A Multiresolution Analysis of Scattering by a Pair of Local Regions of Complex Heterogeneity

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October 1996

Abstract

The scattering by a pair of spatially disjoint local regions of complex (multiscale) stiffness heterogeneity in an unbounded thin linearly elastic plate, is investigated. Issues of the mutual interaction of the spatially disjoint scatterers and its manifestation within a homogenized, or effective, formulation governing large scale response fields, are the focus. It is shown analytically and numerically that while one must incorporate scatterer/scatterer coupling for estimating the large scale response field, one can neglect this coupling in estimating the effective measures, for describing the scatterers.

* **Acknowledgment** This research was supported in part by THE ISRAEL SCIENCE FOUNDATION administered by THE ISRAEL ACADEMY OF SCIENCE AND HUMANITIES

1 Introduction

An investigation of the scattering of time-harmonic, elastic waves by a spatially limited region of highly structured heterogeneity in an otherwise homogeneous plate bounding a fluid half space, was reported in a recent series of papers [1]–[5]. The heterogeneity refers to a variation of a plate property, i.e. the bending stiffness or the surface mass density. Primary attention applied to a specific regime of parameters which characterize the scattering problems, and the research issues relevant to these parameters. To set the issues of the present work in a proper context, we briefly describe the characteristics of the scattering problems and the research issues addressed.

The length, λ, of the flexural waves which would obtain in a homogeneous background plate response is one parameter. This is used to distinguish between a macroscale and a microscale, as applying to a variation that is observed in distances of the order of λ and much smaller than λ (say $\lambda/20$ and smaller), respectively. A *complex scatterer* is described as a local region of heterogeneity which requires description on both the macro and micro scales. Thus, the size of the region of heterogeneity is of a macroscale length. Then, the variation in the heterogeneity can have two components, a macroscale component observed in distances of the order of λ and a microscale component observed in distances much smaller than λ. The plate response in the presence of the articulated complex heterogeneity can similarly be resolved into macroscale and microscale components. The microscale, or "detail," component is a variation which averages to zero over distances of the order of λ. It is only the macroscale component of the plate response, also termed the spatially smoothed plate response, that results in acoustic field which radiates to significant distances from the plate. Variation in the plate response that is described on length scales smaller than λ result in an acoustic field, in the fluid half space, which lies in the evanescent part of the spectrum. The specific research issues addressed in [1]–[5] relate to the effects of the microscale heterogeneity on the macroscale response. A self consistent formulation expressed only in the macroscale component of the plate response was developed. The "footprint" of the small scale interior heterogeneity structure, i.e. the microscale heterogeneity, appears in the obtained formulation as an "across-scale coupling operator", or "effective material operator" (EMO). It is this operator, then, that describes the complex scatterer as determining the far-radiated acoustic field.

The norm of this across-scale coupling operator was shown to vanish in the case of a heterogeneity in the mass density of the plate [1]. The physical implication of this is: a mass density variation which averages to zero over regions of the size of a fraction of a wavelength, has no effect on the component of the plate response which results in a radiation of sound to far distances. The norm of the across-scale coupling operator was shown to be finite in the case of a heterogeneity in the local bending stiffness of the plate [2, 3]. The physical implication of this is: a bending stiffness variation which averages to zero over regions of the size of a fraction of a wavelength, does effect the component of the plate response which radiates as sound to far distances in a fluid bounded by the plate. Now, let $h(x)$ denote a heterogeneity of a complex scatterer character; $h(x) = m(x)$ or $h(x) = q(x)$ for a mass or stiffness heterogeneity, respectively. An effective measure of $h(x)$ is defined as the *smooth* heterogeneity $h^{(\text{eff})}(x)$ that gives rise to a plate response identical to the smooth component of the plate response associated with $h(x)$. Thus, the complex scatterer $h(x)$ and its effective measure $h^{(\text{eff})}(x)$ "look the same" when observed in the far field, but $h^{(\text{eff})}(x)$ is "simpler" then $h(x)$. With the results articulated above, $m^{(\text{eff})}(x)$ is merely the smooth component of $m(x)$, while $q^{(\text{eff})}(x)$ cannot be obtained by a mere smoothing of $q(x)$. The existence of the effective measure $q^{(\text{eff})}(x)$ and its derivation in the context of multiresolution analysis are first treated in [3, 4].

The study reported in this paper is of the effects of a second complex scatterer located some fixed distance from the first. It is shown analytically and numerically that the across-scale coupling operator which applies for the pair of scatterers is equal to the algebraic sum of those which apply for the isolated scatterers. The demonstration will be limited to the case of a bending stiffness variation; the demonstration for a mass variation follows in a similar manner, but is much less interesting in comparing zero with the sum of zeros.

