Propagation of a finite beam through a random medium

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The theoretical investigation of finite-beam propagation through random media has been handled, until now, by various approximate methods. Several investigators\textsuperscript{1,2} have used the Rytov approximation to derive results for the variance and covariance of the log-amplitude fluctuations. Their results are valid only for weak turbulence and do not describe saturation effects. An extension of Rayleigh–Sommerfeld scattering theory was employed by Leader\textsuperscript{3} to obtain results also valid for weak turbulence only. However, Leader’s result for the off-axis behavior of the variance appears more physically reasonable than Ishimaru’s Rytov-based result.\textsuperscript{2} Calculations for moderate to strong levels of turbulence have been carried out using the extended Huygens–Fresnel principle\textsuperscript{4,5} along with several assumptions relating to the statistics of the elementary spherical waves that are involved in the above formalism. Asymptotic techniques have been tried, too. Whitman and Beran\textsuperscript{6} have used the equation for the fourth-order coherence function to derive asymptotic results that predict a normalized variance that grows without limit with range, in very strong turbulence. Others\textsuperscript{7,8} employing alternative asymptotic series as well as the extended Huygens–Fresnel principle, claim that the normalized variance saturates to unity as \(D/\rho_c\)—the ratio of the initial beam diameter to turbulence-induced coherence radius—becomes large.

It is the purpose of this Letter to solve numerically the equation for the fourth-order coherence function without any simplifying assumptions regarding the nature of the wave fluctuations. However, since considerable computer resources are required, we have chosen to follow Brown\textsuperscript{9} and to use a two-dimensional contracted version of the fourth-order equation. We have found that this contracted form of the equation originates from the following two-dimensional physical model: If \(z\) is the propagation direction and \(x\) and \(y\) are transverse coordinates, we assume that the random index of refraction \(n\) is homogeneous in the \(y\) direction, i.e., that \(n(x,y,z) = n(x,z)\). Previous work by Brown\textsuperscript{9} and others\textsuperscript{10} has shown that, for plane-wave propagation, this two-dimensional model predicts both saturation and focusing. It is our hope, therefore, that most of the salient features of finite-beam propagation in three dimensions are present in the two-dimensional model.

The differential equations that govern the second- and fourth-order coherence functions for the two-dimensional model have been derived by us from first principles using the parabolic approximation (for the Helmholtz wave equation)\textsuperscript{11} and the assumption of local independence.\textsuperscript{12} Their form can be obtained from the relevant three-dimensional equations\textsuperscript{11,13,14} simply by omitting the \(y\) dependence. Thus

\[
\frac{\partial |I|}{\partial z} = \left(\frac{i}{2k} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - k^2 (\sigma(0) - \sigma(x_{12})) \right) |I| \right), \quad (1)
\]

where \(|I| = |U(x_{1,2})U^*(x_{2,2})|\), \(U\) is the field variable, and \(|\cdot\|\) and \(*\) denote, respectively, an ensemble average and complex conjugation.

\[
x_{12} = x_1 - x_2,
\]

\[
\sigma(x_{12}) = \int_{-\infty}^{\infty} \sigma(x_{12},s_z)ds_z,
\]

\[
\sigma(x_{12},s_z) = [n'(x,z)n'(x + x_{12},z + s_z)],
\]

\[
n(x,z) = 1 + n'(x,z) \quad \text{(for convenience we set } |n| = 1),
\]

\[
k = \frac{2\pi\nu}{C} \quad \text{ (}\nu\text{ is the radiation frequency);}
\]

\[
\frac{\partial |L|}{\partial z} = \left(\frac{i}{2k} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right) 
+ k^2(\sigma(x_{12}) + \sigma(x_{13}) + \sigma(x_{23}) + \sigma(x_{34}) 
- \sigma(x_{14}) - \sigma(x_{23}) - 2\sigma(0)) \right) |L|, \quad (2)
\]

where \(|L| = |U(x_{1,2})U^*(x_{2,2})U^*(x_{3,2})U(x_{4,2})|\) and \(x_{ij} = x_i - x_j\).

Once \(\sigma(x,z)\) is known, Eq. (1) can be solved by direct integration. We turn our attention, therefore, to the fourth-order coherence function \(|L|\). It is convenient to transform Eq. (2) by\textsuperscript{11}

\[
s = \frac{1}{4}(x_1 - x_2 + x_3 - x_4), \quad p = \frac{1}{4}(x_1 + x_2 - x_3 - x_4), \quad R = \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \quad q = (x_1 - x_2 - x_3 + x_4).
\]

