When is speckle noise multiplicative?

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In coherent illumination, objects with roughness of the order of a wavelength cause speckle to appear in their images as formed by imaging systems which cannot resolve the microscale of the objects' roughness. Thus, when a laser illuminates a composite object composed of a diffuser whose complex amplitude transmittance is \( d(x',y') \) in contact with a transparency \( t(x',y') \) (see Fig. 1) the spatial intensity distribution in the image plane will be very noisy (i.e., speckled), provided that the amplitude point spread function of the optical system is broad by comparison with the microscopic surface variations of the diffuser. Recent publications\(^2\)\(^-\)\(^7\) as well as older ones\(^5\)\(^-\)\(^7\) have assumed that speckle noise is multiplicative, i.e.,

\[
I_{\text{ts}}(x,y) = \alpha \cdot I_{\text{inc}}(x,y) \cdot I_x(x,y),
\]  \hspace{1cm} (1)

where \( I_{\text{ts}}(x,y) \) is the (random) spatial intensity distribution in the speckled image of the transparency \( t(x',y') \), \( I_x(x,y) \) is the intensity distribution in the image of the diffuser alone \( t(x',y') = 1 \), and \( I_{\text{inc}}(x,y) \) is the incoherent image of the transparency \( t(x',y') \). The proportionality factor \( \alpha \) depends on the system parameters. Equation (1) is mainly based on the work of Lowenthal and Arsenault\(^8\) who showed that when \( d(x',y') \) is a stationary, \( \delta \)-correlated random process with independent real and imaginary parts having zero means and the same variances, the mean and standard deviation of \( I_{\text{ts}}(x,y) \) are equal to \( I_{\text{inc}}(x,y) \). That is,

\[
\langle I_{\text{ts}}(x,y) \rangle = I_{\text{inc}}(x,y); \quad \langle I_x^2(x,y) \rangle = 2I_{\text{inc}}^2(x,y),
\]  \hspace{1cm} (2)

where \( \langle \cdot \rangle \) denote an ensemble average over different realizations of the diffuser and/or the illumination. Using Eqs. (1)–(2), we also find that \( \alpha \) in Eq. (1) is given by \( \alpha = \langle I_x \rangle / \sqrt{\langle I_x^2 \rangle} \)\(^-\)\(^1\). While Lim and Nawab\(^2\) specifically restricted their multiplicative model to the case when the degraded image has been sampled coarsely enough so that the degradation at any point can be assumed to be independent of that at all other points, most other investigators have used Eq. (1) without further limitations.

It is the purpose of this Letter to point out that Eq. (1) is only an approximation which is certainly not valid when the transparency \( t(x',y') \) has spatial details which cannot be resolved by the coherent system. This observation is of practical importance since most objects contain fine details well beyond the resolution capabilities of the systems that are used to image them.

Due to the finite resolving power of the imaging system, the complex wave amplitude \( U \) at the image point \( P \) (see Fig. 1)

\[
\text{Fig. 1. Optical system } d(x',y') \text{ and } t(x',y') \text{ are, respectively, the complex transmittances of the diffuser and object transparency.}
\]

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is the result of a coherent addition of contributions from many independent areas within a finite patch $R_{\text{cell}}$ which is of the order of a resolution cell. The (random) intensity at $P = (x_p, y_p)$ is given by

$$I_0(x_p, y_p) = \left| \mathcal{U}(x_p, y_p) \right|^2 = \iint_{R_{\text{cell}}} dx' dy' \times h(x_p - x', y_p - y') \exp[i\varphi(x', y')].$$

(3)

where $h(x, y)$ is the amplitude impulse response of the system (assuming unity magnification), and $\varphi(x', y')$ is a random phase delay which is introduced by the diffuser at the object point $(x', y')$: $d(x', y') = \exp[i\varphi(x', y')]$. Obviously, when $t(x', y')$ does not change appreciably within $R_{\text{cell}}$, Eq. (3) reduces to

$$I_0(x_p, y_p) = \left| t(x_p, y_p) \right|^2 \iint_{R_{\text{cell}}} dx' dy' \times h(x_p - x', y_p - y') \exp[i\varphi(x', y')].$$

(4)

Since in this limiting case $I_{\text{inc}}(x_p, y_p) = \left| t(x_p, y_p) \right|^2$ and the second factor on the right-hand side of Eq. (4) is $I_0(x_p, y_p)$, Eq. (4) has the same form as Eq. (1). This explains why Lee and his associates in image processing were successful in verifying Eq. (1) for the flat areas in his synthetic aperture radar images. However, when $t(x', y')$ has spatial details smaller than or comparable with $R_{\text{cell}}$, Eq. (1) is no longer valid. As an example, consider the case of a sharp-edge opaque object and denote by $I_{\text{s}}$ the speckled (random) intensity at the geometrical image of the edge $G$ (see Fig. 2) and by $I_{\text{ae}}$ the intensity of the speckle at the same point in the absence of the sharp edge. Based on a random walk model, $I_{\text{s}}$ is the intensity of the sum of two independent random walks $Z^+$ and $Z^-$ originated, respectively, from $R_{\text{cell}}^+$ and $R_{\text{cell}}^-$ in Fig. 2. Obviously, $I_{\text{ae}}$ is the intensity of $Z^+$. Assuming exponential distributions for the intensities, it is readily shown that the joint probability density function for $I_{\text{s}}$ and $I_{\text{ae}}$ is given by

$$p(I_{\text{s}}, I_{\text{ae}}) = \frac{1}{(I_{\text{ae}} - (I_{\text{ae}}))} \times \exp \left[ -\frac{(I_{\text{ae}}) + (I_{\text{ae}}) I_{\text{ae}}}{(I_{\text{ae}} - (I_{\text{ae}}))} \right] I_{\text{ae}} \left( 2 \sqrt{I_{\text{ae}}} \right),$$

(5)

where $(I_{\text{ae}})$ and $(I_{\text{ae}})$ are the ensemble averages of $I_{\text{ae}}$ and $I_{\text{ae}}$, and $I_0$ is a modified Bessel function of the first kind, zero-order.