Before presenting the details of the study, the issue to be studied might be further clarified. There is no question that the "smoothed" plate *response field* for the pair of scatterers cannot be estimated by summing the similarly smoothed plate *response fields* obtained for each of the scatterers in isolation. The coupling effects of multiple scattering are well appreciated. The issue is whether the effects of this coupling can be correctly estimated by a formulation expressed in the smoothed response field, in which the rule for obtaining the *across-scale-coupling* operator of the two scatterers need not incorporate any scatterer/scatterer coupling.

A major advantage of the two-step approach, of first estimating the across-scale-coupling operator and then solving for the smoothed plate response field, is the demonstrated conclusion that one can neglect all scatterer/scatterer coupling in the first step. This coupling is only significant in the second step, in a formulation that does not require calculations involving heterogeneity on scales of the order of the interior structure of the scatterers.

The outline of the paper is as follows: In Section 2, the mathematical framework for resolving heterogeneity and response fields according to length scales and the derivation of the formulation governing the smoothed component of the response field in a general scattering formulation, are reviewed. The application of these to scattering in a thin, linearly elastic plate which contains a stiffness heterogeneity of the complex character articulated above, is demonstrated in Sec. 3. Then, in Section 4, the heterogeneity as a pair of scatterers separated by an arbitrary but fixed distance and each containing an interior structure, is introduced. The size of the individual scatterers are restricted to be of the size of a wavelength, defined for a narrow-band excitation and the homogeneous background plate; the interior structures describe a variation on a scale that is a factor of 20 to 50 times smaller. Subject to these restrictions, it is shown that in the asymptotic regime in which 20 to 50 is a large number, the across-scale-coupling operator depends only on the local properties of the plate, i.e. the properties in a neighborhood of a size equal to a fraction of a wavelength. In Section 5, numerical results for the effective measures obtained for the pair of scatterers and obtained by adding the results for the two scatterers in isolation are shown and compared. Also presented are calculations of the smoothed component of the plate response field obtained for the pair of scatterers and obtained by adding the results of response fields for the two scatterers in isolation. Clearly demonstrated is while one must incorporate scatterer/scatterer coupling for estimating the smoothed response field, one can neglect this coupling in estimating the effective measures. Finally, in Section 5, summary remarks are offered.

2 A wavelet based phase-space

The theory of multiresolution decomposition (MRD) and orthogonal wavelets [6]–[7] provides a natural framework for the study of inter-scale coupling processes. There is a rapidly growing literature on the application of wavelet transformations to operators, e.g. the studies reported in [8]–[12]. The main concern the studies reported is computational, to reduce the number of operations required for the inversion of certain classes of operators by applying wavelet transformations and thereby reducing a densely populated matrix to one that is sparsely populated. The approach followed will not reduce the number of operations for the complex scatterer problem of interest here, however, since its effectiveness requires a smoothness in the system heterogeneity. In a study more closely related to the complex scatterer problem [13], the MRD theory was used to achieve a numerical homogenization, i.e. to formulate an equation with slowly varying coefficients whose solution has the same large scale behavior as that of the original equation. While this study has some relationship to previous studies of the present authors, [1, 2], the motivations were clearly different. Thus, the work in [13] is devoted mainly to a sophisticated "decimation" process in which efficient numerical algorithm for estimating the large scale response component is developed. Left unaddressed are all of the questions articulated in the Introduction of the present paper.

Addressing these questions requires a different viewpoint, and subsequently, a different analysis. Below the MRD theory is briefly summarized in a manner tailored to the class of problems of interest.

2.1 Representation of functions

Some terminology is required. A wavelet-based phase-space representation of a function is expressed as a linear sum; i.e., a synthesis; of *shifted* and *dilated* window functions, termed "wavelets" [6]. The term "mother wavelet" is used to identify a particular window function, the shifts and dilations of which result in an appropriate "wavelet system." Then, the term "discrete" applies if the linear sum for representing a function, requires only finite increments of shift and dilation. Finally, the term "orthogonal" applies if the wavelets are mutually orthogonal. Discrete, orthogonal wavelet systems represent a particular class of systems; a mathematical description of the system can be given in terms of the properties of the mother wavelet used for generating it.

