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The plane-wave short-term structure function with finite turbulence scales: an empirical approach

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Abstract. The short-term structure function $D_n(p)$ with arbitrary outer ($L_0$) and inner ($l_0$) turbulence scales is evaluated by two different methods. In the first method, the short-term structure function is assumed to statistically represent all wavefront distortions other than tilt. The second method comprises a novel technique, in which the short-term structure function is evaluated by replacing the outer turbulence scale $L_0$ in the Von-Karman spectrum by an effective short-term scale which also depends on the aperture diameter $D$, $L_0(D, l_0, D) < L_0$, thereby minimizing the contribution of large-size turbulence eddies which only give rise to beam tilts. While both methods give very close results when $p \ll D$, the empirical approach has some computational advantages and a wider regime of validity.

1. Introduction

An optical beam, propagating through the randomly inhomogeneous turbulent atmosphere, experiences both phase and intensity fluctuations which are the result of random scattering and diffraction from the refractive index inhomogeneities in the random medium. Here we consider only the phase fluctuations which give rise to wavefront tilts and distortion. A plane wave focused by a lens will generate a dancing spot where dancing is the result of the random wavefront tilts and the spot size is determined by the smaller-scale phase distortions, superimposed on the tilted wavefronts. While a short-exposure snapshot at the focal plane captures only the dancing spot in one of its everchanging positions, a long-exposure picture averages over all possible tilts, thereby producing a relatively large image which is equivalent to a poorer resolution [1–3].

Long-exposure imaging through the turbulent atmosphere is mathematically analysed via the (long-term) mutual coherence function ($MCF$) at the receiving aperture [2]. For either plane or spherical waves its normalized form is related to the appropriate wave structure function, $D_n$ (lt = long-term), via

$$\text{MCF}(p) = \frac{\langle U(r) U^*(r + p) \rangle}{\langle |U(r)|^2 \rangle} \exp \left[ -\frac{1}{2} D_n(p) \right]$$
$$D_n(p) = \langle |\psi(r) - \psi(r + p)|^2 \rangle$$

where $U(r)$ is the complex amplitude of the electric field, $\psi = \ln(U)$ is the complex phase and $p$ is the separation between two points at the aperture. By analogy, a short-term mutual coherence function and its accompanying short-term structure function, $D_n$, can also be introduced to describe the instantaneous (or short-term) beam spread [1–4].
dancing is the result of wavefront tilts, it is generally agreed that $D_{m}$ could be obtained from its long-term version, $D_{l}$, by excluding the component responsible for the wavefront tilt. It was suggested that this tilt component should be estimated as the best fit of a planar wavefront to the turbulence-induced phase distortions over the receiving aperture \[1\]. The structure function obtained in this way is presented in several papers \[1,3–5\] for the Kolmogorov spectrum ($L_{0} = \infty$ and $l_{0} = 0$) and under range-independent turbulence conditions. The case of a focused beam for a Von-Karman spectrum with either arbitrary $L_{0}$ and $l_{0} = 0$ or arbitrary $l_{0}$ but $L_{0} = \infty$ is discussed in \[6\]. In section 2 of this paper we shall extend the above-mentioned results to include the plane-wave short-term structure function with arbitrary turbulence scales as well.

However, the theoretical treatment of the short-term mutual coherence function and structure function can be also addressed by modifying the turbulence spectrum $\Phi_{\eta}(K)$, which measures the fraction of refractive index fluctuations whose eddy size is of the order of $1/K$. The idea to modify the turbulence spectrum in order to produce the short-term functions is based on physical reasons \[3\]. The turbulence eddies which are greater than the beam diameter $D$ contribute mainly to the deflection (tilt) of the beam without significantly affecting its short-term broadening. It follows that the short-term functions can be obtained by reducing the effect of these eddies in the turbulence spectrum. This idea was implemented in \[7\] where the short-term structure function was expressed as an integral of the Kolmogorov spectrum over $K$, but with a finite lower integration limit $K_{D}$, instead of zero, as in the long-term expression. By a proper choice of $K_{D}$, the resulting short-term structure function does not differ by much from the one obtained using the previously mentioned tilt subtraction technique. The integration limit $K_{0}$ was found to be of the order of the inverse receiver diameter. A similar approach was also used in the theory of intensity scintillations \[8–10\].

