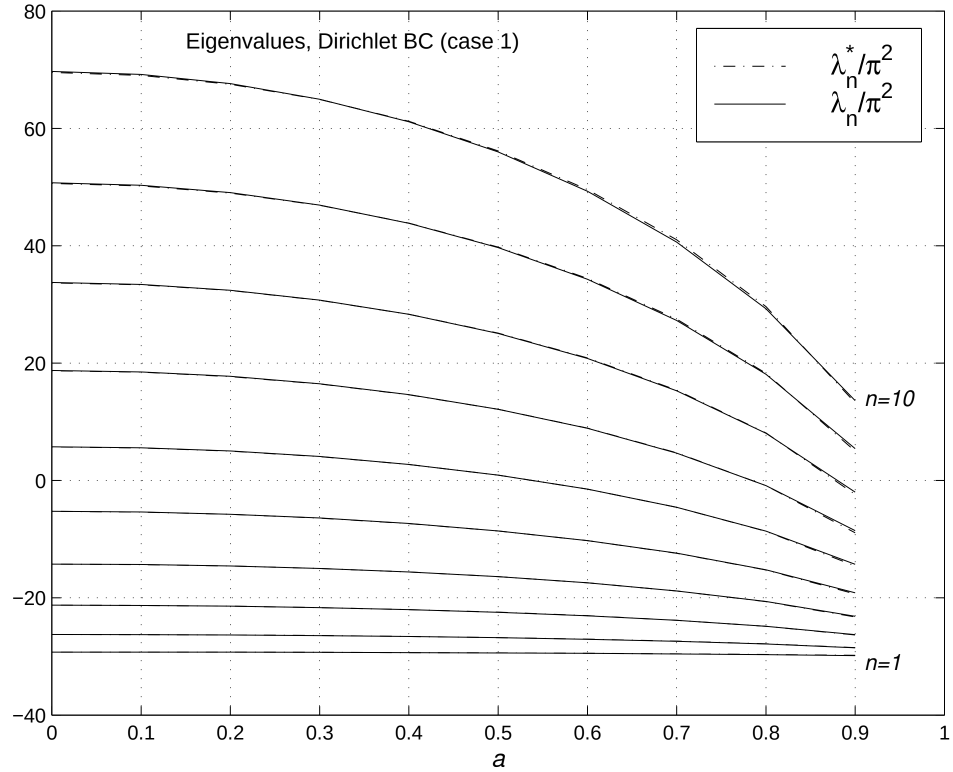

Eigenvalues, Dirichlet BC (case 1)
λ_n^*/π^2
λ_n/π^2
n=10
n=1
80
60
40
20
0
−20
−40
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
a

