TECHNION
Israel Institute of Technology

אוניברסיטת בן-גוריון בנגב
Ben-Gurion University of the Negev

אורט בראודה
ORT Braude College

PTAS for the Bi-Scenario Total Completion Time Trade-Off Problem

Miri Gilenson, Hussein Naseraldin, Dvir Shbtay, Liron Yedidsion

Introduction

<u>Input</u>

Processing times, due dates, weights, etc.

*Information may be **uncertain***

Scheduling with limited recourses

<u>Output</u>

Scheduling decision

<u>Example</u>:
Processing a job by different workers/ on unreliable machines

Introduction

- *Four approaches are commonly used in the literature to capture the uncertainty level:*
  1. *One possible and known value for each parameter (Deterministic approach).*
  2. *Parameters belongs to a given distribution function (Stochastic approach).*
  3. *Parameters belongs to bounded intervals, with lower and upper bounds.*
  4. *Parameters belongs to a set of discrete scenarios.*

Problem Definition

- *A set of n jobs, $\mathcal{J} = \{J_1, \ldots, J_n\}$ is to be scheduled on a single machine so as to minimize the total completion time.*
- *We assume two scenarios, $S = \{s_1, s_2\}$, each of which defines a different possible set of job processing times, such that under scenario $s_i \in S$, $p_j^{(s_i)}$ is the processing time of job J_j .*

Discrete Scenario Uncertainty- An Example

- *Preventive maintenance:*
  - *Processing time of each job consists of inspection and repair.*
  - *The inspection duration is known and represents the minimal processing time.*
  - *Job- dependency: if repair is required, it affects all jobs.*

$Scenario\ (s_1)$: $Inspection \rightarrow p_j^{(s_1)}, \forall j$

$Scenario\ (s_2)$: $Inspection + Repair \rightarrow p_j^{(s_2)}, \forall j$

Problem Definition

- *Let* $\sigma(j)$ *denote the job in the* j^{th} *position in a given schedule* σ.
- *Let* $C_{\sigma(j)}^{(s_i)}$ *be the completion time of job* $\sigma(j)$ *under scenario* s_i.
- *The quality of a solution,* σ*, is measured by a pair solution values:* $\emptyset = \left(\sum_{j=1}^{n} C_{\sigma(j)}^{(s_1)},\ \sum_{j=1}^{n} C_{\sigma(j)}^{(s_2)}\right)$.

