Numerical solutions for the fourth moment of a plane wave propagating in a random medium

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Numerical solutions of the plane-wave fourth-moment differential equation are obtained for a two-dimensional homogeneous and isotropic random medium that is characterized by a Gaussian correlation function. Results show that the range dependence of the variance of the intensity fluctuations exhibits both saturation and focusing. In addition to the correlations of the intensity fluctuations, the full spatial dependence of the fourth moment of the propagating field is also described and compared with the the Born and Rytov approximations (weak scattering) and with recent asymptotic results (strong scattering).

1. INTRODUCTION

Plane-wave propagation in homogeneous and isotropic random media has been extensively investigated by using the Rytov method.1 Although this perturbation technique leads to relatively simple closed-form analytical expressions for such interesting quantities as the variance of the intensity fluctuations and their spatial correlations, the results could not predict the experimentally observed saturation features of the range dependence of the variance.2 Strong fluctuation theory is currently based on moment equations.3 The second-moment equation is easily solved for both plane and spherical waves,5,4 and the results are equivalent to those obtained from the extended Huygens-Fresnel principle.5 However, the equations for the higher moments have not been solved, to date, by exact analytical methods. Numerical solutions exist. Dagkesamanskaya and Shishov5 solved the fourth-moment equation for a three-dimensional Gaussian correlation function. The results exhibit a monotonic increase of the scintillation index \( \sigma_r^4 \) [see Eq. (3)] with range up to a certain maximum that depends on the strength of the index-of-refraction fluctuations. Then \( \sigma_r^4 \) decreases and approaches a saturation value of unity. Brown7 obtained numerical solutions for a two-dimensional random medium with a power-law spectrum for the refractive-index fluctuations. His results are qualitatively similar to those of Dagkesamanskaya and Shishov. Liu et al.8 investigated the free propagation of the fourth moment after passing through a two-dimensional thick phase screen with a Gaussian correlation function. The case of a power-law spectrum and three-dimensional inhomogeneities was successfully treated by Tatarskii et al.9 and Elepov and Mikhailov10 using the Monte Carlo method. Again, the results exhibit saturation and slight focusing (i.e., the peaked behavior of \( \sigma_r^4 \) versus range). Unpublished results by Uscinski and Miller11 include numerical investigations of the dependence of the height and location of the abovementioned focus on the strength of scattering for a two-dimensional geometry with Gaussian and Lorentzian correlation functions. Several investigators have used asymptotic techniques12-14 and physical models15 to describe the correlation properties of saturated intensity fluctuations in random media for the cases of initially plane and spherical waves. Recently, Gurvich et al. first measured16,17 and then applied the asymptotic analysis of Zavorotnyi14 to the study of the full spatial behavior (see below) of the fourth moment for an initially plane wave18.

This work presents numerical solutions for an initially plane wave that propagates in a two-dimensional homogeneous and isotropic random medium with a Gaussian correlation function. For the first time to our knowledge, the full spatial characteristics of the fourth moment are presented. Section 2 deals with the physical significance of the fourth moment for different choices of its transverse coordinates. The differential equation that governs the propagation of the fourth moment is described in Section 3 along with approximate analytical solutions for weak scattering (Section 3.B) and for strong scattering (Section 3.C). The numerical results are presented and discussed in Section 4.

2. PHYSICAL SIGNIFICANCE OF THE FOURTH MOMENT

The fourth moment of a propagating quasi-monochromatic field \( U(x, z) \) with frequency \( \nu \) is defined by3

\[
\Gamma_4(x_1, x_2, x_3, x_4, z) = |U(x_1, z)U^*(x_2, z)U^*(x_3, z)U(x_4, z)|, \tag{1}
\]

where \( x_i \) stands for the transverse coordinates of the \( i \)-th point: \( (x_i, y_i) \) in the three-dimensional case and \( x_i \) in the two-dimensional case (where we assume translational invariance in the \( y \) direction [see Eq. (1.4)]). \(| | \) and \(* \) denote, respectively, an ensemble average and complex conjugation.

\( \Gamma_4 \) is related to several measurable characteristics of the propagating field:

(1) When \( x_1 = x_2 = x_3 = x_4, \) the function

\[
\Gamma_4(x_1, x_1, x_3, x_4, z) = |I(x_1, z)I(x_3, z)| \tag{2}
\]

is the coherence of intensity fluctuations and plays a significant role in space diversity systems. The quantity \( |I(x, z)| \) gives the intensity fluctuations in the re-
ceived signal. In practice, $|I|^2$ is scaled by the average intensity to give the well-known scintillation index $\sigma_I^2$:

$$\sigma_I^2 = \frac{|I|^2 - |\langle I \rangle|^2}{|\langle I \rangle|^2}.$$  (3)

Information about $\sigma_I^2$ is of importance in studying noise in communication systems as well as in interplanetary and interstellar plasma research.\footnote{2}

