The Single and Multi-Item Transshipment Problem with Fixed Transshipment Costs

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Abstract: This article deals with supply chain systems in which lateral transshipments are allowed. For a system with two retailers facing stochastic demand, we relax the assumption of negligible fixed transshipment costs, thus, extending existing results for the single-item case and introducing a new model with multiple items. The goal is to determine optimal transshipment and replenishment policies, such that the total centralized expected profit of both retailers is maximized. For the single-item problem with fixed transshipment costs, we develop optimality conditions, analyze the expected profit function, and identify the optimal solution. We extend our analysis to multiple items with joint fixed transshipment costs, a problem that has not been investigated previously in the literature, and show how the optimality conditions may be extended for any number of items. Due to the complexity involved in solving these conditions, we suggest a simple heuristic based on the single-item results. Finally, we conduct a numerical study that provides managerial insights on the solutions obtained in various settings and demonstrates that the suggested heuristic performs very well. © 2014 Wiley Periodicals, Inc. Naval Research Logistics 61: 637–664, 2014

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1. INTRODUCTION AND LITERATURE REVIEW

Over the past couple of decades, supply chain optimization has become one of the most important topics that a firm must deal with due to globalization, increasing market competition, and accelerated technology development. The main goals of the firm are to maximize profit and provide customers with a high service level. Due to demand variability and market uncertainty, achieving these goals requires flexibility, short response time and development of new innovative solutions along the supply chain.

Consider, for example, a chain of stores that sells various small electrical appliances. On any particular day, customers of a certain branch may request products that are out of stock, but these products may be available in one of the other store's branches. To avoid lost sales, it may be possible to tranship the requested units from the other store. But this entails sending a vehicle from the distant branch, thus, incurring costs such as gas expenses. Due to these costs, transshipping only one unit may not be profitable. However, for a higher number of units that possibly includes different types of items, the additional profit from meeting all customer demands may cover the additional expenses and make the transshipment worthwhile. Similarly, transshipments of multiple items involving fixed transshipment costs may occur when spare parts are transshipped between different repair centers or when power tools and electric parts for automobiles are transshipped between distributors, see Rudi et al. [18]. In these examples, items in large volumes are involved. Therefore, lateral stock transshipments may be an effective strategy for achieving flexibility and improving the supply chain performance, even in the presence of fixed transshipment costs.

Transshipments allow facilities that are at the same echelon in the supply chain to share inventory. This *risk-pooling* strategy helps firms to reduce discrepancies between supply and demand and to consequently avoid some shortage and surplus costs and increase the service level. In addition, when making replenishment decisions while taking into consideration the possibility of performing transshipments at a later stage, retailers may partially rely on the inventory of other retailers. Consequently, the total safety stock in the system may be reduced.

Over the past decades, a considerable amount of research has focused on lateral transshipments. The most common assumption appearing in the literature is that transshipment costs are proportional to the number of units transshipped. However, when different items are transshipped together, a

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joint fixed cost is likely to be incurred, in addition to (or instead of) the variable transshipment cost. This cost may represent the costs of sending a vehicle from one location to another (as in the example above) or the payment to a delivery company that charges a fixed service cost, possibly in addition to a surcharge that depends on the delivery size. In the presence of fixed transshipment costs, transshipping units from a retailer with a surplus to a retailer with a shortage is not always profitable and depends on the surplus/shortage levels.

The contribution of this article is in developing and analyzing single and multi-item transshipment models in which fixed transshipment costs are considered in addition to the commonly used variable transshipment costs. In considering these two types of costs, we extend the transshipment cost structure discussed in the literature. In particular, the contributions of this article are as follows: first, we provide optimality conditions for both the single and multi-item transshipment problems with fixed transshipment costs, in a centralized system with two retailers facing stochastic demand. Second, for the single-item problem, we analyze the expected profit function, characterize the area in which the function is concave, and demonstrate that solving the optimality conditions yields an optimal solution in most, if not all cases. For the multi-item problem, we numerically show similar characteristics. Third, for the multi-item problem, we develop a heuristic, for cases that are computationally hard to solve to optimality. We also develop upper bounds on the optimal solution value. The heuristic is evaluated through a numerical study that demonstrates its effectiveness. Finally, we provide managerial insights into the solution's characteristics in relation to various settings of the problem. In the rest of this section, we review the transshipment literature that is related to our work.