One can truncate the wavelet sum at some arbitrarily chosen dilation, and thereby suppress variability on scales larger than the one that corresponds to the cutoff dilation. This truncation can be identified with a "high-pass" filtering. The truncated representation can once again be made complete by adding its "complement" to it. Moreover, this complement can be given representation as a linear sum of shifted window functions. The window function to be shifted, is a properly dilated function that has a precise relationship to the mother wavelet. This function is termed a "scaling function." Like the mother wavelet, the scaling function is defined independently of an absolute length. For an orthogonal system, the set of shifted, properly dilated, replicas of the scaling function are mutually orthogonal among themselves and are orthogonal to the wavelets.

The complete representation of an arbitrary function in terms of its "smooth" component, represented by a sum of shifted, properly dilated replicas of the scaling function, and its "detail" component, represented by a truncated sum of wavelets is expressed by

$$f(x) = f^s(x) + f^d(x), \tag{2.1}$$

where

$$f^s(x) = \sum_n F^s_{jn}\phi_{jn}(x), \quad F^s_{jn} = \langle f, \phi_{jn}\rangle, \tag{2.1a}$$

and

$$f^d(x) = \sum_{m=j}\sum_n F^d_{mn}\psi_{mn}(x), \quad F^d_{mn} = \langle f, \psi_{mn}\rangle. \tag{2.1b}$$

The $\phi_{jn} = 2^{j/2}\phi(2^j x - n)$ are the shifted (n), properly dilated (j), replicas of the scaling function and the $\psi_{mn} = 2^{m/2}\psi(2^m x - n)$ are the wavelets. $\langle \cdot, \cdot \rangle$ denotes a scalar product. Notice that the wavelet sum is truncated at the dilation level, $m = j$. The described representation applies for discrete, orthogonal wavelet systems.

One particular scaling function and its associated mother wavelet, the cubic-spline Battle-Lemarie system [6, 7], provide the framework for our studies. These are illustrated in Fig. 1.

The partial sum represented by $f^s(x)$ describes that *component* of the variation of $f(x)$ that applies for length scales equal to the chosen reference length scale 2^{-j} or larger. The partial sum of $f^d(x)$, given by a specified value for m, describes that *component* of the variation of $f(x)$ that applies for the specified length scale 2^{-m}. This interpretation suggests the terminology of a "multiscale decomposition," in referring to the representation.

2.2 Representation of operators

The set of functions defined by the wavelets, truncated at an arbitrarily chosen dilation level, (j), and the properly dilated replicas of the scaling function, provide for a matrix representation of an operator. Let $\mathcal{A}$ denote an arbitrary operator. Its matrix representation can be partitioned into four submatrices, $(\mathbf{A}^{(s,s)}, \mathbf{A}^{(s,d)}, \mathbf{A}^{(d,s)}, \mathbf{A}^{(d,d)})$, with elements defined as follows:

$$\begin{aligned} A^{(s,s)}_{jn\,,jn'} &= \langle \mathcal{A}\phi_{jn'}, \phi_{jn} \rangle \\ A^{(s,d)}_{jn\,,m'n'} &= \langle \mathcal{A}\psi_{m'n'}, \phi_{jn} \rangle \\ A^{(d,s)}_{mn\,,jn'} &= \langle \mathcal{A}\phi_{jn'}, \psi_{mn} \rangle \\ A^{(d,d)}_{mn\,,m'n'} &= \langle \mathcal{A}\psi_{m'n'}, \psi_{mn} \rangle \end{aligned} \tag{2.2}$$

There is a dual interpretation to the four submatrices as describing a partitioning of the operator $\mathcal{A}$ defined on a function space which, for the theory of a multiresolution decomposition, is required to be an $L_2(R)$ space. This dual interpretation accepts the four submatrices as representing the action of the $\mathcal{A}$ operator on the two subspaces of $L_2(R)$, the subspace of smooth functions and of detail functions, as this term is determined by the chosen reference scale. Thus, for the submatrices $\mathbf{A}^{(s,s)}$ and $\mathbf{A}^{(d,s)}$, the operator acts on the subspace of smooth functions; for the submatrices, $\mathbf{A}^{(s,d)}$ and $\mathbf{A}^{(d,d)}$, the action is applied to the subspace of detail functions. For the submatrices $\mathbf{A}^{(s,s)}$ and $\mathbf{A}^{(d,d)}$, the result on the action is "projected" onto the same function subspace on which the action was applied. For the submatrices $\mathbf{A}^{(d,s)}$ and $\mathbf{A}^{(s,d)}$, the result of the action is projected on the complement of the function subspace to that on which the action was applied. The suggested terminology is that $\mathbf{A}^{(s,s)}$ and $\mathbf{A}^{(d,d)}$ represent intrascale operators, while $\mathbf{A}^{(d,s)}$ and $\mathbf{A}^{(s,d)}$ represent interscale operators.