(2) Although $|f(x_n, z)|^2 f(x_b, z)$ is derived from $\Gamma_4$ with $x_1 = x_2 = x_3$ and $x_3 = x_4 = x_b$, its propagation for $z_1 > z$ cannot be determined from its behavior at $z$, since $|f(x_n, z)|^2 f(x_b, z)$ does not contain phase information. Rather, the full spatial dependence of $\Gamma_4(x_1, x_2, x_3, x_4, z)$ with $x_1 \neq x_2 \neq x_3 \neq x_4$ must be known before $\Gamma_4(x_1, x_2, x_3, x_4, z)$ and $|f(x_1, z)|^2 f(x_b, z_1)$ can be calculated. Additional symmetries can be invoked to relax this requirement. For example, when $\Gamma_4(x_1, x_2, x_3, x_4, z)$ is invariant under transversal translations (as in the case of an initially plane wave), knowing the spatial dependence of $\Gamma_4(z_1)$ subject to

$$x_1 = x_2 = x_3 = x_4 = \text{constant vector} = q$$  (4)

is sufficient to determine completely the spatial dependence of $\Gamma_4(z_1)$ with the same value of $q$.\footnote{1} Of particular importance is the case $q = 0$, since intensity fluctuations and their correlations are derived from $\Gamma_4(z_1)|_{q=0}$ [see Eqs. (2)-(4)]. This observation has lead most investigators to study the fourth moment under the special condition $q = 0$, which is equivalent to the requirement that the four points $x_1$ lie on the corners of a parallelogram.\footnote{7} Moreover, all the measurements and most of the theoretical analyses of $\Gamma_4$ were restricted to intensity correlations [see characteristic (1) above].

Only recently, Gurvich et al.\footnote{16-18} measured and theoretically analyzed the plane wave $\Gamma_4$ under the more-general condition of $x_1 \neq x_2 \neq x_3 \neq x_4$ with $q = 0$. Experimentally, they determined the amplitude-phase fluctuations in the propagating field as given by Re $\Gamma_4 = |\mathcal{A}|^2 \cos(\phi_1 - \phi_2 - \phi_3 + \phi_4)$ and Im $\Gamma_4 = |\mathcal{A}|^2 \sin(\phi_1 - \phi_2 - \phi_3 + \phi_4)$, where $\phi_1$ and $\phi_2$ are, respectively, the amplitude and phase of $U(x_n)$. Their studies included both weak\footnote{12} and strong fluctuations.\footnote{17} In the former case, satisfactory agreement was obtained with the theoretical predictions of the method of smooth perturbations (MSP).\footnote{1} However, in the strong-fluctuations regime, the measured values for the argument of $\Gamma_4$ were smaller than the corresponding arguments calculated in the MSP approximation, and asymptotic results, based on Feynmann path integrals solutions of the plane-wave fourth-order equation, had to be invoked to fit the experimental data. Their results for Re $\Gamma_4$ and Im $\Gamma_4$ reveal several interesting spatial characteristics (to be discussed later) and justify further study of these quantities.

The presentation of the full spatial dependence of $\Gamma_4$, as obtained from a numerical solution of the fourth-moment differential equation, is considered to be the main contribution of this work.

(3) $q \neq 0$. This is the most general form of $\Gamma_4$. By using an appropriate choice of the transverse $x_n$ coordinates, it is shown elsewhere\footnote{20} that $\Gamma_4$ with $q \neq 0$ contains information on the variance of $U(x_1, z)U^*(x_2, z)$, which is of considerable importance in speckle interferometry\footnote{21} as well as in other short-exposure imaging applications. The analytical properties of $\Gamma_4|_{q=0}$, as well as numerical solutions of the fourth-moment equation for $q \neq 0$, are also discussed in Ref. 20.

### 3. MATHEMATICAL FORMULATION

#### A. Governing Differential Equation

The equation for the fourth moment is well established,\footnote{3} and we write it here in its two-dimensional [see Eq. (14) below] plane-wave form (indicated by the superscript $p$):

$$\frac{\partial \Gamma_4(\xi, \eta, q, \psi)}{\partial \xi} = \gamma f(\xi, \eta) \rho \Gamma_4(\xi, \eta, q, \psi) + i \frac{\delta^2 \rho \Gamma_4(\xi, \eta, q, \psi)}{\delta \xi \delta \eta}.$$  (5)

For an initially plane wave propagating in a two-dimensional random medium with a single scale of inhomogeneities, it is customary\footnote{6-12} to write the fourth moment, Eq. (1),

$$\rho \Gamma_4(\xi, \eta, q, \psi) = |U(x_1, z)|^2 U^*(x_2, z)U^*(x_3, z)U(x_4, z)$$  (6)

in terms of the dimensionless propagation distance

$$\xi = z/kl_n^2$$  (7)

and transverse variables\footnote{1}

$$\xi = (x_1 - x_2 - x_3 + x_4)/2l_n,$$  (8)

$$\eta = (x_1 + x_2 - x_3 - x_4)/2l_n.$$  (9)