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Also, we introduce the following scaling:
\[ s = l_n \xi, \quad p = l_n \eta, \quad q = l_n \vartheta, \quad R = D \rho, \]
\[ z = k l_n^2 \gamma, \quad \alpha = \beta \gamma l_n^2, \quad \beta = D / l_n, \]
where \( D \) measures the initial width of the beam and \( l_n \) is a characteristic correlation distance for index-of-refraction fluctuations. Finally we get
\[
\frac{\partial L(\xi, \eta, \vartheta, \varrho)}{\partial L} = \left[ \frac{\partial^2}{\partial \zeta^2} + \frac{1}{\beta} \frac{\partial^2}{\partial \vartheta^2} \right] L(\xi, \eta, \vartheta, \varrho) + \gamma f(\xi, \eta, \vartheta) \]
\[ f(\xi, \eta, \vartheta) = \delta(\xi + \vartheta / 2) + \delta(\xi - \vartheta / 2) + \delta(\eta + \vartheta / 2) + \delta(\eta - \vartheta / 2) - \delta(\zeta + \eta) - \delta(\zeta - \eta) - 2, \]
\[ \hat{\sigma}(\xi) = \sigma(l_n, \xi) / \sigma(0). \]
For a plane wave, Eq. (5) reduces to
\[
\frac{\partial L}{\partial \xi} = \frac{\partial L}{\partial \zeta} + \gamma f(\xi, \eta, \vartheta) L. \]
Following Tatarski,\textsuperscript{11} we define \( M(\xi, \eta, \lambda, \vartheta) \) by
\[
M(\xi, \eta, \lambda, \vartheta) = \frac{1}{2\pi} \int \left\{ L\left( \xi, \eta, \vartheta, \lambda \right) \right\} \exp[-i\lambda \rho] d\rho. \]
\[ M(\xi, \eta, \lambda, \vartheta) \]
\[ = \int \frac{\partial^2}{\partial \xi^2} + \gamma f(\xi, \eta, \vartheta, \lambda) \right\} M(\xi, \eta, \lambda, \vartheta). \]
Equation (8) has only three independent variables, \( \xi, \eta, \) and \( \xi, \) and two parameters, \( \vartheta \) and \( \lambda. \) It is also similar in form to the plane-wave equation.
Since we are principally interested in irradiance fluctuations, we shall assume that \( \vartheta = 0. \) To find \( L(\xi, \eta, \rho, 0) \) at an arbitrary range \( \zeta = \zeta_0 \), given \( L(\xi_0, \eta, \rho, 0) = 0 \), we first calculate \( M(\xi_0, \eta, \lambda, \vartheta) \). Then we march in the \( \xi \) direction, using Eq. (8), until we find \( M(\xi, \eta, \lambda, \vartheta) \). This is repeated done for as many values of \( \lambda \) as are required for the inversion of Eq. (7). Finally
\[
L(\xi_0, \eta, \rho, 0) = \int \exp[i\lambda \rho] d\lambda. \]
If \( I(x_1, z) \) and \( I(x_2, z) \) are the intensities at points \( (x_1, z) \) and \( (x_2, z) \), then the normalized covariance of intensity fluctuations at the above two points is
\[
C(x_1, x_2, z) = \frac{\left[ I(x_1, z) I(x_2, z) \right] - \left[ I(x_1, z) I(x_2, z) \right]}{\left[ I(x_1, z) I(x_2, z) \right]}. \]
Using the scaling in Eq. (4), we choose
\[ x_1 = \rho D + \frac{l_n \eta}{2}, \quad x_2 = \rho D - \frac{l_n \eta}{2}, \quad \xi = 0. \]
Then
\[
C(\rho, \eta, \xi) = C(x_1, x_2, k l_n^2 \xi) = \frac{\left[ I(\xi, 0, \eta, 0) \right] - \left[ I(x_1, \xi) I(x_2, \xi) \right]}{\left[ I(x_1, \xi) I(x_2, \xi) \right]}. \]
\[ \left[ I(x_1, \xi) \right] \] can be computed from Eq. (1).

The results presented in this Letter have been obtained for a Gaussian correlation function
\[
\sigma(x) = \sigma(0) \exp[-(x/l_n)^2] \]
and for a moderate level of turbulence given by \( \gamma = 0.5. \) Although the Gaussian model is a poor description of the index-of-refraction fluctuations in the atmosphere, it was believed that the first application of our method should involve a smooth and well-behaved correlation function. Moreover, the Gaussian correlation function is realistic in several other cases.

Details of the numerical algorithm appear in the Appendix.

Figure 1 shows the fluctuation index \( \sigma_1 = \left[ C(0,0,0) \right]^{1/2} \) as a function of range \( \xi \) for two values of \( \beta \) as well as for a plane wave. Also shown are perturbation results derived from a first Born approximation. It is seen that for \( D/l_n = 1, \) \( \sigma_1 \) rises quite slowly but eventually crosses the plane-wave curve. As \( D/l_n \) increases, the range dependence of \( \sigma_1 \) approaches that of a plane wave \( (D/l_n = \infty) \). However, for large enough \( \xi, \) \( \sigma_1 \) for the finite beam again exceeds the plane-wave saturation level.