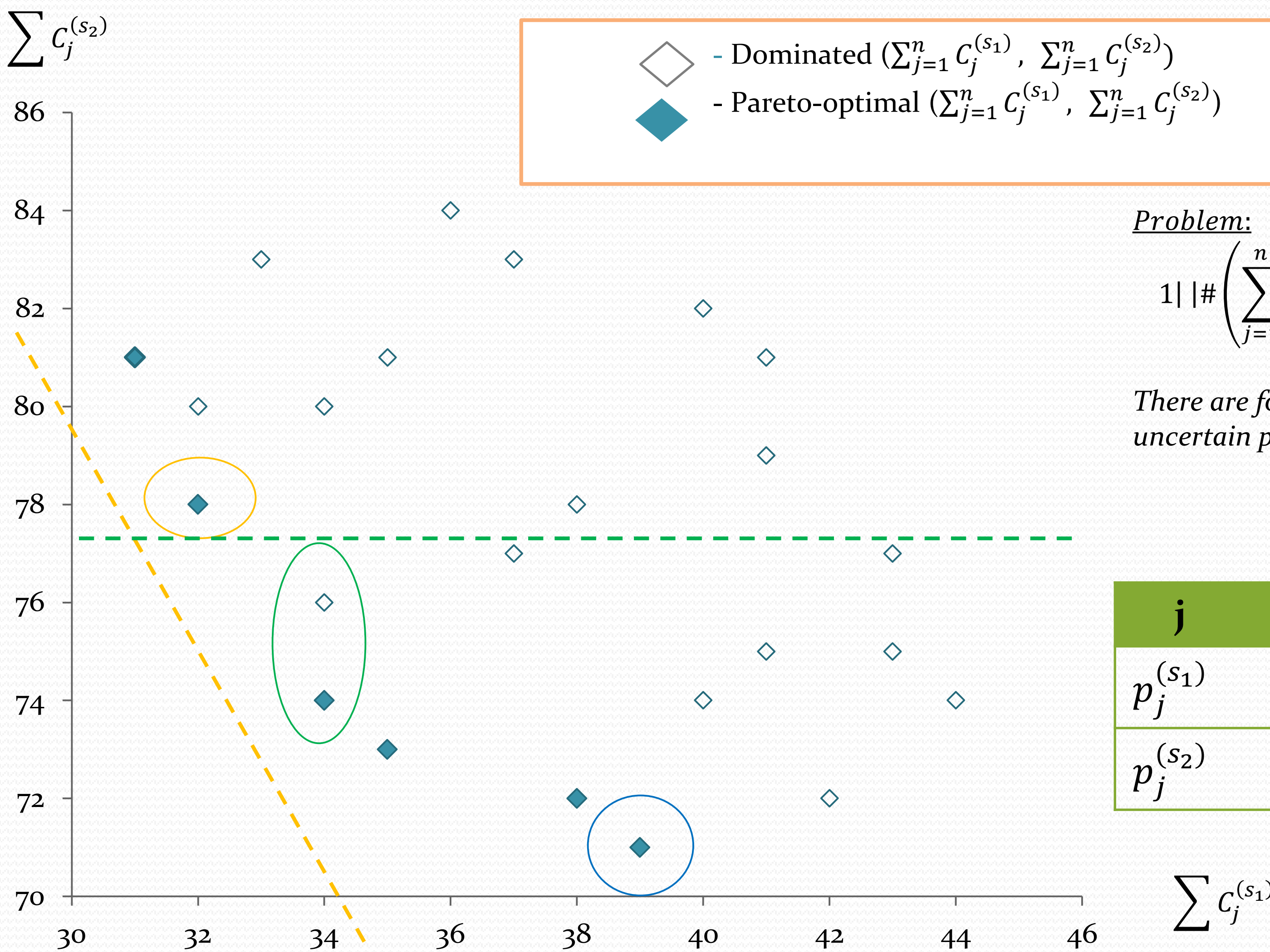
Problem Definition

- ***Objective:*** *Solve the* $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)},\ \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem*

Definition of #: *Identify a single Pareto-Optimal (PO) schedule (also known as efficient) for each PO point, where schedule σ is considered to be a PO schedule with respect to* $\sum C_j^{(s_1)}, \sum C_j^{(s_2)}$ *if there does not exist another schedule σ′ such that* $\sum_{j=1}^{n} C_{\sigma(j)}^{(s_i)} \leq \sum_{j=1}^{n} C_{\sigma\prime(j)}^{(s_i)}$ *for i=1,2 with at least one of these inequalities being strict.*

- *No preemption*
- *Offline scheduling*

Pareto-Optimal Set



Problem:

$$1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$$

There are four jobs with uncertain processing times:

j	1	2	3	4
$p_j^{(s_1)}$	2	4	6	3
$p_j^{(s_2)}$	10	6	8	7

Results

Theorem : *The* $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem is ordinary NP-hard (based on Yang and Yu (2002)).*

- *Thus, for the* $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem, we provide:*
  - *A constant factor approximation algorithm that provides, in polynomial time, (2,1)-approximation and (1,2)-approximation ratios for the* ***entire set*** *of PO solutions (based on the idea of* Angels et al. (2005));
  - *A proof that the above approximation ratios are asymptotically tight;*
  - *A data-dependent analysis of the approximation ratios; and*
  - *A PTAS for the* $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem.*

(ρ_1, ρ_2)-Approximation

- ***Definition:*** *A set Π^A of feasible solutions for a 1| |#$\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ problem with a corresponding set of solution points, $APOS$, is a (ρ_1, ρ_2)-approximation to the Pareto set of optimal solutions (set Π^E with it's corresponding set of solution points POS),* ***if*** *for any point $\left(\sum_{j=1}^{n} C_{\sigma(j)}^{(s_1)}, \sum_{j=1}^{n} C_{\sigma(j)}^{(s_2)}\right) \in POS$ there exists a point $\left(\sum_{j=1}^{n} C_{\sigma'(j)}^{(s_1)}, \sum_{j=1}^{n} C_{\sigma'(j)}^{(s_2)}\right) \in APOS$ in which the condition that*

$$\sum_{j=1}^{n} C_{\sigma'(j)}^{(s_i)} \leq \rho_i \sum_{j=1}^{n} C_{\sigma(j)}^{(s_i)}$$

  holds for $i = 1,2$.
- *An algorithm that provides such an approximate set Π^A is referred to as (ρ_1,ρ_2)-approximation algorithm.*

Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

We present an approximation algorithm that constructs a set of feasible supported solutions, $\Pi^S \equiv \Pi^A$, for the $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ problem, that provides a $(2,1)$-approximation and a $(1,2)$-approximation to set Π^E in a polynomial time.