Unfortunately, when applied to turbulence spectra with finite $L_{0}$, the truncated spectrum, $\Phi_{\eta}(K < K_{D}) = 0$, can violate the physically obvious monotonous behaviour of the structure function. This disadvantage restricts the applicability of this approach to the Kolmogorov spectrum only. In order to avoid this disadvantage we propose to modify the Von-Karman spectrum, with finite $L_{0}$ and $l_{0}$, by replacing $L_{0}$ with an effective (short-term) outer scale $L_{st}$, having the following two properties: (a) $L_{st} \approx L_{0}$ for large aperture diameters, $D \gg L_{0}$; and (b) $L_{st}$ is of the order of the aperture diameter $D$ when $D \ll L_{0}$. In the last case the much smaller effective outer scale renders the low-wavenumber fluctuations (i.e. large-scale eddies) to be relatively unimportant. While achieving the same goal, our novel approach does not sacrifice the physical consistency of the results.

In section 3 we study this approach and compare the results with those of section 2. The concluding section will enumerate some of the advantages of this approach.

2. Short-term structure function determined by tilt exclusion

It has been shown by Fried \[1\] that the short-term structure function can be expressed as the difference between the long-term structure function, $D_{hl}(\rho)$, and a term which represents the effect of the tilt component. In the case of near field propagation of plane waves the short-term structure function is (the superscript $p$ denotes the planar case) \[1,11\]

$$pD_{st}(\rho) = pD_{h}(\rho) - \frac{32\rho^{2}}{D^{4}} \int_{0}^{D} d\rho' \rho' F(\rho' / D) pD_{h}(\rho')$$ \[(2)\]
The plane-wave short-term structure function

where $D$ is the aperture diameter and

$$F(u) = \frac{2}{\pi} \left[ -2 \cos^{-1} u + 2u(3 - 2u^2)(1 - u^2)^{1/2} \right] \quad u = \rho'/D. \quad (3)$$

In the case of homogeneous and isotropic turbulence conditions along the propagation path, the long-term structure function for a plane wave is expressed by [12]

$$pD_{lt}(\rho) = 8\pi^2 k^2 Z \int_0^\infty dK K \Phi_n(K)[1 - J_0(K\rho)] \quad (4)$$

where $Z$ is the propagation distance, $k$ is the wavenumber and $J_0$ is the zero-order Bessel function. $\Phi_n(K)$ is the spatial power spectrum on the inhomogeneities. For the Von-Karman spectrum [13] it is given by

$$\Phi_n(K) = 0.033C_n^2[K^2 + (2\pi/L_0)^2]^{-11/6} \exp\left[-(KL_0/5.91)^2\right] \quad (5)$$

where $C_n^2$ is the index of refraction structure constant.

Fried [1] has evaluated (2) only for the Kolmogorov inertial subrange ($\Phi_n(K) \propto K^{-11/3}$)

$$pD_{lt}(\rho) = 2\left(\frac{\rho}{\rho_0}\right)^{5/3} \left[ 1 - \left(\frac{\rho}{D}\right)^{1/3} \right] \quad \rho < D \quad (6)$$

where $\rho_0 = (1.46k^2ZC_n^2)^{-3/5}$, and $2(\rho/\rho_0)^{5/3}$ is the long-term structure function for a pure Kolmogorov spectrum. Since Fried's formulation [11] is based on the mathematical expansion of the wavefront phase over the aperture $D$, the validity of (6) becomes questionable near $\rho \approx D$.

When both the outer and inner scales assume finite values, the long-term structure function can be conveniently expressed as the product of the Kolmogorov long-term structure function (4) and a function $P_{lt}$ which includes the turbulence scales [14]

$$pD_{lt}(\rho, L_0, L_0) = 2\left(\frac{\rho}{\rho_0}\right)^{5/3} pP_{lt}\left(\frac{\rho}{L_0}, \frac{L_0}{L_0}\right) \quad pP_{lt} < 1. \quad (7)$$

Substituting this expression into (2) one obtains the short-term structure function with both $L_0$ and $L_0$

$$pD_{sl}(\rho, D, L_0, L_0) = 2\left(\frac{\rho}{\rho_0}\right)^{5/3} pP_{lt}\left(\frac{\rho}{D, L_0, L_0}\right) \quad (8)$$

where

$$pP_{sl} = pP_{lt} - \frac{\left(\rho/D\right)^{1/3}}{G\left(D/L_0, L_0/L_0\right)} \quad (9)$$

$$G\left(D/L_0, L_0/L_0\right) = 32 \int_0^{1} du u^{8/3} F(u) pP_{lt}\left(u D/L_0, L_0/L_0\right) \quad u = \frac{\rho}{D} \quad (10)$$

For Kolmogorov spectrum ($L_0 = \infty, L_0 = 0, pP_{lt} = 1$) one obtains $G = 1$ and (8) reduces to (6).