2.3 Representations of scattering problems

The scattering problem can be expressed as an integral equation with the algebraic structure,

$$w = w_0 + \mathcal{G}\mathcal{H}w, \tag{2.3}$$

where w is a measure of the wavefield in the structure, i.e. the plate; w_0 is the same measure for a structure with the region of heterogeneity absent; $\mathcal{H}$ is an operator that describes the interaction of the wavefield with a scatterer, and $\mathcal{G}$ is an operator that describes the distribution of the effects of this interaction throughout the structure; it is a propagator. Depending on the specifics of the scatterer, the $\mathcal{H}$ operator can be algebraic, differential, or perhaps require an integral, i.e. nonlocal, representation. The $\mathcal{G}$ operator is nonlocal.

Applying the Galerkin method to an integral equation represented by Eq. (2.3), results in a system of algebraic equations on the coefficients for describing $w(x)$ in a wavelet-based phase-space [1]–[4]. This system of algebraic equations can be interpreted, then, as providing a phase-space representation of a scattering problem. To "solve" the scattering problem, one first inverts the constructed system of algebraic equations to obtain the coefficients of $w(x)$, followed by the synthesis of Eqs. (2.1)–(2.1b).

The system of algebraic equations governing the coefficients of $w(x)$ can be written as follows:

$$\begin{aligned} (I-\Phi)\vec{s} - C\vec{d} \quad &= \vec{s}_0 \\ -\bar{C}\vec{s} + (I-\Psi)\vec{d} \quad &= 0. \end{aligned} \tag{2.4}$$

The coefficients of $w(x)$ which apply to the properly dilated replicas of the scaling function are collected in $\vec{s}$ and those which apply to the wavelet functions are collected in $\vec{d}$. Thus, $\vec{s}$ represents $w^s(x)$ and $\vec{d}$ represents $w^d(x)$.

The matrix operators $\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{C}, \bar{\mathbf{C}}$ are the four submatrices that represent the combined operator $\mathcal{G}\mathcal{H}$ via a decomposition similar to (2.2) with $\mathcal{A} = \mathcal{G}\mathcal{H}$: $\mathbf{\Phi} = \mathbf{A}^{ss}, \mathbf{\Psi} = \mathbf{A}^{dd}, \mathbf{C} = \mathbf{A}^{sd}, \bar{\mathbf{C}} = \mathbf{A}^{ds}$. The matrix elements are given by

$$
\begin{aligned}
\Phi_{n',n} &= \langle \mathcal{G}\mathcal{H}\phi_{jn}, \phi_{jn'} \rangle \\
\Psi_{m'n',mn} &= \langle \mathcal{G}\mathcal{H}\psi_{mn}, \phi_{m'n'} \rangle \\
C_{n',mn} &= \langle \mathcal{G}\mathcal{H}\psi_{mn}, \phi_{jn'} \rangle \\
\bar{C}_{m'n',n} &= \langle \mathcal{G}\mathcal{H}\phi_{jn}, \psi_{m'n'} \rangle .
\end{aligned}
\tag{2.5}
$$

A formulation tuned to govern the smoothed response measure $\vec{s}$ can be derived from (2.4) via a two steps procedure. In the first step one uses the lower half of (2.4) to formally solve for $\vec{d}$ in terms of $\vec{s}$,

$$
\vec{d} \,=\, (\mathbf{I} - \mathbf{\Psi})^{-1}\, \bar{\mathbf{C}}\, \vec{s}. \tag{2.6}
$$

This last result states that the relation between $\vec{d}$ and $\vec{s}$ (or the relation between the small scale and large scale responses) is independent of the problem forcing, and it proves to be useful in subsequent derivations. An exact, self consistent formulation governing $\vec{s}$ is obtained now by substituting Eq. (2.6) back in the upper half of Eq. (2.4),

$$
\left(\mathbf{I} \,-\, \mathbf{\Phi} \,-\, \mathbf{C}\,(\mathbf{I} - \mathbf{\Psi})^{-1}\, \bar{\mathbf{C}}\right)\, \vec{s} \,=\, \vec{s}_0. \tag{2.7}
$$