Fig. 4a

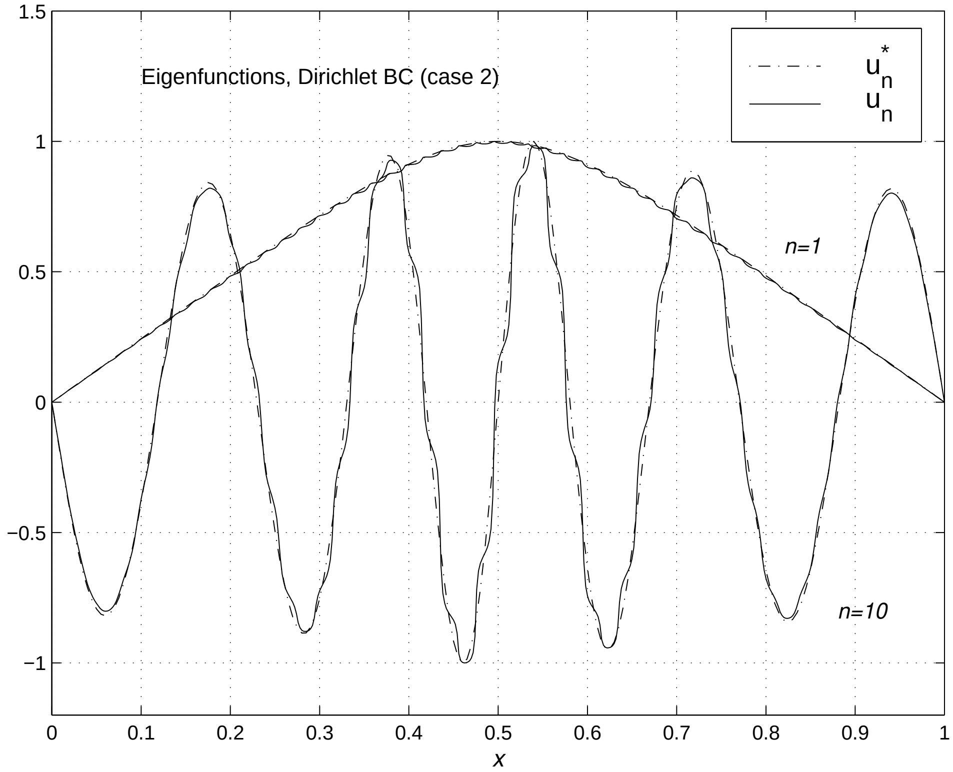

Eigenfunctions, Dirichlet BC (case 2)
u_n^*
u_n
n=1
n=10
x

Fig. 4b

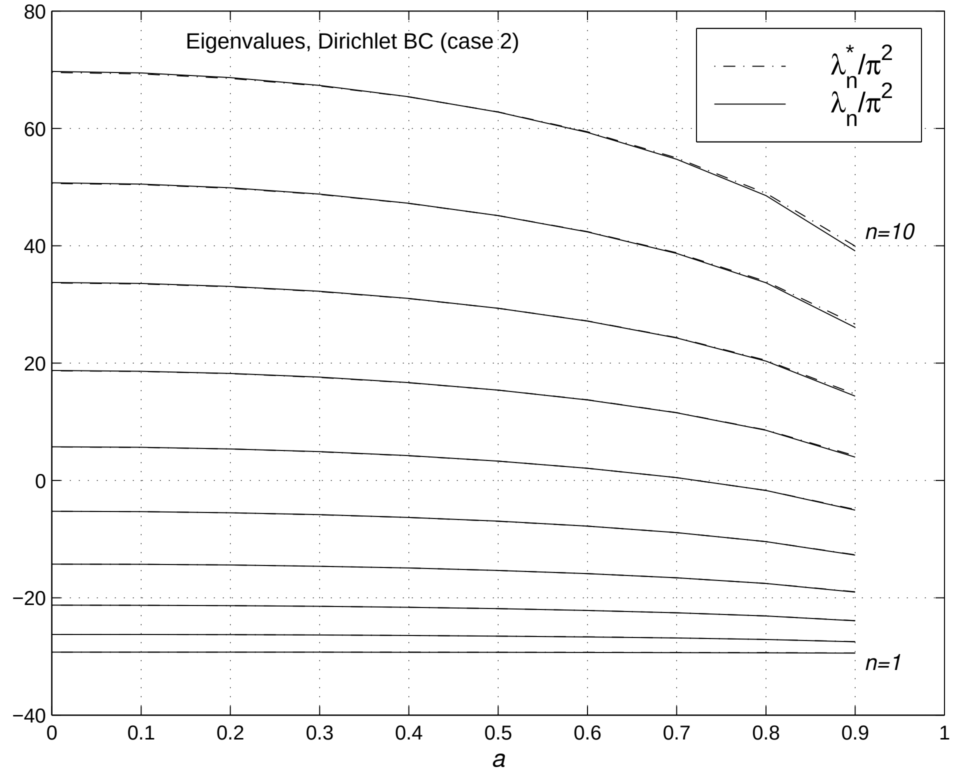

Eigenvalues, Dirichlet BC (case 2)
λ_n^*/π^2
λ_n/π^2