The particular case of $L_0 = 0$ and arbitrary $L_0$ can be also treated analytically, i.e.

$$pP_{lt}\left(\frac{\rho}{L_0}\right) = 0.54y^{-5/3} \left[ 1 + 2^{1/6} y^{5/6} K_{5/6}(y) / \Gamma\left(\frac{5}{6}\right) \right] \quad y = 2\pi \frac{\rho}{L_0} \quad (11)$$
where $K_{5/6}$ is the modified Bessel function [15] and $\Gamma(5/6) = 1.12879$ is the gamma function. Under normal conditions of interest $p$ is much smaller than $L_0$, so that $K_{5/6}(y)$ can be expanded in a Taylor series [15]. Using the first four terms in the above expansion we obtain

$$pP_l\left(\frac{\rho}{L_0}\right) = 1 - 1.48 \left(\frac{\rho}{L_0}\right)^{1/3} + 1.71 \left(\frac{\rho}{L_0}\right)^2.$$  \tag{12}

Substituting (12) into (9) and (10) one finds that only the third term in (12) contributes to the dependence of $pP_{st}$ on $L_0$

$$pP_{st} = 1 - \left(\frac{\rho}{D}\right)^{1/3} \left[ 1 + 1.23 \left(\frac{D}{L_0}\right)^2 \right].$$  \tag{13}

While $P_{lt}$ directly depends on $L_0$ via the $1.48(\rho/L_0)^{1/3}$ of (12), the effect of $L_0$ on the short-term structure function becomes of importance only when $D$ approaches a substantial fraction of $L_0$.

In order to calculate $G(D/L_0, l_0/L_0)$ of (10) for arbitrary $D/L_0$ and $l_0/L_0$, one first needs to evaluate $pP_{lt}(\rho/L_0, l_0/L_0)$. For efficient calculations, it is useful to employ the following (empirical) algebraic approximation [14]:

$$pP_l\left(\frac{\rho}{L_0}, \frac{l_0}{L_0}\right) = \left\{ (1 + Q^6)^{1/6} + \left[ pP_{lt}\left(\frac{\rho}{L_0}, \frac{l_0}{L_0} = 0\right) \right]^{-1} \right\}^{-1}. \tag{14}$$

$pP_{lt}(\rho/L_0, l_0/L_0 = 0)$ is analytically given by (11), and $Q$ contains the effect of the inner scale $l_0$

$$Q = 0.887 \left(\frac{l_0}{\rho}\right)^{1/3} \left[ 1 + 1.32 \left(\frac{l_0}{L_0}\right)^{1/3} + 2.1 \left(\frac{l_0}{L_0}\right)^{2/3} + 18 \left(\frac{l_0}{L_0}\right)^2 + 6.94 \left(\frac{l_0}{L_0}\right)^{11/3} \right]. \tag{15}$$

The fitting function practically coincides with the exact $pP_{lt}$ function except for an interval $0.1l_0 \lesssim \rho \lesssim l_0$ where the discrepancy is of the order of a few percents only (figure 1).

![Figure 1. The long-term $pP_{lt}$ functions for $l_0 = 0.005L_0$: --- exact, by numerical integration of equation (4); - - - approximate (equations (14), (15)).](image-url)
The plane-wave short-term structure function

Figure 2. The function \( G(D/L_0, l_0/L_0) \) (equation (10)).

The results of this calculation for \( l_0 = 0, l_0 = 0.001l_0 \) and \( l_0 = 0.005l_0 \) are presented in figure 2. Only when \( l_0 = 0 \) does the outer-scale \( L_0 \) monotonically reduce the difference between the long- and short-term structure functions. For practical apertures of 10–20 cm and a near ground value of \( L_0 \approx 2 \) m, \( G \approx 0.4 \). Naturally, non-zero values of \( l_0 \) are of importance only for small \( D \) of the order or smaller than \( l_0 \) itself.

3. Short-term structure function defined by a modified turbulence spectrum

As described in section 1 we propose to express \( pD_{st} \) using the same expression as for \( pD_{lt} \), but with a modified Von-Karman spectrum. Rather than truncating the spectral summation over \( K \), starting it from some \( K_D > 0 \) \[8\], we suggest the reduction of the relative weight of large eddies (small \( K \)) by replacing \( L_0 \) in (5) by an effective outer scale \( L_{st}(L_0, D) < L_0 \) which is a function of both the true outer scale and the aperture diameter \( D \). Obviously, \( L_{st}(L_0, D) \to L_0 \) when \( D \to \infty \) and \( L_{st}(L_0, D) \to L_{st}(D) \) when \( L_0 \to \infty \). Among the many possible dependences which satisfy the expected asymptotic behaviour, this paper assumes that

\[
\frac{1}{L_{st}^2} = \frac{1}{L_0^2} + \frac{1}{(\gamma D)^2}
\]

with \( \gamma \) serving as a fitting parameter. Note that (16) predicts a similar dependence of \( L_{st} \) on either \( L_0 \) or \( D \).