It is also assumed that the four points obey the relation

$$\psi = (x_1 - x_2 + x_3 + x_4)/2l_n = 0.$$  (10)

$k$ is the wave number ($=2\pi c/\lambda$) and $l_n$ is the characteristic correlation length for the refractive-index fluctuations. As before, $\Gamma$ and $\ast$ denote, respectively, an ensemble average and complex conjugation. The scattering function $f$ is defined in terms of the index of refraction correlation function $\theta(x, z)$:

$$f(\xi, \eta) = |\overline{\mathcal{S}}(\xi + \eta) - \overline{\mathcal{S}}(\xi - \eta)|$$

$$\overline{\mathcal{S}}(\xi) = \int_{-\infty}^{\infty} \sigma(l_n, \xi, s_2)ds_2,  \tag{12}$$

$$\sigma(l_n, \xi, s_2) = |n'(x, z)n'(x + l_n, \xi, s_2)$$

$$n(x, z) = n(x, z) = 1 + n'(x, z)$$

(for convenience we set $|n| = 1$), where $n(x, z)$ is the random refractive index of the medium (assumed to be two-dimensional, i.e., invariant to translations in the $y$ direction) and the parameter $\gamma = \overline{\mathcal{S}}(0)l_n^{-1}$ measures the relative strength of scattering and diffraction.\footnote{21}

The correlation function $\theta(x, z)$ can assume various functional forms depending on the nature of the random medium. In this paper attention is focused on a Gaussian correlation function\footnote{16} with the following two-dimensional form\footnote{11,22}:

$$\theta(x, z) = \theta_0 \exp[-(x^2 + s_z^2)/l_n^2].$$  (15)

which yields

$$\overline{\mathcal{S}}(\xi) = \overline{\theta}(0) \exp[-\xi^2].  \tag{16}$$
Finally, we follow Liu et al. and convert Eq. (5) to
\[ \rho \Gamma_4(\xi, \eta, \xi', \eta') = \rho \Gamma_4(\xi, \eta, \xi', \eta') \exp[\gamma f(\xi, \eta)(\xi - \xi') + i \frac{\partial}{\partial \xi} \rho \Gamma_4(\xi, \eta, \xi', \eta') \frac{\partial}{\partial \eta} \rho \Gamma_4(\xi, \eta, \xi', \eta')] d\xi. \]

This integral equation is the starting point of the numerical algorithm of Section 4.

B. Perturbation Solution [Weak Scattering]
A plane wave that enters the medium at \( \xi = 0 \) will be gradually perturbed by the medium. As long as the disturbance imposed by the medium randomness is small, it is expected that a perturbation solution will provide adequate results. Perturbation results are important not only because they are usually given by closed-form analytical expressions but also because they can be used to check the consistency of the numerical algorithm in the region in which the results of both methods overlap.

In the Born perturbation formulation, \( \rho \Gamma_4(\xi, \eta, \xi', \eta') = 0 \) is approximated by the first term of a power series in the small parameter \( \sigma(0) \):
\[ \rho \Gamma_4 = \rho \Gamma_4(0) + \rho \Gamma_4(1) + \rho \Gamma_4(2) + \rho \Gamma_4(3) + \ldots. \]

\( \rho \Gamma_4(0) \) is the solution of Eq. (5) with \( f = 0 \) such that
\[ \frac{\partial}{\partial \xi} \rho \Gamma_4(0) = i \frac{\partial}{\partial \xi} \rho \Gamma_4(0). \]

The equation for \( \rho \Gamma_4(1) \),
\[ \frac{\partial}{\partial \xi} \rho \Gamma_4(1) = \gamma f(\xi, \eta) \rho \Gamma_4 + i \frac{\partial}{\partial \xi} \rho \Gamma_4, \]
is easily solved to give (assuming that \( \rho \Gamma_4(0) = 1 \))
\[ \rho \Gamma_4(1) = \gamma f(\xi, \eta) \rho \Gamma_4 + i \frac{\partial}{\partial \xi} \rho \Gamma_4. \]

\( \rho \Gamma_4(2) \) is the spatial spectrum of \( \sigma(0) \),
\[ \Phi(K) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi \sigma(\xi) \exp[-iK\xi]. \]

In the Rydov perturbation formulation, it is \( \Psi = \ln \rho \Gamma_4(\xi, \eta, \xi', \eta') \) that is approximated by a similar power series. For an initially plane wave, \( \Psi_0 = 0 \) and \( \Psi_1 \) obeys the equation
\[ \frac{\partial}{\partial \xi} \Psi_1 = \gamma f(\xi, \eta) + i \frac{\partial}{\partial \xi} \Psi_1, \]
with the initial condition \( \Psi_1(\xi = 0) = 0 \). Since, for an initially plane wave, \( \rho \Gamma_4(0) \) is a constant independent of \( \xi, \xi', \) or \( \eta, \eta' \), Eqs. (20) and (23) have identical forms, and we may conclude that the first Born approximation and the first Rydov approximation are related to each other by
\[ \rho \Gamma_4(\xi, \eta, \xi', \eta') = \rho \Gamma_4(0) \rho \Gamma_4(1) \rho \Gamma_4(2) \ldots. \]