n=10
n=1
a

Fig. 5a

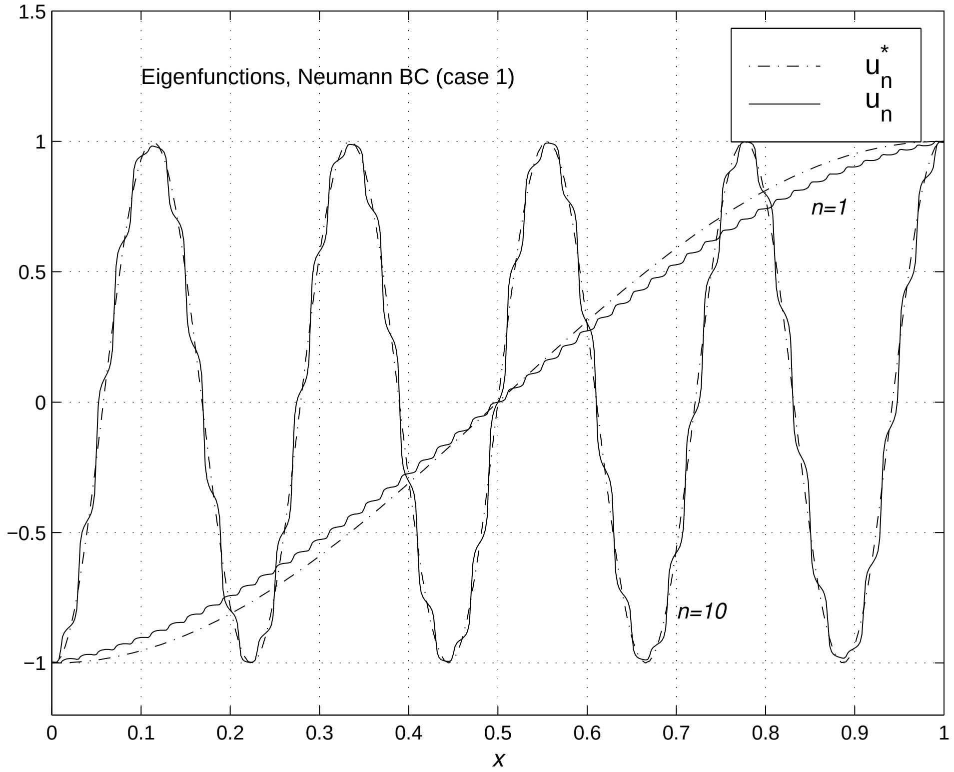

Eigenfunctions, Neumann BC (case 1)
u_n^*
u_n
n=1
n=10
x

Fig. 5b

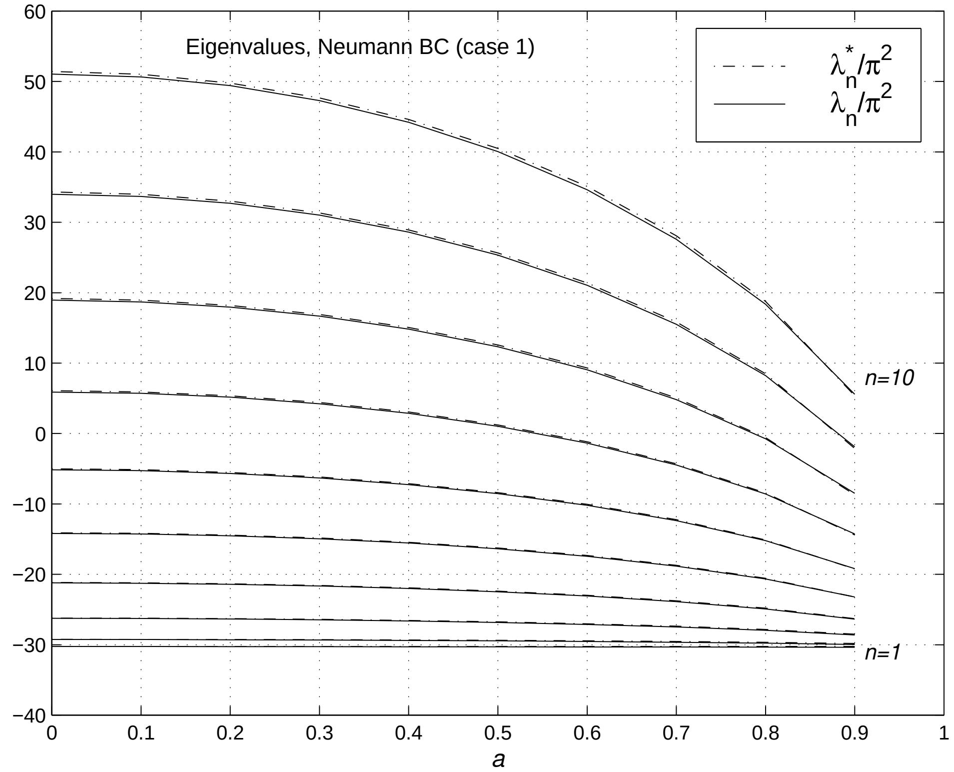

Eigenvalues, Neumann BC (case 1)