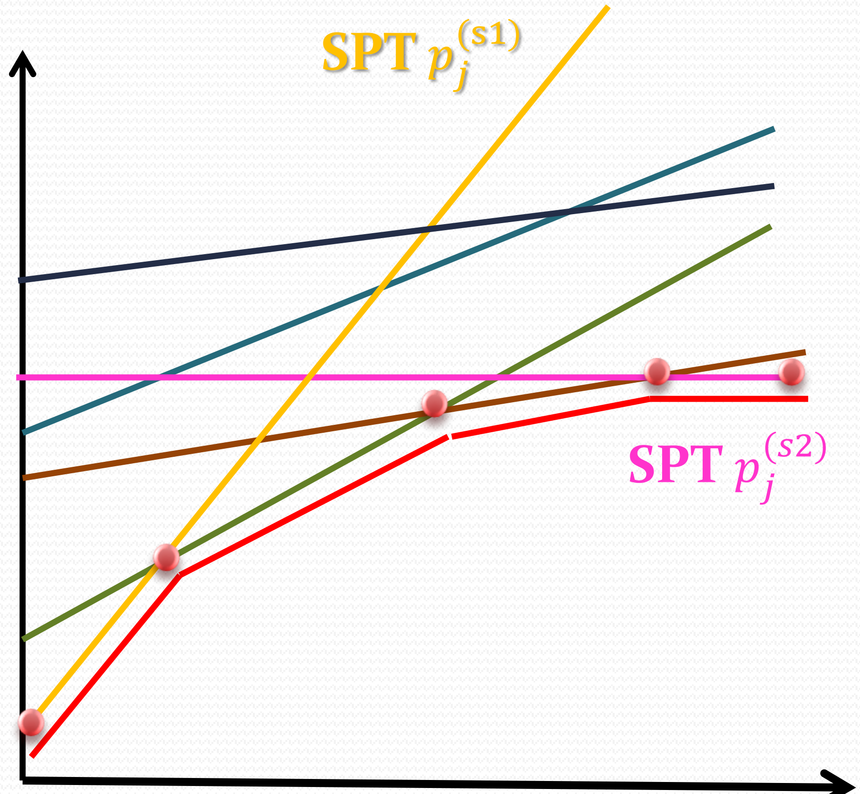
Definition: *A solution σ for a bi-scenario problem is called "supported", if there exists a pair of (θ_1, θ_2) values such that σ is an optimal solution the $1||\sum_{i=1}^{2} \theta_i \sum_{j=1}^{n} C_j^{(s_i)}$ problem.*

Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

Corollary: *Given θ, the optimal solution, σ_θ^*, for the $1||\left(\sum_{j=1}^{n} C_j^{(s_1)} + \theta \sum_{j=1}^{n} C_j^{(s_2)}\right)$ problem is to sequence the jobs in a non-decreasing order of $p_j^{(s_1)}+\theta p_j^{(s_2)}$.*