The normalized covariance of intensity fluctuations at the two points \( x_1 = l_n \eta/2 \) and \( x_2 = -l_n \eta/2 \) appears in Fig. 2 for various ranges and values of \( D/l_n. \) The characteristic transverse length for correlation of intensity fluctuations is seen to be of order \( l_n \) regardless of \( D/l_n \) and range. This behavior is typical of our correlation function, Eq. (13), which is characterized by a single correlation length.

The dependence of \( \sigma_1 \) and beam intensity on \( \rho \) is depicted in Fig. 3. \( \sigma_1 \) increases toward the edge of the beam, in agreement with previously published perturbation-based results.\textsuperscript{17,18}

Even our two-dimensional computations rely heavily

![Fig. 1. \( \sigma_1 \) versus \( \xi \) for various values of \( D/l_n. \) Also shown are the Born approximations for a plane wave and for \( D/l_n = 1. \)](image)

![Fig. 2. The normalized covariance \( C(x_1, x_2, k l_n^2 \xi) \) of intensity fluctuations at two points, \( x_1 = x = (l_n \eta/2) \) and \( x_2 = -x, \) symmetrically located around \( \rho = 0. \)](image)
on computer resources, both in processing time and in core-memory allocations. Therefore, results emerge at a slow rate. Using the above algorithm, we are currently calculating \( \sigma_f \) for various levels of the turbulence and plan to extend the calculations to other correlation functions as well as to initially focused or partially coherent beams.

Appendix: Numerical Algorithm

Equation (8) is converted into an integral equation by using an integration factor:

\[
M(\xi, \eta, \lambda, \phi) = ig(\xi_0, \eta, \lambda, \phi) \times \int_{\xi_0}^{\xi} \left[ g(\xi_0, \xi', \eta, \lambda, \phi) \right]^{-1} \frac{\partial^2 M(\xi', \eta, \lambda, \phi)}{\partial \xi' \partial \eta} d\xi' + M(\xi_0, \eta, \lambda, \phi) g(\xi_0, \xi, \eta, \lambda, \phi),
\]

(A1)

\[
g(\xi_0, \xi, \eta, \lambda, \phi) = \exp \left[ \gamma \int_{\xi_0}^{\xi} \left( \xi', \eta, \frac{\lambda}{\beta} \right) d\xi' \right].
\]

(A2)

To find \( M(\xi + \Delta \xi, \eta, \lambda, \phi) \) from \( M(\xi, \eta, \lambda, \phi) \), we iterate Eq. (A1) until proper convergence is achieved (usually after three to four steps). It is assumed that \( M \) vanishes on the boundaries. Since

\[
M(\xi, \eta, \lambda, \phi) = M(\xi, \eta, \xi', \phi) = M^*(\xi', \eta, \lambda, \phi)
\]

(* denotes complex conjugation), all calculations are confined to a right-angle triangle in the \( \xi, \eta \) plane. Grid spacings are \( \Delta \xi = \Delta \eta = 0.125 - 0.25 \) and \( \Delta \xi = 0.005 - 0.025 \), and boundaries lie at \( \xi_{\text{max}} = \eta_{\text{max}} = 40 \). The numerical scheme was successfully checked for stability, convergence, and consistency.

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References

15. This equation was first derived in Ref. 11 [p. 427, Eq. (23)], with two apparent mistakes: \( \lambda / \beta \) does not appear in \( f \), and there is also a sign mistake.
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\times \int_{\xi_0}^{\xi} [g(\xi', \eta_0, \lambda, \varphi)]^{-1} \frac{\partial^2 M(\xi', \eta_0, \lambda, \varphi)}{\partial \xi' \partial \eta'} d\xi' \\
+ M(\xi_0, \eta_0, \lambda, \varphi) g(\xi_0, \eta, \lambda, \varphi),
$$

(A1)

$$
g(\xi_0, \xi, \eta_0, \eta, \lambda, \varphi) = \exp \left[ \gamma \int_{\xi_0}^{\xi} \frac{\xi, \eta, \lambda, \varphi}{\beta} d\xi' \right].
$$

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To find $M(\xi + \Delta \xi, \eta_0, \lambda, \varphi)$ from $M(\xi, \eta_0, \lambda, \varphi)$, we iterate Eq. (A1) until proper convergence is achieved (usually after three to four steps). It is assumed that $M$ vanishes on the boundaries. Since

$$
M(\xi, \eta, \lambda, \varphi) = M(\xi, \eta, \lambda, \varphi) = M(\xi, -\xi, \eta, \lambda, \varphi)
$$

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