Krishnan and Rao [12] introduced the basic two-retailer inventory problem with transshipments. They considered a periodic review model with identical cost parameters for each retailer and later extended it to a general number of retailers. Tagaras [20] generalized the two-retailer model presented in Krishnan and Rao by allowing different cost parameters for the retailers. Robinson [16] extended the multiple retailer transshipment model by considering multiple periods with nonidentical cost parameters and developed a heuristic to solve the problem. Herer et al. [10] investigated the same multiretailer problem with an infinite horizon and provided an algorithm, based on an infinitesimal perturbation analysis method, that finds an optimal solution. Rudi et al. [18] analyzed and derived optimality conditions for the same two-retailer centralized system as in Tagaras [20], as a basis for examining and analyzing a decentralized system. Shao et al. [19] and Rong et al. [17] also considered transshipments in a decentralized system. The former extended Rudi et al. [18] by considering both the supplier and retailer's decisions, while the latter considered preventive transshipments (transshipments that take place before the demand is realized) in a system with two retailers and two demand subperiods. Zhao and Atkins [25] considered two competing retailers who cooperate to establish transshipment agreements. They demonstrated that competition can reduce the effectiveness of transshipments. Other examples of single-item transshipment models can be found in Dong and Rudi [4], Zhang [24], Wee and Dada [21], Gong and Yücesan [6], and Özdemir et al. [14].

In all of the above models, it was assumed that fixed costs for inventory replenishment and transshipment are negligible. This was also mentioned as a common assumption in the transshipment literature in a recent review by Paterson et al. [15]. In the absence of fixed costs, decisions regarding a specific unit in the system are made individually, irrespective of other units. As a result, the transshipment policy presented in Krishnan and Rao [12], later named by Tagaras [20] as the complete pooling policy, is optimal and a replenishment decision following a base stock policy is also optimal. Adding fixed costs to the above models has a significant effect on the complexity and the structure of the optimal solution, as is the case in inventory problems without transshipments. Herer and Rashit [8] considered non-negligible, fixed, and joint replenishment costs in the two-location, single period problem with transshipments. They showed that while the complete pooling transshipment policy is still optimal, the base-stock policy is no longer the optimal replenishment policy. Indeed, the replenishment policy may have quite a complex structure. Another work that considers fixed replenishment costs is that of Herer and Tzur [9]. They, however, assume deterministic and dynamic demand.

Some studies specifically consider fixed transshipment costs, but not in a periodic review model with stochastic demand. That of Herer and Tzur [9] mentioned above dealt with a deterministic environment. Two other studies by Axsäter [2] and Kukreja and Schmidt [13] deal with a stochastic environment but with continuous review. In Axsäter's study, transshipment costs are represented by a general function that allows the consideration of fixed transshipment costs; in Kukreja and Schmidt, transshipment costs are fixed, regardless of the number of units transshipped.

A single-item periodic transshipment model with fixed transshipment costs, a model that closely resembles ours, was developed by Estrugo [5]. Estrugo showed that the complete pooling policy is no longer optimal and described an optimal transshipment policy. For the replenishment policy, a branch-and-bound algorithm provides heuristic replenishment quantities. None of the existing literature provides an optimal replenishment policy to the single-item periodic transshipment problem with fixed transshipment costs.