2.4 The footprint of small scale structure in large scale response

The interpretation of a scaling function as a low pass filter with a cutoff spatial frequency determined by the scale 2^{-j}, conveys the interpretation of the matrix operator $\mathbf{\Phi}$ as a representation of $\mathcal{G}\mathcal{H}$ smoothed on a reference scale 2^{-j}. Thus, the small scale structure information in $\mathbf{\Phi}$ has been eliminated. The effect of the small scale structure on the large scale response is expressed via the presence of the operator $\mathbf{C}\,(\mathbf{I} - \mathbf{\Psi})^{-1}\, \bar{\mathbf{C}}$, termed the *across scales coupling operator* or the *effective material operator* (EMO). A general measure of the potential degree of this effect is the EMO norm; if $\|\text{EMO}\| \ll \|\mathbf{\Phi}\|$, then the small scale structure has practically no effect on the large scale response. If these norms are comparable, then a small scale heterogeneity can have a significant effect on the large scale response. An estimate of the EMO norm requires one to specify the structure of the operators $\mathcal{G}$ and $\mathcal{H}$.

Of special interest is the case in which $\mathcal{H}$ is an algebraic multiplication operator and $\mathcal{G}$ is an integral operator. Then, for every $f(x)$ one has

$$
\mathcal{H}f(x) \,=\, h(x) f(x) \tag{2.8}
$$

$$\mathcal{G}f(x) = \int \Gamma(x,y) f(y)\, dy, \tag{2.9}$$

provided the integral exists. Here $h(x)$ and $\Gamma(x,y)$ are known functions, the former represents a heterogeneity field and the latter represents a response measure associated with an appropriately defined background problem (usually a one defined by a constant coefficients differential equation.) It has been shown [1]–[4] that when $\mathcal{H}$ and $\mathcal{G}$ take the form of Eqs. (2.8)–(2.9), the EMO norm is essentially determined by the irregularity order of $\Gamma(x,y)$ at the origin $x = y$. For highly irregular kernels the EMO norm can be $O(1)$, whereas for smooth kernels $\|\text{EMO}\| \ll 1$. The former represents the cases for which the small scale heterogeneity can have a significant effect on the large scale response, and the latter represents cases in which it has practically no effect. Another result that holds in the latter case follows directly from Eq. (2.6),

$$\left\| \vec{d} \right\| \ll \| \vec{s} \| \quad \Longleftrightarrow \quad \left\| w^d \right\| \ll \| w^s \| . \tag{2.10}$$

This result will prove to be extremely important.

3 Scattering on a thin linearly elastic plate

The differential equation governing the motion of a thin linearly elastic plate with a one-dimensional stiffness heterogeneity, $Q(x)$, subject to a time harmonic line forcing, $F(x)$, is given by

$$\frac{d^2}{dx^2}\left[Q(x) \frac{d^2}{dx^2} w(x) \right] - \omega^2 m w(x) = F(x), \tag{3.1}$$

where $w(x)$ is the flexural (transverse displacement) response of the plate, m is the surface mass density of the plate, and ω is the radian frequency. The time harmonic dependence $e^{-i\omega t}$ has been suppressed. We shall define the local normalized (nondimensional) fluctuating component of $Q(x)$ via the relation,

$$Q(x) = Q_0 \left[1 + q(x) \right], \tag{3.2}$$

in which the constant Q_0 is interpreted as the stiffness associated with the homogeneous "background" plate. The constants Q_0, m and ω define the background wavenumber,

$$K_{f_0} = \left(\frac{\omega^2 m}{Q_0} \right)^{1/4} . \tag{3.3}$$

It is the wavenumber of the free waves that propagate in the homogeneous background plate. For convenience, we shall normalize the physical coordinate x with respect to the background wavenumber via

$$x \mapsto \frac{K_{f0}x}{2\pi} = \frac{x}{\lambda_{f0}}. \tag{3.4}$$

Here λ_{f0} is the wavelength associated with K_{f0}. Expressed in the new dimensionless variable, x, the equation of motion takes the form

$$\frac{d^2}{dx^2}\left[(1+q(x))\,\frac{d^2}{dx^2}w(x)\right] - (2\pi)^4 w(x) \;=\; f(x), \tag{3.5}$$

where $f(x) = \frac{\lambda_{f0}^4}{Q_0}F(x)$ is the normalized forcing. A formulation governing the background plate is readily obtained from Eq. (3.5) by setting $q(x) = 0$. Similarly, the corresponding Green function $G(x)$ satisfies

$$\frac{d^4}{dx^4}G(x-x') \;-\; (2\pi)^4 G(x-x') \;=\; -\delta(x-x'), \tag{3.6}$$

where $\delta(x)$ is the Dirac delta function. The last equation can be solved analytically. The result is

$$G(x) \;=\; -\frac{1}{32\,\pi^3}\left(ie^{i2\pi|x|} \;-\; e^{-2\pi|x|}\right). \tag{3.7}$$