Using our Gaussian correlation function [Eq. (16)], we find from Eqs. (3) and (21) that, in the perturbation region, the variance of the intensity fluctuations is given by
\[ \sigma f^2(Born) = 2\gamma \zeta - \gamma(1 + (4\zeta)^2)^{1/4} \sin(1/2 \tan^{-1}(4\zeta)). \]

Two limiting cases can be analyzed:

1. Geometrical Region: \( \zeta \ll 1 \). Here
\[ \sigma f^2(Born) = 4\gamma \zeta. \]

2. Diffraction Region: \( \zeta \gg 1 \)
\[ \sigma f^2(Born) = 2\gamma \zeta = 2k^2\sigma(0)\zeta. \]

Remember, however, that Eqs. (26) and (27) are valid only when \( \sigma f^2 \ll 1 \). The Rydov equivalents of Eqs. (26)–(27) are directly obtained from Eq. (24).

Some analytical insight into the spatial behavior of \( \rho \Gamma_4 \) can be obtained in the geometrical region (\( \zeta \ll 1 \)). Here, Eq. (21) reduces to
\[ \rho \Gamma_4(\xi, \eta, \xi', \eta') = \gamma f(\xi, \eta) \zeta \]
\[ + \frac{1}{2} \frac{\partial}{\partial \xi} \rho \Gamma_4 + i \gamma f(\xi, \eta) \frac{\partial}{\partial \xi} \rho \Gamma_4. \]

When either \( \xi = 0 \) or \( \eta = 0 \), \( f(\xi, \eta) = 0 \) and the first lowest-order nonzero corrections for \( \rho \Gamma_4 \) and \( \Im \rho \Gamma_4 \) must be derived, respectively, from the \( i^2 \) and \( i^2 \) terms in Eq. (28). As long as these corrections are small, the Born and Rydov approximations will not differ substantially from each other. However, when either \( \xi = 0 \), \( \eta = 0 \), \( f(\xi, \eta) = 0 \) and we may expect significant differences between the above two approximations.

Returning now to our Gaussian correlation function [Eq. (16)], we note from Eqs. (26) and from Eqs. (8)–(10) that, in the limit of \( \xi = \eta = \omega \),
\[ \rho \Gamma_4(\xi, \eta, \xi', \eta') = 1 - 3\gamma \zeta - i\gamma \zeta^2 - 2\gamma \zeta^3. \]

Equation (29) measures the fourth-order statistical moment of the complex wave amplitude at four collinear points \((x_1, z), \ldots, (x_4, z)\) with the following (transversal) geometrical relationship: \( 1 - 2z, 3z - 4 \) [the asterisk denotes complex conjugation at \((x_2, z)\) and \((x_3, z)\) as well]. Also \( x_2 = x_3 \).

C. Asymptotic Solution [Strong Scattering]
In the region of strong fluctuations, a rigorous solution of the fourth-moment differential equation has not yet been obtained. Using Feynmann path integral techniques, Zavorotniy et al. were able to derive an asymptotic solution for \( \rho \Gamma_4 \). By adapting their method to our two-dimensional random medium, we could derive the following asymptotic expression for \( \rho \Gamma_4 \):
\[ \rho \Gamma_4(\xi, \eta, \xi', \eta') = \rho \Gamma_4(2\xi, \eta) + \rho \Gamma_4(2\eta, \xi) + \rho \Gamma_4(2\xi', \eta') + \rho \Gamma_4(2\eta', \xi'). \]
\[ g(\xi, \eta, \theta) = 4\pi k_n^3 \int_0^1 d\alpha \int_{-\alpha}^\alpha d\xi \Phi_n(0, \theta) \times [1 - \cos(\xi \theta - \eta \alpha)] \times \exp \left[ i \theta \eta - 2\gamma \xi (1 - \alpha) D(\xi - \eta \alpha) \right] - 2\gamma \xi \int_0^\alpha d\alpha' D(\xi - \eta \alpha') \right], \] (31)

\[ \Phi_n(0, \theta) = \frac{1}{4\pi^2} \int_{-\alpha}^\alpha d\xi \int_{-\alpha}^\alpha d\eta \sigma(l_n, \xi, l_n \eta) \times \exp(-i\eta \theta), \] (32)

\[ D(\xi) = 1 - \frac{\sigma(\xi)}{\sigma(0)}, \] (33)

and

\[ p\Gamma_2(\xi, \eta) = \exp(-\gamma \xi D(\xi)) \] (34)

is the second-order coherence function of a unit intensity propagating plane wave.

Equation (30) represents an asymptotic expansion of \( p\Gamma_4 \) with respect to the small parameter \( \delta = \xi_{\text{coh}} / \xi \), where \( \xi_{\text{coh}} \) the coherence radius, is found from the quality \( |\ln p\Gamma_4(\xi, \eta)| = 1 \).