In multi-item systems, where items may share a mutual capacitated resource or carry joint fixed costs, a significant cost reduction can be achieved by considering the inventory decisions regarding all items together. Very few papers address the transshipment problem in a multi-item environment. Archibald et al. [1] studied the multi-item inventory system with lateral transshipments in a periodic model with two retailers and limited capacity. In their model, transshipments can occur at any time during the period and the total demand in the period is unknown when the transshipment occurs. Wong et al. [23] considered a two-location, multi-item repairable spare parts inventory system in a continuous review model. Wong et al. [22] subsequently extended this model to include multiple locations. Their objective was to minimize total costs subject to a joint set of service level constraints. While Wong et al. [22] assume a complete pooling transshipment policy, Kranenburg and Van Houtum [11] apply a network structure that allows for partial pooling in a multi-item, multilocation system. As evident from the above review, none of the multi-item transshipment models investigated in the literature considered non-negligible, fixed transshipment costs, as we do in this article.

The rest of the article is organized as follows. In Section 2, we describe the problem's formulation and notation and in Section 3, we present the problem's optimal transshipment policy. In Section 4, we derive optimality conditions for the replenishment policy in the single-item problem, discuss the characteristics of the expected profit function (Section 4.1) and perform a numerical analysis on the problem's parameters (Section 4.2). In Section 5, we extend the optimality conditions to the multi-item problem, develop a heuristic (Section 5.1) as well as upper bounds on the expected profit (Section 5.2). In Section 6, we present a numerical study that demonstrates the heuristic's effectiveness and Section 7 contains our conclusions.

2. PROBLEM FORMULATION AND NOTATION

We consider a system consisting of two retailers facing stochastic demand for n items. We analyze the single period problem where the demand is independent among different retailers and items. Extensions to multiple periods and dependent demands are briefly discussed in the Conclusions section. At the beginning of the period, each retailer places an order to the supplier. The order is received and then the demand at each retailer for each item is realized. Before demand is satisfied, it is possible to transship units from retailers with surplus units to retailers with a shortage. At the end of the period, after demand has been fulfilled (directly or through transshipments), revenues and costs are incurred and the total profit in the system is calculated. There are revenues obtained for each item sold and a salvage value for each item left at the end of the period. The cost components include purchase costs from the supplier and a penalty cost for each unit of unmet demand. All parameters may be retailer and item-dependent. The goal is to determine replenishment and transshipment policies such that the total centralized expected profit of both retailers is maximized. This model generalizes the work of Tagaras [20] by relaxing the assumption of negligible fixed transshipment costs and by considering multiple items.

Our notation is similar to that of Rudi et al. [18], generalized to the case with fixed transshipment costs and multiple items, in a centralized system. We refer to the two retailers as 1 and 2 or i and j, when discussed in general and use the index k to denote the item type. In a system with n different items, each retailer $i (i = 1,2)$ is characterized by:

- c_{ik} Replenishment cost per unit of item k purchased from the supplier
- r_{ik} Revenue per unit sold of item k
- p_{ik} Penalty cost for every unit of unmet demand of item k
- s_{ik} Salvage value per unit of item k left at the end of the period
- D_{ik} A random variable that represents the demand of item k
- d_{ik} Demand realization of item k
- $f_{ik}(.)$ Demand density function of item k
- $F_{ik}(.)$ Demand cumulative distribution function of item k

In addition, for $i \neq j$, $(j = 1,2)$, we define:

- τ_{ijk} Cost per unit of item k transshipped from retailer i to retailer j
- A_{ij} Fixed cost for transshipping items from retailer i to retailer j

For each retailer i, we also denote:
$v_{ik} \equiv r_{ik} + p_{ik}$ as the marginal value of an additional sale of item k.

To avoid trivialities, we assume (as in previous studies on transshipments) the following relationships with respect to the parameters of each retailer i and each item k:

$$r_{ik} > c_{ik}$$
$$p_{ik} \geq 0$$

and the following inequalities for all $i,j \neq i$ and all k:

$$\tau_{ijk} \geq 0$$
$$c_{ik} \leq c_{jk} + \tau_{jik}$$
$$s_{jk} \leq s_{ik} + \tau_{ijk}$$
$$v_{jk} \leq v_{ik} + \tau_{ijk}$$

The second inequality ensures that it is not profitable for one retailer to purchase units through another retailer. The third and fourth inequalities ensure that transshipment will occur only when one retailer has surplus units and the other is facing a shortage. We also assume that the fixed transshipment costs are non-negative, that is, $A_{ij} \geq 0$ for all $i,j \neq i$.