Therefore, the quotient $Q = \sqrt{I_{\text{ae}}/I_{\text{inc}}}$, assumed constant in Eq. (1), is instead a continuously distributed quantity with a probability density given by

$$p(Q) = 2Q(B - 1)Q^2 + (B - 1) \frac{[Q^2 + 2(B - 2)Q^2 + B^2]^{1/2}}{[Q^2 + 2(B - 2)Q^2 + B^2]^{3/2}}.$$

(6)

where $B = (I_{\text{ae}})/I_{\text{inc}}$. (In the simplest case, when $R_{\text{cell}}$ is equally divided by the edge, $B = 2$.) Since for large values of $Q, p(Q) \approx Q^{-3}$, $Q$ has an infinite variance which means that the zeroes of $I_{\text{ae}}$ occur independently of those of $I_{\text{inc}}$ in contradiction to Eq. (1).

To further validate our assertion, we have carried out 1-D numerical simulations of the image forming system of Fig. 1 for slanted objects with different slopes. The speckled image was generated using convolution techniques rather than Fourier domain methods. Results for $L_{\text{inc}}/L_{\text{inc}}$ near the point $G$ in Fig. 2 show that for two different realizations of the diffuser, the steeper the edge, the larger the difference between the two realizations. We have also confirmed that for $B = 2$, $(I_{\text{ae}}/I_{\text{inc}})^{1/2}$ assumes the value 1.91 which is the first moment of the distribution $p(Q)$, Eq. (6).

A statistical estimate for the error introduced by the multiplicative model can be obtained from the mean square error $M(x, y)$:

$$M(x, y) = \left( |I_{\text{inc}}(x, y) - \alpha \cdot I_{\text{inc}}(x, y) - I_{\text{ae}}(x, y)| \right)^2.$$

(7)

To evaluate $M$ we follow the assumptions and results of Lowenthal and Arsenault6 so that Eq. (7) reduces to

$$M(x, y) = 2L_{\text{inc}}(x, y) \left[ 1 - \frac{(I_{\text{inc}}(x, y) - I_{\text{inc}}(x, y))}{I_{\text{inc}}(x, y) - (I_{\text{inc}}(x, y))} \right].$$

(8)

Using the above assumptions together with the additional fact that the complex wave amplitudes $U_{\text{inc}}$ and $U_{\text{ae}}$ at the image point $(x, y)$ originate from the same diffuser, we may conclude that $U_{\text{inc}}(x, y)$ and $U_{\text{ae}}(x, y)$ are two complex mutually circular Gaussian random variables, and therefore

$$\langle I_{\text{inc}}(x, y) - I_{\text{ae}}(x, y) \rangle = \langle U_{\text{inc}}(x, y) U_{\text{inc}}^*(x, y) \rangle \langle U_{\text{ae}}(x, y) U_{\text{ae}}^*(x, y) \rangle = I_{\text{inc}}(x, y) - I_{\text{inc}}(x, y) + \langle U_{\text{inc}}(x, y) U_{\text{inc}}^*(x, y) \rangle.$$

(9)

The star denotes conjugate substitution. Substituting Eq. (9) into Eq. (8), we find

$$M(x, y) = 2L_{\text{inc}}(x, y) \left[ 1 - \frac{(I_{\text{inc}}(x, y) - I_{\text{inc}}(x, y))}{I_{\text{inc}}(x, y) - (I_{\text{inc}}(x, y))} \right].$$

(10)

Recalling that $\langle d(x_1, y_1) d(x_2, y_2) \rangle = \delta(x_1 - x_2) \delta(y_1 - y_2)$ and that $\langle d(x, y) \rangle = 1$, Eq. (10) may be rewritten as

$$\text{Fig. 3. Spatial variation of } M(x, y) \text{ for a sharp edge object. The incoherent image of the sharp edge is also included. The abscissa is scaled by the transverse dimension of } R_{\text{cell}} \text{ and the wavelength, } \lambda, \text{ is the focal length of the imaging system, and } a \text{ is the size of its square aperture.}$$

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\[
M(x, y) = 1 - \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' |t(x', y')|^2 h(x - x', y - y')^2}{2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' [t(x', y')]^2 [h(x - x', y - y')]^2}
\] (11)

Due to Schwarz's inequality, \(0 \leq M(x, y) \leq 1\). Moreover, using the condition for equality in Schwarz's inequality, the multiplicative model holds, i.e., \(M = 0\), if and only if \(t(x', y')\) is constant over the resolution cell of the system (where \(h(x - x', y - y') \neq 0\)).

Figures 3 and 4 describe the spatial distributions of \(M(x, y)\) and \(I_0(x, y)\) for sharp-edge as well as for slanted (see insert in Fig. 4) objects, as obtained from an imaging system with unit magnification and a square aperture. It is readily seen that far from the edge, \(M\) approaches zero. Also, as the slope of the object decreases, so does \(M\).

We have thus shown that the multiplicative model fails when the object contains fine details which cannot be resolved by the imaging system.

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References