$\theta_1 = 1$ *and* $\theta_2 = \theta \geq 0$

$$\left(\sum_{j=1}^{n} C_j^{(s_1)} + \theta \sum_{j=1}^{n} C_j^{(s_2)}\right)$$

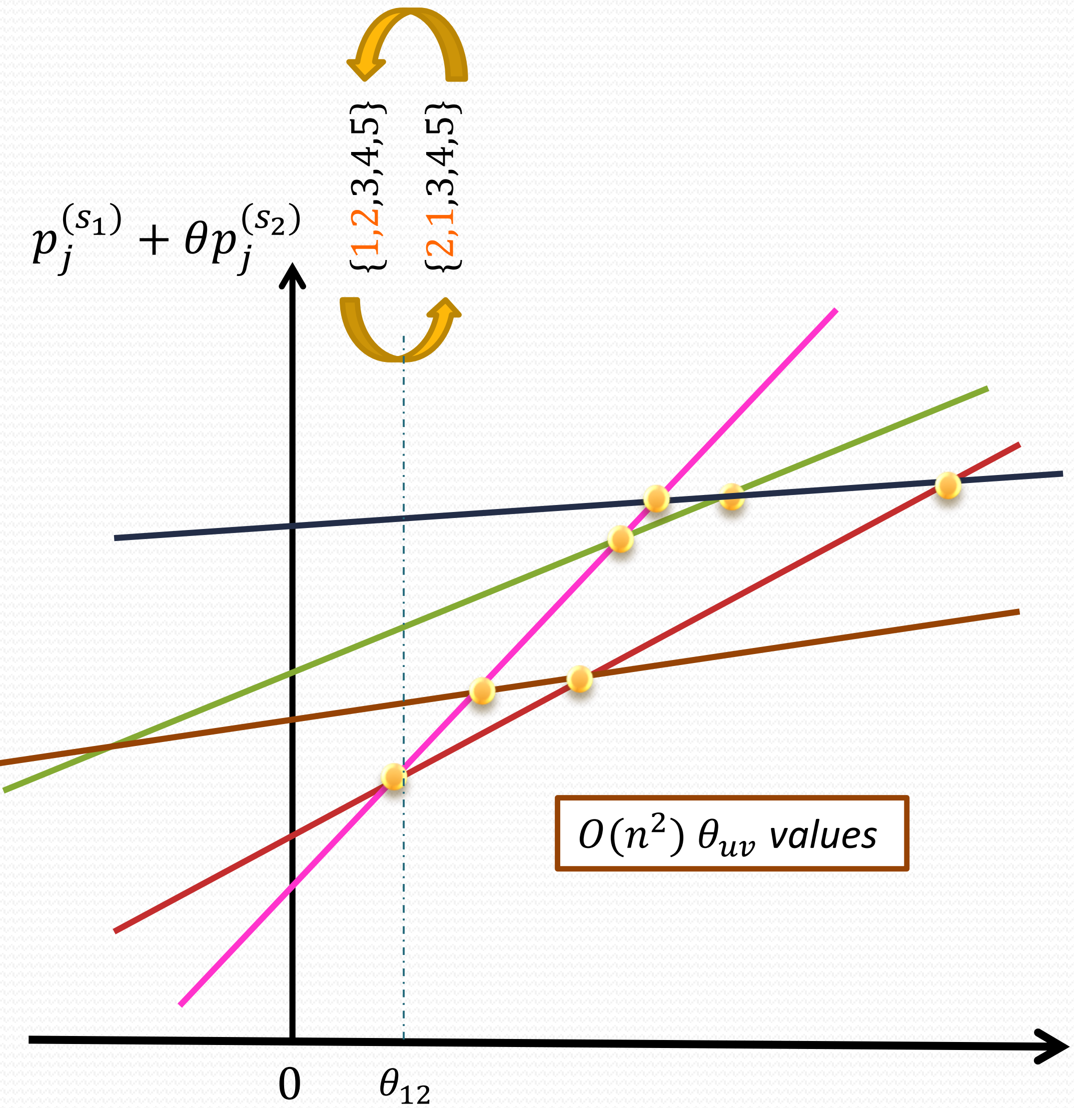


Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

Algorithm Outline:

1. *Initiate the algorithm by finding the first supported solution, which is the optimal solution for* $1||\left(\sum_{j=1}^{n} C_j^{(s_1)} + \theta \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem for* $\theta = 0$.
2. *Find the set* Π^S *of supported solutions, by finding consecutive supported solutions, i.e., finding the* $\theta > 0$ *values for which the optimal solution for* $1||\left(\sum_{j=1}^{n} C_j^{(s_1)} + \theta \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *changes.*

Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$



- *Consider an arbitrary pair of jobs* $(J_u, J_v) \in \mathcal{J}$.
- *Assume, w.l.o.g., that* $p_u^{(s_1)} \leq p_v^{(s_1)}$.