In the case of a Gaussian correlation function [Eq. (15)]

\[ p\Gamma_2(\xi, \eta) = \exp(-\gamma \xi (1 - \exp(-\xi^2))) \] (35)

and

\[ \xi_{\text{coh}} = \sqrt{\left[ \ln \left( 1 + \frac{1}{\gamma \xi} \right) \right]}^{1/2} \rightarrow (\gamma \xi)^{-1/2}. \] (36)

Since \( p\Gamma_2(\xi, |\xi| \rightarrow \infty) \rightarrow \exp(-\gamma \xi) \) represents the (yet) unscattered (coherent) part of the propagating wave, \( \gamma \xi \) can serve as a dimensionless measure of the strength of the scattering suffered by the wave, and strong scattering conditions will be defined by \( \gamma \xi \gg 1 \).

Gurvich et al.\(^{16}\) have studied their asymptotic solution for three-dimensional plane-wave propagation in a random medium with power-law spectra. They found that in strong scattering the spatial structure of \( p\Gamma_4 \) is characterized by two scales: the coherence radius that describes the initial transversal falloff of \( p\Gamma_4 \) and \( \delta^{-1} \), which is responsible for the long tail of the covariance of the intensity fluctuations.\(^{7} \)

In Section 4 we compare our numerical results with the predictions of the perturbation and asymptotic theories.

4. NUMERICAL RESULTS

Our starting point is the integral equation [Eq. (17)] with the initial condition

\[ p\Gamma_4(\xi_0, \eta, \eta, 0) = 1, \] (37)

which corresponds to an initially plane wave with unit amplitude and boundary conditions of the form

\[ p\Gamma_4(\xi, \eta, \eta, 0) = \exp[\gamma f(\xi = \pm \xi_{\text{max}}, \eta)] \xi. \] (38)

\[ p\Gamma_4(\xi, \eta, \eta, 0) = \exp[\gamma f(\xi, \eta = \pm \eta_{\text{max}})] \xi. \] (39)

These boundary conditions reduce to those of Brown\(^{7} \) when either \( \eta = 0 \) or \( \xi = 0 \) and their real parts agree with the perturbation results to first order in \( \xi \). Although they erroneously predict that \( \text{Im} \ p\Gamma_4(\xi, \eta, 0) = 0 \), as well as \( \text{Im} \ p\Gamma_4(\xi, \eta, 0) = 0 \), is zero [in contradiction to Eq. (29)], the results of this section support these approximations.

A numerical algorithm that uses bicubic splines to approximate the transverse partial derivatives and Simpson’s rule to recover a cubic \( \xi \) dependence in a single \( \Delta \xi \) step has been described in Refs. 22 and 25. The program was executed by using the Gaussian correlation function [Eq. (16)] and various values of \( \gamma \). Although the Gaussian model is a poor description of the index-of-refraction fluctuations in the atmosphere, it was believed that this smooth and well-behaved function would reduce numerical errors (the zero inner scale Kolmogorov-type correlation function that was previously used by Brown\(^{7} \) has divergent high-order derivatives and therefore can induce numerical errors in our derivative-based algorithm). Also, the Gaussian correlation function is realistic in several other cases.\(^{6,8,11} \)

Figure 1 describes the range dependence of \( \sigma_1^2 \) [Eq. (3)] for various values of \( \gamma \) as obtained from the numerical solution of Eq. (5). Table 1 gives the grid parameters. Coarser grids for the cases \( \gamma = 0.5, 2.5, 10 \) gave the same results as the grids of Table 1. Also shown in Fig. 1 are the results of the perturbation solutions of Section 3. As long as \( \sigma_1^2 \ll 1 \), good agreement is obtained between the numerical solution of Eq. (5) and the perturbation results (see also Figs. 6, 8, and 10).

This agreement constitutes a good test for the consistency of our numerical scheme, at least in the perturbation region.

Figure 1. Range dependence of \( \sigma_1^2 \) for several values of \( \gamma \). Also included are the Born and Rytov approximations.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \Delta \xi = \Delta \eta )</th>
<th>( \xi_{\text{max}} = \eta_{\text{max}} )</th>
<th>( \Delta \xi )</th>
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<td>50</td>
<td>0.0625</td>
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<td>0.005</td>
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</table>
the quantity

\[ S(\xi) = \int_{-\eta_{\text{max}}}^{\eta_{\text{max}}} \rho B_{1}(\xi, \eta) \, d\eta. \]

(42)

For far-enough boundaries, \( \rho B_{1}(\xi, \eta_{\text{max}}) = 0 \), and it follows from Eq. (42) and the initial conditions that \( S(\xi) = 0 \). Initially (i.e., for small \( \xi \)), \( S(\xi) \) is of the order of \( 10^{-8} \). As the wave propagates into the medium, a significant amount of residual correlation is developed and \( \rho B_{1}(\xi, \eta) \) does not vanish on the boundary \( \eta = \eta_{\text{max}} \). As a result, the infinite integration limits of Eq. (40) cannot be approximated, and \( S(\xi) \) assumes higher values. When the value of \( S(\xi) \) is of order unity, the results for \( \delta_{\rho}^{2} \), as obtained from grids with different values of \( \xi_{\text{max}} \), depart from each other.