λ_n^*/π^2
λ_n/π^2
n=10
n=1
a

Fig. 6a

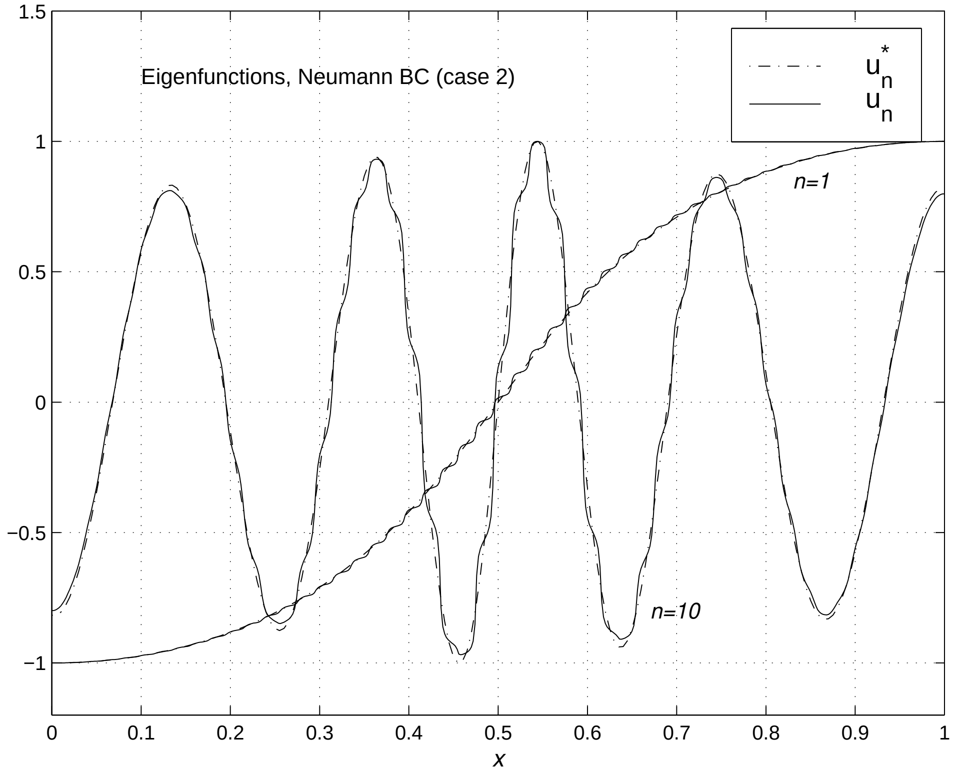

Eigenfunctions, Neumann BC (case 2)
u_n^*
u_n
n=1
n=10
1.5
1
0.5
0
−0.5
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Fig. 6b