Case 1: $p_u^{(s_2)} \leq p_v^{(s_2)} \Rightarrow$

$$p_u^{(s_1)} + \theta p_u^{(s_2)} \leq p_v^{(s_1)} + \theta p_v^{(s_2)}, \forall \theta;$$

Case 2: $p_u^{(s_2)} > p_v^{(s_2)} \Rightarrow$

$$p_u^{(s_1)} - \theta p_u^{(s_2)} = p_v^{(s_1)} - \theta p_v^{(s_2)}$$

$$\theta_{uv} = \frac{p_v^{(s_1)} - p_u^{(s_1)}}{p_v^{(s_2)} - p_u^{(s_2)}}$$

Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

Proving the approximation ratio:

1) $\Delta = \sum_{j=1}^{n} C_j^{(s_1)}(\sigma') - \sum_{j=1}^{n} C_j^{(s_1)}(\sigma) =$

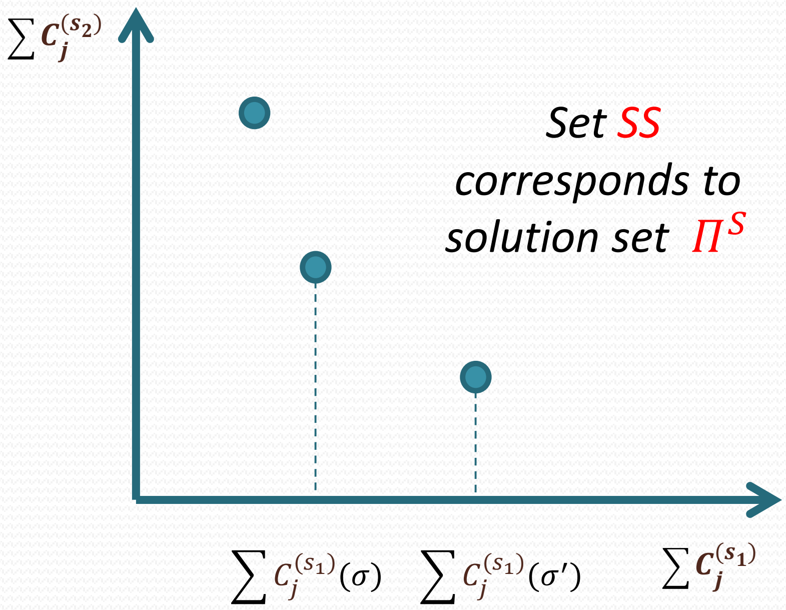
$$\underbrace{p_v^{(s_1)} - p_u^{(s_1)}}_{\text{Swapping adjacent jobs}} \leq p_{max}^{(s_1)}$$

Swapping adjacent jobs

2) $\sum_{j=1}^{n} C_j^{(s_1)}(\sigma') = \sum_{j=1}^{n} C_j^{(s_1)}(\sigma) + \Delta$

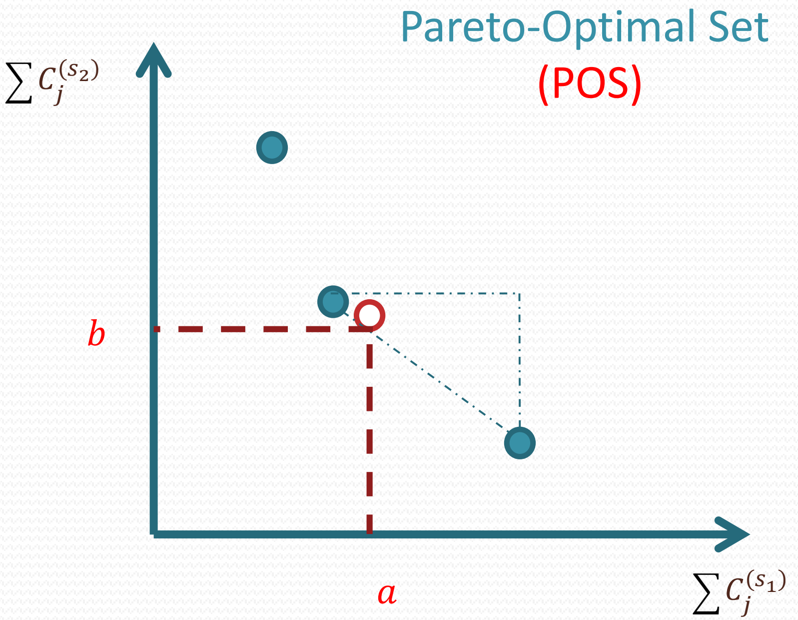
$\leq \sum_{j=1}^{n} C_j^{(s_1)}(\sigma) + p_{max}^{(s_1)}$

