Ex. 2 - Digital image processing

1) For each of the following linear systems, determine the transfer function, frequency response, the OTF and the MTF:

   a) \( y(m,n)-a_1 y(m-2,n)- a_2 y(m,n-2)=x(m,n) \)
   b) \( y(m,n)-a_1 y(m-1,n)- a_2 y(m,n-1)+ -a_1 a_2 y(m-1,n-1)=x(m,n) \)

2) An image given by the two dimensional function

   \[ f(x,y) = 2 \cos[2\pi(4x + 6y)] \]

   is sampled on an infinite grid of pixels with separation \( \Delta x = 0.1 \) and \( \Delta y = 0.2 \), in the \( x \) and \( y \) directions respectively.

   We form the continuous FT of this sampled image; multiply by the ideal low pass spatial filter with cutoff frequencies at half the sampling frequency in each spatial frequency axis; and then inverse Fourier transform to obtain a reconstructed image. Determine:

   a) The spatial spectrum of the sampled image before low pass filtering.
   b) The spectrum of the image after after it has been low-pass filtered.
   c) The reconstructed image.
   d) In order to reconstruct without distortion the original image from sampled data what is the maximum sizes of \( \Delta x \) and \( \Delta y \) that can used, and the minimum and maximum bandwidths of the low pass reconstruction filters that can be used?

3) A two dimensional continuous image is given by:

   \[ f(x,y) = \sin^2 \left( 2\pi(ax + by) \right) \]

   a) sketch the function for \( a=4; b=4 \) & \( a=1; b=8 \)
   b) We want to correctly sample this function/ what is the Nyquist rate on the \( x \) and \( y \) axis? (as a function of \( a \) & \( b \)).
   c) Is the picture’s “actual” frequency higher or lower then the one you gave at section b?
   d) For \( a=2 \) and \( b=4 \), suggest the optimal sampling grid.
   e) For \( a=2 \) and \( b=2 \), \( \Delta x=1/8 \) and \( \Delta y=1/8 \) the image has been sampled. We want to create the image from its samples. However our new display has pixels spaced \( \Delta x=\Delta y=1/16 \). For zero hold and 1st order hold interpolators, what will be the value of the reconstructed picture on pixels located at:

   \( (x,y)=((2n+1)/16, (2m+1)/16), n,m=0,1,2, .... \)
   f) Is there a way to exactly reconstruct the image if the original picture \( f(x,y) \) is bounded in the region \([-5:5,-5:5]\)?
4) An image is samples using a HEXCOMB function, which is a function of \( \delta(x, y) \)'s, each in the center of a hexagonal covering the whole plane:

![Hexagonal sampling](image)

Figure: Hexagonal sampling

a) Express the HEXCOMB function, using comb functions or \( \delta(x, y) \) functions.

b) Given \( g(x, y) \) which Fourier transform satisfies

\[
F[g(x, y)] = G(\xi_1, \xi_2) = \begin{cases} 0 & \sqrt{\xi_1^2 + \xi_2^2} > B \\ f(x, y) & \text{otherwise} \end{cases}
\]

Prove that sampling \( g(x, y) \) using the HEXCOMB function allows a complete reconstruction, and requires less sampling points than the COMB function.

5) Given a function whose Fourier transform is given in the figure below:

![Function diagram](image)

a) This function is sampled by a rectangular grid (COMB). Find the minimum sampling resolution (in samples per meter) to allow a full reconstruction of the function.

b) Find a different sampling grid that gives optimal sampling resolution. What is the sampling grid? What is the sampling resolution?