With the help of Eqs. (3.6)–(3.7), Eq. (3.5) can be transformed to an integral equation formulation. To achieve this, the term containing the heterogeneity, $\frac{d^2}{dx^2}\left[q(x)\frac{d^2}{dx^2}w(x)\right]$, is moved to the right hand side, and the solution, $w(x)$, written as a convolution of the background problem Green function, $G(x)$, with the extended source term, $f(x) - [q(x)w''(x)]''$. The result is

$$w(x) \;=\; w_0(x) \;+\; \int G(x-y)\,[q(y)w''(y)]''\;dy \tag{3.8}$$

where $w_0(x)$ is the background system response to the forcing $f(x)$,

$$w_0(x) \;=\; -\int G(x-y)f(y)\,dy. \tag{3.9}$$

Equation (3.8) is integro-differential. It can be transformed to a Fredholm integral equation of the second kind governing $w''(x)$—the inverse of the local radii of curvature of the plate. Thus, taking a second derivative of Eq. (3.8) with respect to x obtains

$$u(x) \;=\; u_0(x) \;+\; \int G''(x-y)\,[q(y)u(y)]''\; dy, \tag{3.10}$$

where $u(x) = w''(x)$ and $u_0(x) = w_0''(x)$. Next the rhs is integrated by parts twice, and Eq. (3.6) used to express the fourth derivative of $G(x)$ in terms of $G(x)$. The result is

$$u(x) \;=\; u_0(x) \;+\; \int \Gamma(x,y)q(y)u(y)\,dy, \tag{3.11}$$

where $\Gamma(x,y)$ is a kernel function given by

$$\Gamma(x,y) \;=\; (2\pi)^4 G(x-y) - \delta(x-y). \tag{3.11a}$$

This is a second kind Fredholm integral equation scattering formulation, identical to that in Eq. (2.3) with the special forms in Eqs. (2.8)–(2.9). It is seen in Eq. (3.11a) that the kernel associated with the integral equation formulation is highly irregular. In light of the discussion in Sec. 2.4 and [2]–[4], the conclusion follows that a small-scale stiffness heterogeneity (small-scale variation of $q(x)$) can have a significant effect on the large-scale response $u^s(x)$. This is demonstrated numerically in Sec. 5.

An alternative formulation, similar in form to that of $u(x)$ in Eq. (3.11), but with a smoother kernel can be derived. Performing the integration over $\delta(x-y)$ and moving the result to the left hand side obtains

$$u(x)\,[1+q(x)] \;=\; u_0(x) \;+\; (2\pi)^4 \int G(x-y)q(y)u(y)\,dy. \tag{3.12}$$

This suggests a rewrite the scattering formulation in terms of the normalized local bending moment, $T(x)$:

$$T(x) \;=\; u(x)\,[1+q(x)]\,, \tag{3.13}$$

which yields ($u_0(x) = T_0(x)$)

$$T(x) \;=\; T_0(x) \;+\; (2\pi)^4 \int G(x-y)r(y)T(y)\,dy, \tag{3.14}$$

with

$$r(y) \;=\; \frac{q(y)}{1+q(y)} \;=\; 1 - \frac{1}{1+q(y)}. \tag{3.14a}$$

This scattering formulation, written now on the bending moment $T(x)$, has the same standard form of Eq. (2.3) with Eqs. (2.8)–(2.9)–a property shared also by the scattering formulation (3.11). However, unlike the formulation in (3.11), the integral operator kernel function $(2\pi)^4 G(x-y)$ as well as its first two derivatives are continuous at the origin $(x=y)$. Thus, the corresponding EMO norm is very small compare to $\|\mathbf{\Phi}\|$ (see for example [2]–[4]). Thus, small scale variation of $r(x)$ has practically no effect on large scale response. As a consequence, the scattering formulation governing the large scale component of $T(x)$ is identical to Eq. (3.14), but with $r(x)$ replaced by its large scale component $r^s(x)$,

$$T^s(x) \;=\; T_0(x) \;+\; (2\pi)^4 \int G(x-y) r^s(y) T^s(y)\, dy, \tag{3.15}$$

where, with Eq. (3.14a),

$$r^s(y) \;=\; 1 - \left(\frac{1}{1+q(y)}\right)^s. \tag{3.15a}$$

Furthermore, the local constitutive relation (3.13), in conjunction with (2.10) can be approximated by $(\|T^s\| \gg \left\|T^d\right\|)$