$\leq \sum_{j=1}^{n} C_j^{(s_1)}(\sigma) + \sum_{j=1}^{n} C_j^{(s_1)}(\sigma) \leq 2\sum_{j=1}^{n} C_j^{(s_1)}(\sigma)$



Approximation for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

Proving the approximation ratio:



- *Consider an arbitrary PO point (a,b) that is included in POS but not in SS (thus, point (a,b) is not a supported point).*
- $\sum_{j=1}^{n} C_j^{(s_1)}(\sigma) < a < \sum_{j=1}^{n} C_j^{(s_1)}(\sigma')$ *and* $\sum_{j=1}^{n} C_j^{(s_2)}(\sigma) > b > \sum_{j=1}^{n} C_j^{(s_2)}(\sigma')$.

- $\sum_{j=1}^{n} C_j^{(s_1)}(\sigma) < a < \sum_{j=1}^{n} C_j^{(s_1)}(\sigma') < 2\sum_{j=1}^{n} C_j^{(s_1)}(\sigma) < 2a$
- $\sum_{j=1}^{n} C_j^{(s_1)}(\sigma') < 2a$ *and* $\sum_{j=1}^{n} C_j^{(s_2)}(\sigma') < b.$

⇓

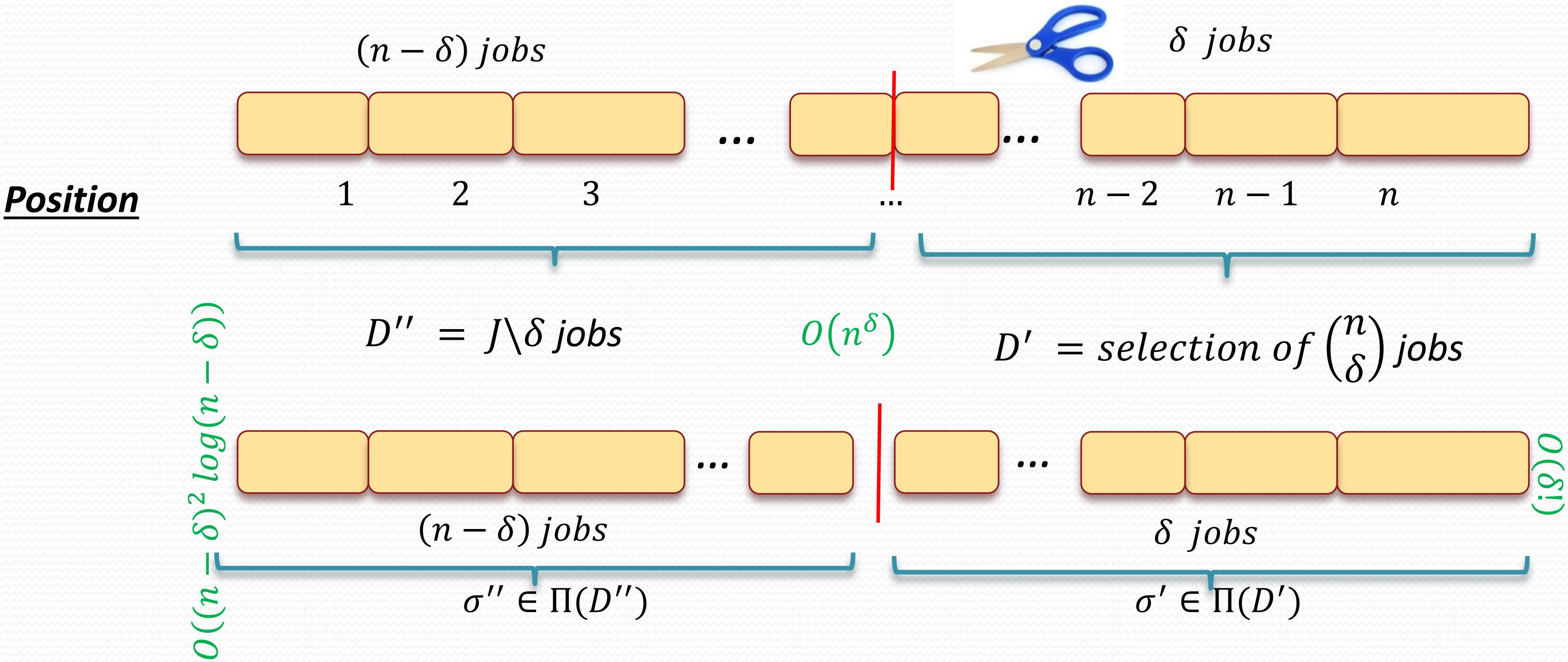
(2,1)-approximation

PTAS for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