Fig. 2. Dependence of \( \rho \Gamma_{4}(\xi = 0.5, \xi, \eta, \tilde{\eta} = 0) \big|_{\gamma = 0.5} \) on \( \xi \) and \( \eta \), as determined from the numerical solution of Eq. (5).

Fig. 3. Dependence of \( \rho \Gamma_{4}(\xi = 0.15, \xi, \eta, \tilde{\eta} = 0) \big|_{\gamma = 0.5} \) on \( \xi \) and \( \eta \), as determined from the numerical solution of Eq. (5).

The accuracy of the numerical results, and in particular the effects of the finite grid size, was checked against Tatarkii's conservation law

\[ \frac{d}{d\xi} \int_{-\xi}^{\xi} \rho B_{1}(\xi, \eta) \, d\eta = 0, \]

(40)

where

\[ \rho B_{1}(\xi, \eta) = \left[ I(x_{a}, z)I(x_{b}, z) \right] - 1 \]

\[ = \rho \Gamma_{4}(\xi, 0, \eta = (x - x_{b})/l_{n}, \tilde{\eta} = 0) - 1 \]

(41)

is the covariance of the intensity fluctuations. Equation (40) follows directly from Eqs. (5), (11), and (39). At each range \( \xi \geq 0 \), we have used the generated numerical data to evaluate

Fig. 4. Dependence of \( \rho \Gamma_{4}(\xi = 0.27, \xi, \eta, \tilde{\eta} = 0) \big|_{\gamma = 0.5} \) on \( \xi \) and \( \eta \), as determined from the numerical solution of Eq. (5).

Fig. 5. Dependence of \( \rho \Gamma_{4}(\xi = 0.5, \xi, \eta, \tilde{\eta} = 0) \big|_{\gamma = 0.5} \) on \( \gamma \) and \( \eta \), as determined from the numerical solution of Eq. (5).
It is seen from Fig. 1 that, as \(\gamma\) increases, a focusing behavior is developed.\textsuperscript{6,11} The ratio \(\frac{\sigma_f}{\sigma_{\text{max}}} \) is 1.5, which is in fair agreement with the theoretical study of Shishov\textsuperscript{23} and Uscinski,\textsuperscript{24} who predicted a dependence of the form \(\frac{\sigma_f}{\sigma_{\text{max}}} \approx \gamma^{-1/2}\). An extensive numerical study of the location and height of the focus can be found in the work of Uscinski and Miller.\textsuperscript{11}

We also note that a local minimum exists after the peak. For more details see Tur et al.\textsuperscript{25}

Also shown in Fig. 1 is \(\sigma_f^2\) as obtained from Eq. (30) with \(\xi = \eta = \theta = 0\). The resulting curve gives incorrect values for the heights of the maximum and does not predict the existence of the additional minimum. It should be remembered, however, that Eq. (30) is asymptotic in the small parameter \(\delta\), which is only \(\approx 0.1\) for \(\gamma = 50\) and \(\xi = 0.5\).

The spatial dependence of \(\rho\Gamma_4(\xi, \eta, \varphi = 0)\) on \(\xi\) and \(\eta\) appears in Figs. 2–5. These three-dimensional plots not only present a perspective view of \(\rho\Gamma_4\) but also give quantitative information. For example, in order to know the value of \(\text{Re} \rho\Gamma_4\) at \(\xi = 0.5, \eta = 0.625\) for \(\gamma = 0.5\), we first locate this point on Fig. 2 and denote it by P. The location of its projection Q on the \(\xi, \eta\) plane can be found in several ways. The segment PQ is now overlaid on the vertical annotated axis with Q at the origin of that axis. The requested value of \(\text{Re} \rho\Gamma_4\) is therefore 0.92. Note that, in Figs. 2 and 3, \(-\text{Im} \rho\Gamma_4\) is plotted instead of \(+\text{Im} \rho\Gamma_4\). These figures reveal several interesting features of the spatial dependence of \(\rho\Gamma_4\):

1. \(\text{Re} \rho\Gamma_4\) has a characteristic shape that, apart from its scale, does not appreciably change with either \(\xi\) or \(\gamma\). Its shape will be further discussed below by using perpendicular cuts along different directions.