The decision variables include the replenishment quantities of retailer i for items 1,...,n, denoted by $\vec{Q}_i \equiv (Q_{i1}, \ldots, Q_{in}) \forall i$ and the quantities of items 1,...,n transshipped from retailer i to retailer j, denoted by $\vec{T}_{ij} \equiv (T_{ij1}, \ldots, T_{ijn}) \forall i, j \neq i$. Thus, Q_{ik} is the replenishment quantity of retailer i for item k and T_{ijk} is the quantity of item k transshipped from retailer i to retailer j, $(\forall i, j \neq i)$. We also denote:

$\boldsymbol{Q_k} \equiv (Q_{1k}, Q_{2k})$	Replenishment quantities of item k at the two retailers
$\vec{\boldsymbol{Q}} \equiv (\vec{Q}_1, \vec{Q}_2)$	Replenishment quantities of all items at the two retailers
$\vec{\boldsymbol{T}} \equiv (\vec{T}_{12}, \vec{T}_{21})$	Transshipment quantities of all items in both directions

For each retailer i, we define the following auxiliary random variables:

$R_{ik} = \min(D_{ik}, Q_{ik} + T_{jik} - T_{ijk})$	Number of units sold of item k
$Z_{ik} = (D_{ik} - Q_{ik} - T_{jik} + T_{ijk})^+$	Unmet demand of item k
$U_{ik} = (Q_{ik} - D_{ik} + T_{jik} - T_{ijk})^+$	Surplus units of item k

We also define for all i and $j \neq i$, the binary variable: $X_{ij} = \{ {1 \text{ if } \sum_{k=1}^{n} T_{ijk} > 0 \atop 0 \quad \text{otherwise}} \}$, which indicates whether a transshipment is performed from retailer i to retailer j, regardless of the items' identity and quantity. Then, the expected system profit is given by:

$$\pi(\vec{\boldsymbol{Q}}, \vec{\boldsymbol{T}}) = \sum_{k=1}^{n} \sum_{i=1}^{2} [E(r_{ik} R_{ik} - p_{ik} Z_{ik} + s_{ik} U_{ik} - \sum_{\substack{j=1 \\ j \neq i}}^{2} \tau_{ijk} T_{ijk}) - c_{ik} Q_{ik}] - \sum_{i=1}^{2} \sum_{\substack{j=1 \\ j \neq i}}^{2} E(A_{ij} X_{ij}) \tag{1}$$

and the objective is to maximize this profit, that is, max $\pi(\vec{\boldsymbol{Q}}, \vec{\boldsymbol{T}})$.

An optimal solution of this problem consists of interrelated optimal transshipment and replenishment policies. We present the former in Section 3 and the latter in Sections 4 and 5.

3. OPTIMAL TRANSSHIPMENT POLICIES

As mentioned in the previous section, transshipments occur after orders have been placed and received and demand realized. Thus, an optimal transshipment policy is a function of the replenishment quantities for all items of retailers 1 and 2 $(\vec{Q}_1, \vec{Q}_2)$ and the demand realization for all items of these retailers $(d_{1k}, d_{2k}, k = 1, \ldots, n)$. Although the replenishment and transshipment decisions are interrelated, the transshipment policy presented below is optimal for any given replenishment quantities, thus, enabling a sequential analysis. As stated in the previous section, the conditions assumed regarding the problem's parameters ensure that transshipments will occur only when one retailer has surplus units while the other is facing a shortage. Consequently, in the analysis below, we only consider transshipments of this type.