We present an approximation algorithm that constructs a set Π^A *of feasible solutions for the* $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$ *problem, that provides a* $(1+\varepsilon, 1)$*-approximation and a* $(1, 1+\varepsilon)$*-approximation to set* Π^E *in a time that is polynomial in n (but not in* $\frac{1}{\varepsilon}$*). Thus we present a Polynomial Time Approximation Scheme (**PTAS**).*

Denote: $\delta = \left\lceil \frac{1}{\varepsilon} - 1 \right\rceil$

PTAS for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$



Approximation algorithm
$(n-\delta)^2$ *possible permutations*
for each $\Pi(D'')$

Full enumeration of D'
$\delta\,!$ *possible permutations*
for each $\Pi(D')$

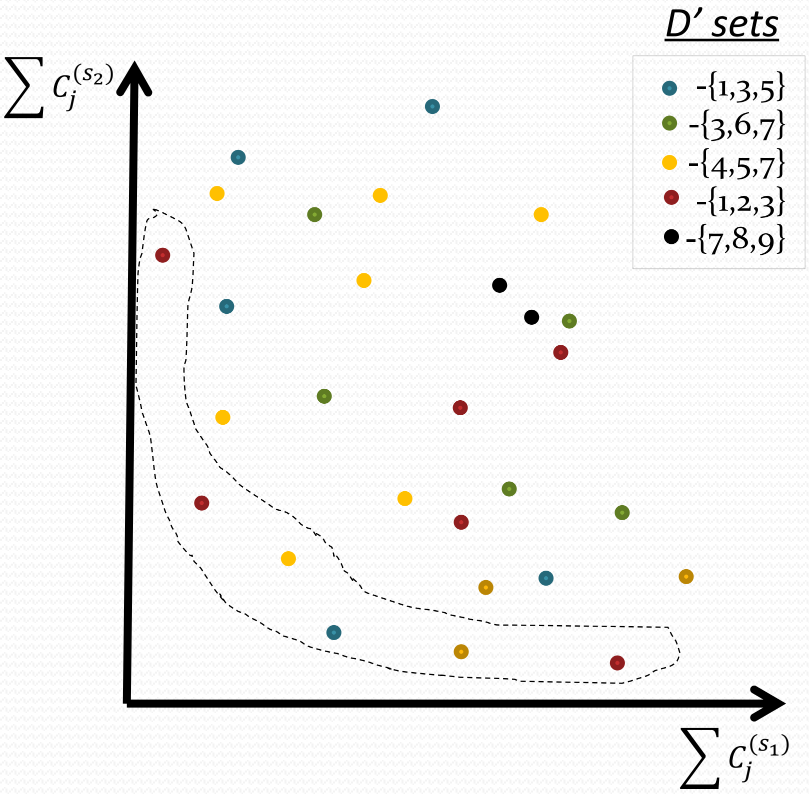
$[\sigma'', \sigma'] = \sigma \in \Pi^A$

n *jobs in each* σ

$$O\left((\delta\,! + n^2 \log n)n^{\delta}\right) = \boldsymbol{O(n^{2+\delta} \log n)}$$

