Exercise No. 3 in Image Processing

1. Can two monochromatic sources of different wavelength appear like the same color? Prove your answer.

2. Prove: $T_k(C) = \int C(\lambda) R_k(\lambda) d\lambda$

3. In this problem we show that any two tristimulus coordinate systems based on different sets of primary sources are linearly related. Let $\{P_k(\lambda)\}$ and $\{P'_k(\lambda)\}$, $k=1,2,3$, be two sets of primary sources with corresponding tristimulus coordinates $\{T_k\}$ and $\{T'_k\}$ and reference white sources $W(\lambda)$ and $W'(\lambda)$. If a color $C(\lambda)$ is matched by these sets of sources, then show that:

$$\sum_{k=1}^{3} a_{i,k} w_k T_k(C) = \sum_{k=1}^{3} a'_{i,k} w'_k T'_k(C).$$

Express this in matrix form and write the solution for $\{T'_k\}$.
4. A bandlimited image acquired by a practical sensor is observed as
\[ g(x, y) = f(x, y) + n(x, y) \]
where \( \xi_{x0} = \xi_{y0} = \xi_f \) and \( n(x, y) \) is wideband noise whose spectral density function \( S_n(\xi_1, \xi_2) = \frac{n}{4} \) is bandlimited to \( -\xi_n \leq \xi_1 \), \( \xi_n = 2\xi_f \), \( \xi_2 \leq \xi_n \). The random field \( g(x, y) \) is sampled without prefiltering and with prefiltering at the Nyquist rate of the noiseless image and reconstructed by an ideal low-pass filter whose bandwidths are also \( \xi_{x0} = \xi_{y0} = \xi_f \) see figure below:

![Figure: A practical sensor](image)

Show that:

a) The SNR of the sensor output \( g \) over its bandwidth is \( \frac{\sigma_f^2}{4\eta_x^2} \) where \( \sigma_f^2 \) is the image power.

b) The SNRs of the reconstructed image with and without prefiltering are \( \frac{\sigma_f^2}{\eta_x^2} \) and \( \frac{\sigma_f^2}{4\eta_x^2} \) respectively. What would be the SNR of the reconstructed image if the sensor output were sampled at the Nyquist rate of the noise without any prefiltering? Compare the preceding sampling schemes and recommend the best way for sampling noisy images.
5. Define \( w \equiv f(u) = \begin{cases} \int_0^u p_u \, du & u > 0 \\ -f(-u) & u < 0 \end{cases} \)

Suppose the compandor transformation in the upper definition is \( g(x) = f^{-1}(x) \) and \( p_u(u) = p_u(-u) \). This transformation (also called histogram equalization) causes \( w \) to be uniformly distributed over the interval \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \). The uniform quantizer is now optimized for \( w \). However the overall quantizer need not be optimized for \( u \).

a) Let \( p_u(u) = \begin{cases} 1-|u| & -1 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \) and the quantizer levels be 4. What are the decision and reconstruction levels for the input \( u \)? Calculate the MSE.

b) Show that this compandor is sub optimal compared to the one discussed in the text.

6. a) Show that the random variable \( v \) defined via \( v \equiv F_U(u) = \int_0^u p_u \, du \) satisfies the condition \( \Pr(V \leq v) = \Pr(U \leq F^{-1}(v)) = F(F^{-1}(v)) = v \), where \( 0 < v < 1 \).

This means \( v \) is uniformly distributed over \((0,1)\).

b) On the other hand, show that the transformation

\[ p_v(x_i) = \frac{h(x_i)}{\sum_{i=0}^{L-1} h(x_i)} \quad i = 0,1,\ldots,L-1 \]


gives \( p_v(v_k) = p_u(x_k) \) where \( v_k \) is given by \( v_k = \sum_{i=0}^k p_u(x_i) \).

7. The PDF of an image is given by \( \Pr(r) \) as shown below. Find the transform to convert the image's PDF to \( \Pr(z) \). Assume continuity, and find the transform in terms of \( r \) and \( z \). Explain the transformation.