$$u^s(x) \;\simeq\; T^s(x) \left(\frac{1}{1+q(x)}\right)^s. \tag{3.16}$$

¿From Eqs. (3.15)–(3.16) it is seen that the only heterogeneity measure relevant for the computation of $u^s(x)$ is $\left(\frac{1}{1+q(x)}\right)^s$. In other words,

$$\frac{1}{1+q^{(\mathrm{eff})}(x)} \;=\; \left(\frac{1}{1+q(x)}\right)^s \quad\Longrightarrow\quad q^{(\mathrm{eff})}(x) \;=\; \left[\left(\frac{1}{1+q(x)}\right)^s\right]^{-1} - 1, \tag{3.17}$$

where $q^{(\mathrm{eff})}(x)$ is the effective measure of the heterogeneity variation $q(x)$. The computational role of $q^{(\mathrm{eff})}$ is clarified by the following steps. Use Eq. (3.17) to express the relation in Eq. (3.16) in terms of $q^{(\mathrm{eff})}(x)$,

$$T^s(x) \;\simeq\; u^s(x)\left[1 + q^{(\mathrm{eff})}(x)\right]. \tag{3.18}$$

Similarly, express $r^s(x)$ in Eq. (3.15a) in terms of $q^{(\mathrm{eff})}(x)$,

$$r^s(x) \;=\; \frac{q^{(\mathrm{eff})}}{1+q^{(\mathrm{eff})}(x)}. \tag{3.19}$$

Use the last two expressions to rewrite the integral equation formulation (3.15) in terms of $u^s(x)$ and $q^{(\mathrm{eff})}(x)$,

$$u^s(x)\left[1+q^{(\mathrm{eff})}(x)\right] \;=\; u_0(x) \;+\; (2\pi)^4 \int G(x-y)q^{(\mathrm{eff})}(y)u^s(y)\,dy, \tag{3.20}$$

an alternative form of which is

$$u^s(x) \;=\; u_0(x) \;+\; \int \Gamma(x,y)q^{(\mathrm{eff})}(y)u^s(y)\,dy \tag{3.21}$$

where $\Gamma(x,y)$ is the same kernel as that of the complete formulation written for $u(x)$ in Eqs. (3.11)–(3.11a). Thus, the formulation written directly for the large scale response measure $u^s(x)$ is identical to that written for $u(x)$, with the heterogeneity $q(x)$ replaced by its effective measure $q^{(\mathrm{eff})}(x)$. Certainly, one can use now Eq. (3.21) as a starting point, and reverse the steps that lead from the differential equation (3.1) to the integral equation formulation. The resultant differential equation formulation is then

$$\frac{d^2}{dx^2}\left[Q^{(\mathrm{eff})}(x)\frac{d^2}{dx^2}w^s(x)\right] - \omega^2 m w^s(x) \;=\; F(x) \tag{3.22}$$

where $w^s(x)$ is the large scale response component, and $Q^{(\mathrm{eff})}(x) = 1 + q^{(\mathrm{eff})}(x)$.

4 The two scatterers experiment

The only nonlocal operation involved with the computation of $q^{(\mathrm{eff})}(x)$ is the smoothing. The rest are algebraic. The smoothing operator, however, becomes local when observed on length scales that are large compared to the smoothing length scale 2^{-j} (reference scale). One usually chooses the reference scale to faithfully describe variations on the length scale of the background wavelength λ. Thus, the reference scale 2^{-j} shall be, say, $\lambda/4$ or $\lambda/8$. It follows that the effective property measure is local with respect to the wavelength.

Let $q_1(x)$ and $q_2(x)$ two highly structured stiffness heterogeneity functions, and let $q_3(x)$ be the stiffness heterogeneity that corresponds to the sum of the two heterogeneity measures,

$$q_3(x) \;=\; q_1(x) + q_2(x). \tag{4.1}$$

Let I_1 and I_2 be the intervals in space in which $q_2(x) \neq 0$ and $q_1(x) \neq 0$, respectively. Assume the intervals do not overlap. If the distance between I_1 and I_2 is measured on the large scale, the local property of the effective measure yields,

$$q_3^{(\mathrm{eff})}(x) \;=\; q_1^{(\mathrm{eff})}(x) + q_2^{(\mathrm{eff})}(x). \tag{4.2}$$

That is: *effective measures can be superimposed, if their corresponding scatterers are isolated when observed on the large scale.*