2. Unlike \(\text{Re} \rho\Gamma_4\), \(\text{Im} \rho\Gamma_4\) changes its shape considerably as the wave propagates into a medium with strong refractive-index fluctuations. The minimum in Fig. 3 (\(\gamma = 50\)) is converted into a maximum in Figs. 4 and 5 (note the inverted vertical scale in Fig. 3 and see also Fig. 11). In order to understand this behavior, we rewrite Eq. (5) in the form

\[
\frac{\partial \text{Re} \rho\Gamma_4}{\partial \xi} = \frac{\partial^2 \text{Im} \rho\Gamma_4}{\partial \xi^2} + \gamma \frac{\partial \text{Re} \rho\Gamma_4}{\partial \eta},
\]

(43)

\[
\frac{\partial \text{Im} \rho\Gamma_4}{\partial \xi} = \frac{\partial^2 \text{Re} \rho\Gamma_4}{\partial \xi^2} + \gamma \frac{\partial \text{Im} \rho\Gamma_4}{\partial \eta}.
\]

(44)

When \(\xi = \eta = 0\), we have, from Eqs. (43) and (11),

\[
\frac{\partial \sigma_f^2}{\partial \xi} = \frac{\partial^2 \text{Im} \rho\Gamma_4(\xi, \xi = 0, \eta = 0, \varphi = 0)}{\partial \xi^2}.
\]

(45)

Looking back at Fig. 1, we see that \(\sigma_f^2\) increases with \(\xi\) both for \(\gamma = 0.5, \xi = 0.5\) (Fig. 2) and for \(\gamma = 50, \xi = 0.15\) (Fig. 3). This requires \(\text{Im} \rho\Gamma_4\) to have a negative \(\partial^2 \text{Im} \rho\Gamma_4/\partial \xi^2\) derivative (see Eq. (45)). However, since \(\text{Im} \rho\Gamma_4\) is antisymmetric in \(\xi\) and \(\eta\), \(\text{Im} \rho\Gamma_4\) must be negative near \(\xi = \eta = 0\). At the peak (\(\xi = 0.27\) in Fig. 4) and at the minimum (\(\xi \approx 0.5\) in Fig. 5), the left-hand side of Eq. (45) vanishes and the mixed \(\xi, \eta\) derivative is zero, giving a flat appearance to \(\text{Im} \rho\Gamma_4\) near the origin of the \(\xi, \eta\) plane. This behavior is clearly observed in Figs. 4 and 5. As the wave propagates from \(\xi = 0.27\) (peak) to \(\xi = 0.5\) (minimum), the minima along the lines \(\xi \approx 0.125\) and \(\xi = 0.125\) (Fig. 4) are converted into maxima (Fig. 5). These changing characteristics of \(\text{Im} \rho\Gamma_4(\xi), \gamma = 1\) are not unique either to the two-dimensional character of our formulation or to the Gaussian correlation function. In fact, Gurvich et al.\textsuperscript{18} observed the same behavior in their asymptotic solutions for a three-dimensional Kolmogorov-type medium.

When \(\gamma = 0.5\), saturation is obtained only for \(\xi \approx 1.5\) and therefore, in the range \(\xi \geq 0.5\), \(\text{Im} \rho\Gamma_4\) changes only in mag-

Fig. 6. Dependence of \(\rho\Gamma_4(\xi, \xi = 0, \eta, \varphi = 0)|_{\gamma = 0.5}\) on \(\eta\), as determined from the numerical solution of Eq. (5). Although not shown, the Born and Rytov results for \(\xi = 0.15, 0.27\) are in good agreement with the numerical data.

(a)

(b)

Fig. 7. (a) Dependence of \(\rho\Gamma_4(\xi, \xi = 0, \eta, \varphi = 0)|_{\gamma = 50}\) on \(\eta\), as determined from the numerical solution of Eq. (5). (b) Dependence of \(\rho\Gamma_4(\xi, \xi = 0, \eta, \varphi = 0)|_{\gamma = 50}\) on \(\eta\), as determined from the asymptotic expression of Eq. (30).
Figures 6–11 represent cuts of Figs. 2–5 (Figs. 6, 8, and 10 also contain information on $p\Gamma_4(\xi, \eta, q = 0)$ with $\bar{z} = 0.15, 0.27$). Also included are perturbation results for $\gamma = 0.5$ and asymptotic results for $\gamma = 50$.

Since $p\Gamma_4(\xi, \eta, q = 0)$ and $pB_1(\xi, \eta)$, Figs. 6 and 7(a) describe the correlations of intensity fluctuations at several ranges. When $\gamma \ll 1$, $|p\Gamma_4(\xi, \eta, q = 0) - 1| \ll 1$, and both the Rylov and Born solutions are in close agreement with the numerical results. Of particular interest is the shape of the $\xi = 0.5$ curve in Fig. 7(a) ($\gamma = 50$), which is characterized by two length scales: After a steep decrease of the order of the coherence radius $\xi_c$, the curve changes its slope and approaches its asymptotic value of 1 [i.e., $pB_1(\xi, \eta - \rightarrow \infty) \rightarrow 0$] with a long tail. We also observe that the characteristic correlation length for the intensity fluctuations decreases as $\gamma$ increases, in accordance with previous predictions. According to Ref. 18, the scale of the long tail is of the order of $\delta^{-1} = f/\sigma_c^2$, and therefore it grows with $\gamma$. When $\gamma = 50$ and $\xi = 0.5$, $\delta^{-1} \approx 10$. Extrapolation of the $\xi = 0.5$ curve in Fig. 7(a) gives $\approx 7$. The asymptotic results of Eq. (30) with $\xi = 0$ appear in Fig. 7(b). It is evident that the $p\Gamma_2$ term

Fig. 6. Dependence of $\Re p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$ as determined from the numerical solution of Eq. (5). Also shown are the Born and Rylov approximations.