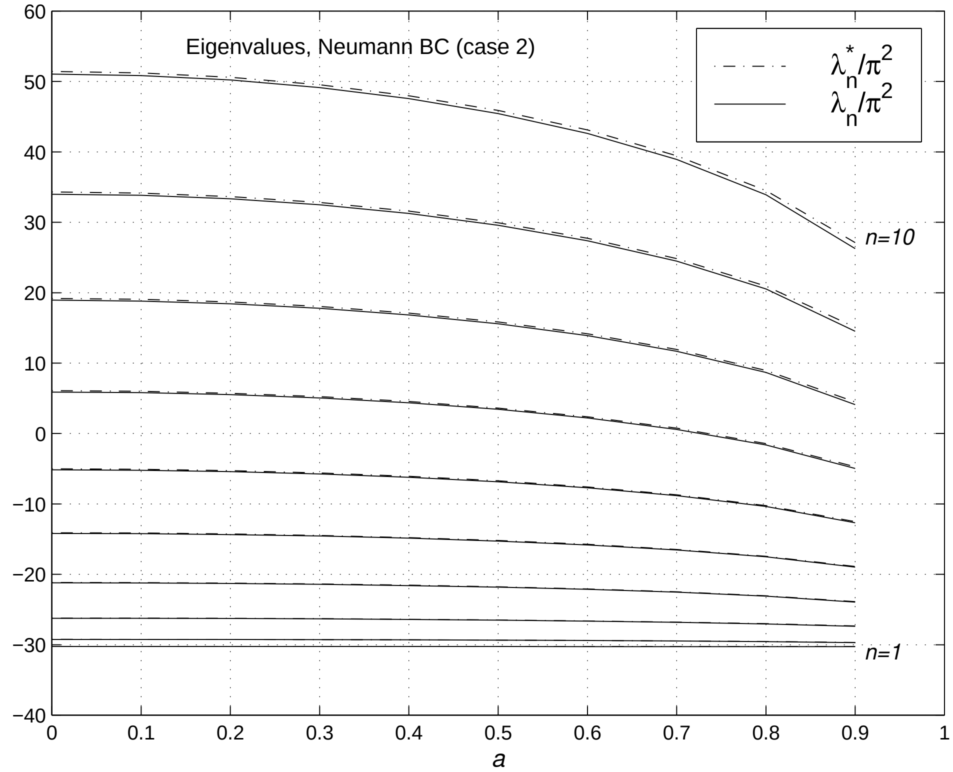

Eigenvalues, Neumann BC (case 2)
λ_n^*/π^2
λ_n/π^2
n=10
n=1
60
50
40
30
20
10
0
−10
−20
−30
−40
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
a