5 Numerical example and conclusions

The non-dimensional (normalized) formulation (3.5) with the normalized space coordinate system denoted by (3.4) is to be used throughout this section. Thus, the length scale associated with the background system response equals one unit (see (3.7)). The reference, or smoothing, length scale, 2^{-j}, is chosen to correspond to $\lambda/8$, thus $j = 3$. Two complex scatterers were synthesized with a random number generator. The corresponding stiffness distributions $q_1(x)$ and $q_2(x)$ are shown in Fig. 2a and Fig. 2b. Their supports are spatially disjoint. A third complex heterogeneity $q_3(x)$ given by the sum of $q_1(x)$ and $q_2(x)$ (see Eq. (4.1)) is shown in Fig. 2c. In all three complex scatterers, the length scale for observing the inner structure is in the order of $\lambda/20$ ($\lambda = 1$). The outer dimensions of $q_1(x)$ and of $q_2(x)$ as well as the spatial separation between them are in the order of a few λ. With the chosen reference scale, the smooth component of the heterogeneity vanish,

$$q_i^s(x) \;=\; 0 \qquad \text{for } i = 1, 2, 3. \tag{5.1}$$

Thus, the scatterers under consideration possess only microscale heterogeneity.

Assume the system is subject to a point forcing $f(x) = \delta(x - x')$ located at $x' = -10$. The background system response to this forcing is given by $u_0(x) = G(x - x')$, where $G(x)$ is given in Eq. (3.7). This background response is shown in Fig. 3. Denote by $u_i^s(x), u_i^d(x)$, and $u_i(x)$ the macroscale, microscale, and complete responses associated with the heterogeneity $q_i(x)$, subject to the forcing $f(x)$.

Consider the following demonstration of a footprint of the microscale heterogeneity in the macroscale response. Figures 4a, 4b, and 4c apply to the plate responses associated with $q_1(x)$,$q_2(x)$, and $q_3(x)$, respectively. Shown are the magnitudes of the complete response $u_i(x)$ and the magnitudes of its macroscale and microscale components $u_i^s(x)$ and $u_i^d(x)$. Clearly, $u_i^s(x)$ in all cases differ considerably from the background system response shown in Fig. 3. Thus, as anticipated by the presented analysis, small scale stiffness heterogeneity can have a significant effect on large scale response.

Figure 5 allows a comparison of $u_3^s(x)$ with the synthetic field $\bar{u}(x) = \frac{1}{2}u_1^s(x) + \frac{1}{2}u_2^s(x)$. The multiplication factor $1/2$ has been inserted in the latter to make the total source intensities of $u_3^s(x)$ and of $\bar{u}(x)$ equal. It is seen that the magnitudes differ considerably. Thus, *one must incorporate scatterer/scatterer coupling for estimating the macroscale response component.*

Finally, Fig. 6 contains a demonstration that the effective measures for a pair of spatially disjoint scatterers is correctly given by the algebraic sum of the effective measures for the isolated scatterers. Thus, the lines representing the effective measure of $q_3(x)$ and the sum of the effective measures of $q_1(x)$ and $q_2(x)$, in Fig. 6a, are indistinguishable to within the graphic resolution of the figure. Figure 6b contains a demonstration of the fidelity of the effective measure in determining the macroscale response, $u_3^s(x)$, in reproducing the *complete* response associated with $q_3^{(\mathrm{eff})}(x)$. The curves are in good agreement. These last results are consistent with the assertion in Sec. 4 that *one can neglect scatterer/scatterer interaction in estimating the effective measures.*

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Figure captions

Figure 1. The cubic spline Battle-Lemarie multiresolution system. (a) The scaling function $\phi(x)$. (b) The wavelet $\psi(x)$.

Figure 2. Examples of stiffness heterogeneity that constitute complex scatterers. (a) Complex scatterer $q_1(x)$. (b) Complex scatterer $q_2(x)$. (c) Complex scatterer $q_3(x) = q_1(x) + q_2(x)$.

Figure 3. The background system response $u_0(x) = G(x - x')$.

Figure 4. Magnitudes of $u^s(x)$, $u^d(x)$, and $u(x)$ associated with (a) $q_1(x)$, (b) $q_2(x)$, (c) $q_3(x)$.

Figure 5. Magnitudes of the macroscale response $u_3^s(x)$ and the synthetic field $\bar{u}(x) = \frac{1}{2}u_1^s(x) + \frac{1}{2}u_2^s(x)$.

Figure 6. Effective measures. (a) $q_3^{(\mathrm{eff})}(x)$ - the effective measure of $q_3(x)$ - compared to $q_1^{(\mathrm{eff})}(x) + q_2^{(\mathrm{eff})}(x)$. (b) The field associated with the stiffness distribution $q_3^{(\mathrm{eff})}(x)$ compared to the macroscale component of the field associated with $q_3(x)$.