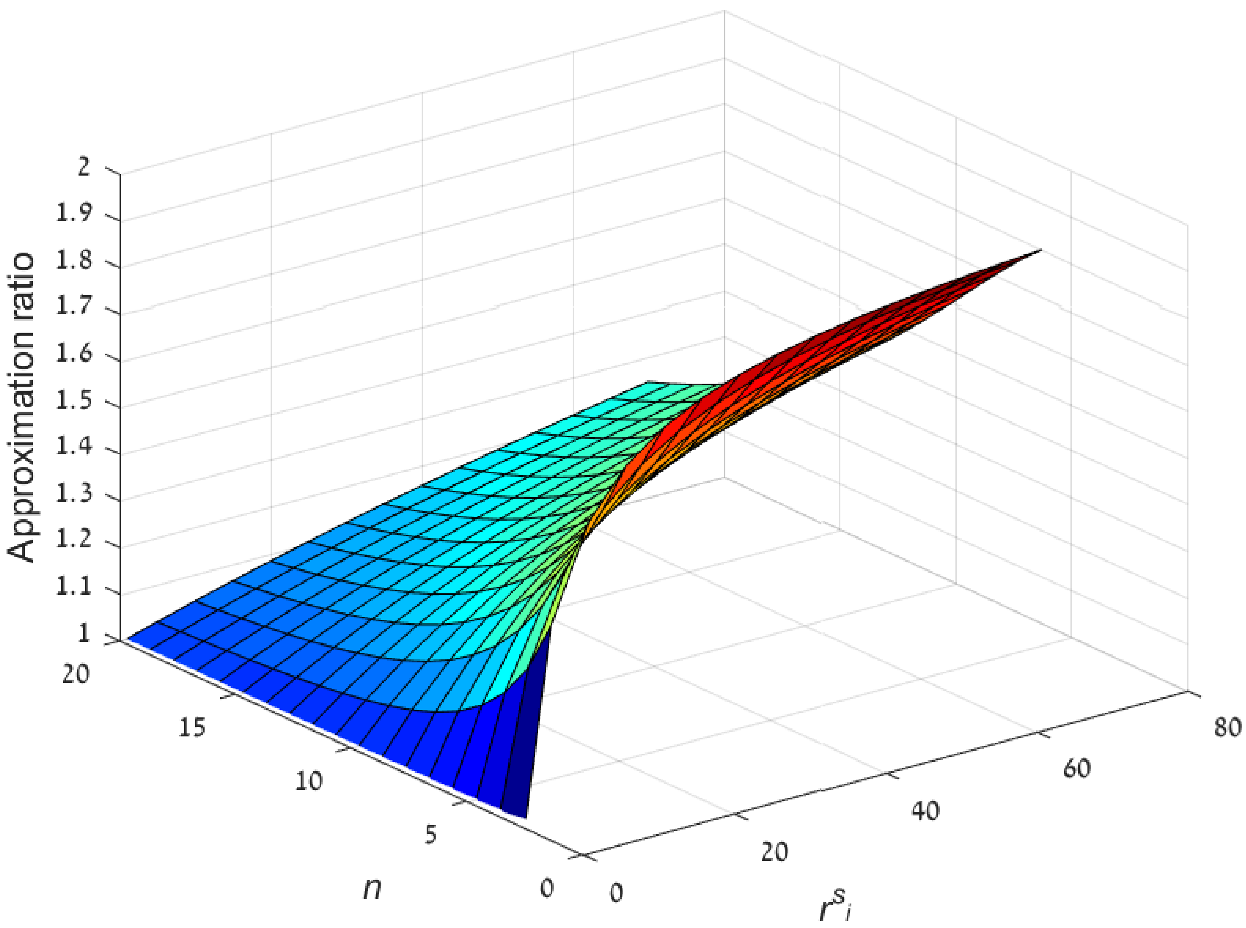
PTAS for $1||\#\left(\sum_{j=1}^{n} C_j^{(s_1)}, \sum_{j=1}^{n} C_j^{(s_2)}\right)$

We may eliminate the dominated points in $O(\delta n^{\delta+2} \log n)$



Set *APOS* corresponds to solution set Π^A

Data-dependent Analysis



Definition: $r^{(s_i)} = \frac{p_{max}^{(s_i)}}{p_{min}^{(s_i)}}$

An example:

- *N=20*
- $1 \leq r^{(s_i)} \leq 64$

Work in Process

- *Exploring multi-scenario scheduling problems with rejection option, for both fixed and arbitrary number of scenarios.:*
  - *The* $1|reg|(C_{max}^{(s_1)}(A) + RC, \dots, C_{max}^{(s_q)}(A) + RC)$
  - *The* $1|reg|\left(\sum_{J_j \in A} w_j C_j^{(s_1)} + RC, \dots, \sum_{J_j \in A} w_j C_j^{(s_q)} + RC\right)$
- *Exploring the multi-scenario problem of maximizing the weighted number of JIT jobs in two-machine flow-shop system*
  - *The* $F2|p_j^{(s_i)}, d_j^{(s_i)} \in S|\sum_{J_j \in E} w_j^{(s_i)}$ *problem for different uncertain parameters (proceeding times, weights, due-dates).*

Thank you

gilenson@post.bgu.ac.il