Fig. 7a

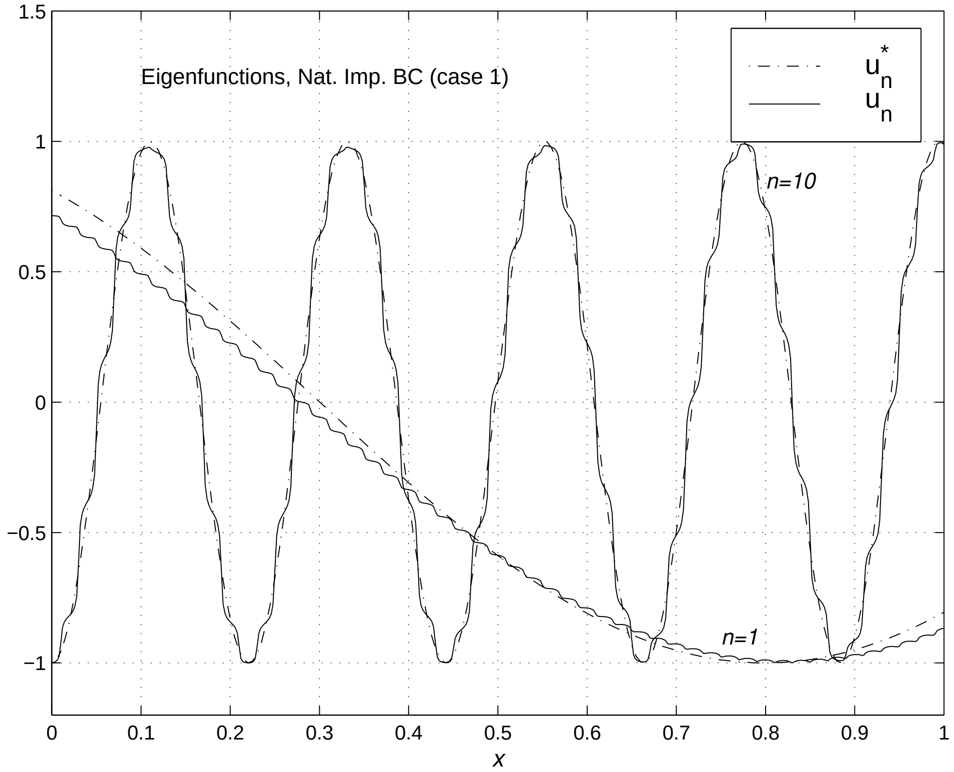
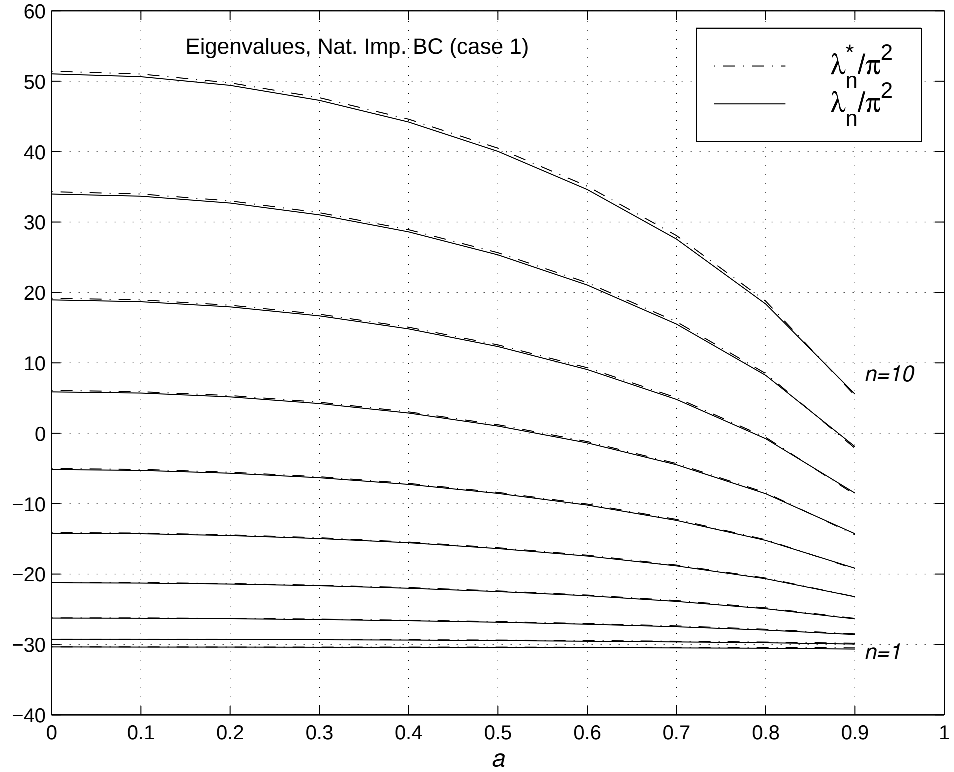

Eigenfunctions, Nat. Imp. BC (case 1)
u_n^*
u_n
n=10
n=1
x

Fig. 7b


Eigenvalues, Nat. Imp. BC (case 1)
λ_n^*/π^2
λ_n/π^2
n=10
n=1
a