Fig. 7. Dependence of $\Im p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$ as determined from the numerical solution of Eq. (5). Also shown are the Born and Rytov approximations.

Fig. 8. Dependence of $\Re p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$, as determined from the numerical solution of Eq. (5). Also shown are the Born and Rytov approximations.

Fig. 9. (a) Dependence of $\Re p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$, as determined from the numerical solution of Eq. (5). (b) Dependence of $\Re p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$, as determined from the asymptotic expression of Eq. (30).

Fig. 10. Dependence of $\Im p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$, as determined from the numerical solution of Eq. (5). Also shown are the Born and Rytov approximations.

Fig. 11. Dependence of $\Im p\Gamma_4(\xi, \eta, q = 0)$ on $\eta$, as determined from the numerical solution of Eq. (5). The asymptotic expression [Eq. (30)] are also shown.
terminates the steep decrease, whereas the \( g(\cdot) \) functions generate the long tail.

Re \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \) versus \( \eta \) appears in Figs. 8 and 9(a). When \( \gamma = 0.5 \), the curves approach nonzero straight asymptotes in qualitative agreement with the predictions of the geometrical region solution [Eq. (29)]. Whereas the Rytov results are practically indistinguishable from the numerical solutions [except for small differences, of the order of \( \sigma_{t}^{2} \) (Rytov) - \( \sigma_{t}^{2} \) (numerical), near \( \eta \approx 0 \)], the Born results approximate the numerical solutions only for small values of \( \eta \).

Figure 9(b) shows the asymptotic results of Eq. (30) for Re \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \). Figures 10 and 11 describe Im \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \). Again, the Rytov approximation, in the \( \gamma = 0.5 \) case, is far better than the Born approximation. Figure 11 shows the dependence of Im \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \) on \( \eta \) for ranges before the peak (\( \xi = 0.15 \)), near the peak (\( \xi = 0.27 \)) and near the minimum. The deep minimum changes to a shallow maximum and the \( \eta \) derivative near \( \eta = 0 \) increases from its negative value at \( \xi = 0.5 \) up to approximately zero. Here too the asymptotic results of Eq. (30) show qualitative agreement with the numerical data.

Finally, we note that, in the asymptotic region of Figs. 8–11 (but far enough from the actual boundaries), either (Figs. 8 and 10)

\[
\text{Im} \, \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) < \text{Re} \, \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \quad (46)
\]

or [Figs. 9(a) and 11]

\[
\text{Re} \, \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) = 0,
\]

\[
\text{Im} \, \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) = 0,
\]

and the approximations involved in Eqs. (38) and (39) are thus justified.

5. CONCLUSION

This work has presented the range evolution of the full spatial coherence of the fourth moment, \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \), of an initially plane wave, as obtained from a numerical solution of the appropriate partial differential equation. Although the medium has been assumed to be two-dimensional with a Gaussian correlation function, the results, especially in the important strong-scattering regime, bear a striking similarity to previously obtained asymptotic solutions for a three-dimensional, power-law random medium. It has been shown that the topographical dependence of Re \( \rho \Gamma_{4}(\xi, \eta, \eta, \varphi = 0) \) on \( \xi \) and \( \eta \) is characterized by a peak at the origin (i.e., where all four points coincide) and by two symmetrical and perpendicular ridges above the \( \xi \) and \( \eta \) axes (see Figs. 2–5). The widths of the ridges and the shapes of their crests vary as the wave propagates deeper and deeper into the medium. Under strong-scattering conditions, the crests are characterized by a steep decrease with a scale of the order of the coherence radius, followed by a long tail. Unlike Re \( \rho \Gamma_{4} \), Im \( \rho \Gamma_{4} \) changes its shape considerably as the wave propagates into a medium with strong refractive-index fluctuations. In the weak-scattering regime, excellent agreement was obtained between the numerical data and the predictions of the Rytov approximation. The Born approximation was satisfactory only for extremely weak scattering. In the strong-scattering regime, the numerical data were compared with the asymptotic theory of Zavorotnyi et al. and found to be in good qualitative agreement.

The interesting spatial structures of Re \( \rho \Gamma_{4} \) and Im \( \rho \Gamma_{4} \) and, in particular, their range dependence, suggest that further studies of the full spatial behavior of \( \Gamma_{4} \) will contribute to a better understanding of the range dependence of \( \sigma_{t}^{2} \) (see Ref. 26) and to future efforts to find analytical solutions of the fourth-moment equation.

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