In the following we study a few cases in some detail.

(a) \( l_0 = 0, L_0 = \infty \). For this pure Kolmogorov spectrum

\[
L_{st} = \gamma D.
\]

Substituting \( L_{st} \) for \( L_0 \) in the long-term expression of (12), neglecting the small last term, one finds:

\[
pD_{st}(\rho) = 2 \left( \frac{\rho}{\rho_0} \right)^{5/3} \left[ 1 - 1.48\gamma^{-1/3} \left( \frac{\rho}{D} \right)^{1/3} \right]
\]

which coincides with the short-term expression (6) if \( \gamma = 3.24 \). This value of \( \gamma \) will be used in all subsequent calculations.
(b) \( I_0 = 0, \rho, D \ll L_0 \). Expanding \( L_{st} \) of (16) in a Taylor series in \( D/L_0 \) and substituting the result in (12) we get

\[
p_{st} = 1 - \left( \frac{\rho}{D} \right)^{1/3} \left[ 1 + 1.75 \left( \frac{D}{L_0} \right)^2 \right]. \tag{19}
\]

Note that (19) and (13) share the same functional dependence on \( \rho/D \) and \( D/L_0 \) although the coefficients of \( (D/L_0)^2 \) differ somewhat from each other.

(c) \( I_0 = 0 \) but arbitrary \( L_0, \rho \) and \( D \). Here \( p_{st} \) can be calculated by using (11) with \( L_0 \) replaced by \( L_{st} \) (16). This \( p_{st} \) is shown in figure 3(a) for \( L_0 = \infty \) and in 3(b) for \( L_0 = 5D \) and is compared with the results of Fried’s technique, (9–11). Note that when \( L_0 = \infty \), \( p_{st} = 1 \). For small \( \rho \), roughly \( \rho < 0.4D \), both methods give practically the same result. Deviations occur near \( \rho \approx D \), where the modified spectrum technique provides more physically reasonable results compared with Fried’s \( p_{st} \) which goes to zero and even to negative values. Even in the case of finite \( I_0 \) one can calculate \( p_{st} \) analytically by the use of the approximate formula (14). The results for finite \( I_0 \) and \( L_0 \), e.g. \( L_0 = 5D, I_0 = 0.005L_0 = 0.025D \) are presented in figure 3(b). As expected, the inner scale effect is strong when \( \rho \) is smaller than \( I_0 \). The long-term \( P \)-function (not shown in figure 3(b)) is only marginally affected by \( I_0 \), at least for \( \rho \) not much smaller than \( I_0 \).

It is also important to evaluate the short-term \( \mathcal{M} \mathcal{F}_{st}(\rho) \), not only because it measures the correlation properties of the optical field, but mainly due to the fact that the short-term optical transfer function of the turbulent atmosphere is given by [13]

\[ \mathcal{H}_{st}(\nu) = \mathcal{M} \mathcal{F}_{st}(\lambda f \nu), \]

where \( \lambda \) is the optical wavelength, \( f \) is the focal length of the system, and \( \nu \) is the spatial frequency in units of cycles per unit length. Following our method,
we calculate $\text{mcf}_n(\rho)$ by evaluating the long-term structure function $D_n(\rho)$ in (1), (14), (15) with $L_0$ replaced by $L_4$. The dependence of the mcf on $\rho/\rho_0$ is presented in figure 4 for the Kolmogorov spectrum ($l_0 = 0, L_0 = \infty$) and an aperture diameter of $D = 4\rho_0$, under both short-term and long-term exposures. The short-term mcf obtained by the modified spectrum technique has the following appealing properties: (a) it is a decreasing function of $\rho$; (b) it is always larger than the long-term mcf; (c) it coincides with Fried's expression for $\rho/D << 1$. On the other hand, since Fried's $\rho D_0(\rho)$ vanishes for $\rho = D$, the corresponding mcf does not appear to be physically meaningful for $\rho/D > 0.5$ (for the Kolmogorov spectrum).

4. Conclusions

A novel technique has been presented in this paper for the evaluation of the short-term structure function. Rather than being based on Fried's spectral analysis of the wavefront, it reduces the role of large eddies by replacing the outer scale with a new effective one which depends on both the true outer scale and the aperture diameter. For $\rho < D$, good agreement is obtained with Fried's results, which were extended to include expressions for finite $l_0$ and $L_0$. However, the results of our modified spectrum technique appear to be more physically meaningful as $\rho$ approaches $D$. Other advantages of this technique include the ability to describe $D_n$ for arbitrary $\rho$, $D$, $l_0$, and $L_0$, and also in the simplicity of the calculation, which can be performed analytically. The short-term structure function for spherical waves, based on this modified spectrum technique, will be presented in a forthcoming paper.

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