We define a_{ijk} as the marginal profit per unit of item k (k=1...n) transshipped from retailer i to retailer j given that the unit transshipped is a surplus at retailer i and covers a shortage at retailer j:

$$a_{ijk} \equiv r_{jk} + p_{jk} - s_{ik} - \tau_{ijk} = v_{jk} - s_{ik} - \tau_{ijk} \quad i \neq j$$

Thus, the additional system profit from transshipping T_{ijk} units of item k from i to j is $a_{ijk} T_{ijk}$. Summing this expression over all items and subtracting the fixed transshipment cost (paid once for all items), the additional system profit is:

$$\textstyle\sum_{k=1}^{n} a_{ijk} T_{ijk} - A_{ij} \quad i \neq j$$

Therefore, it is profitable to transship units from i to j only when the above expression is non-negative, that is, when:

$$\textstyle\sum_{k=1}^{n} a_{ijk} T_{ijk} - A_{ij} \geq 0 \quad i \neq j \tag{2}$$

For simplicity of analysis and w.l.o.g., we assume $a_{ijk} > 0 \; \forall k, \forall i \neq j$. If $a_{ijk} < 0$ for some k and $i \neq j$, it is clearly not desirable to transship item k from i to j, and if $a_{ijk} = 0$, we are indifferent as to whether the item should be transshipped or not. The implications of these cases on the replenishment policy can be handled with a few model adaptations, but they complicate the analysis, and thus, w.l.o.g. are ignored.

We define for each item k:

$$q_{ijk} \equiv \frac{A_{ij}}{a_{ijk}} \tag{3}$$

Note from Condition (2) that q_{ijk} represents the minimum transshipment quantity of item k from i to j, *if transshipped* *alone*. Thus, this quantity also represents the minimum transshipment quantity in the single-item case (when dropping the index k). In the multi-item case considered here, due to possible transshipments of other items, a lower quantity of item k may be transshipped from i to j.

There exists a connection between the optimal transshipment policy in our case and the policy when no fixed transshipment costs exist (in which case the solution for each item is obtained independently of the other items). In the absence of fixed transshipment costs, the optimal transshipment policy was defined for the single-item case by Krishnan and Rao [12] and named the *complete pooling policy* by Tagaras [20]. Let $\hat{T}_{ij}$ denote the transshipment quantity according to the complete pooling policy in the single-item case. Then, omitting the index k from Q_{ik} and d_{ik} $\forall$ i to represent the order quantity and the demand in the single-item case, respectively, the transshipment quantity is the minimum between the surplus of one and the shortage of the other:

$$\hat{T}_{ij} \equiv \min[(Q_i - d_i)^+, (d_j - Q_j)^+] \quad i \neq j \tag{4}$$

In the presence of fixed transshipment costs, it is not always profitable to transship units of a certain item even if one retailer has surplus units and the other is facing a shortage. As a result, the complete pooling policy is no longer optimal even in the single-item case. Nevertheless, the quantity defined by the complete pooling policy plays an important role in the definition of the optimal transshipment policy when fixed transshipment costs are present. Specifically, for the multi-item problem with fixed transshipment costs, we define the multi-item constrained complete pooling policy, as an extension of the complete pooling policy incorporating both the existence of multiple items and fixed transshipment costs. A special case of this policy was introduced by Estrugo [5] for the single-item problem with fixed transshipment costs. We prove in Theorem 1 below that the multi-item constrained complete pooling policy is optimal. The policy is defined as follows:

Let $\hat{T}_{ijk}$ denote the optimal transshipment quantity of item k from i to j according to the complete pooling transshipment policy:

$$\hat{T}_{ijk} \equiv \min[(Q_{ik} - d_{ik})^+, (d_{jk} - Q_{jk})^+] \quad i \neq j \tag{5}$$

Then, the transshipment quantities according to the multi-item constrained complete pooling policy are:

$$T_{ijk} = \begin{Bmatrix} \hat{T}_{ijk} & \text{if } \sum_{k=1}^{n} a_{ijk}\, \hat{T}_{ijk} - A_{ij} \geq 0 \\ 0 & \text{otherwise} \end{Bmatrix} \quad \forall i, \forall j \neq i, \forall k \tag{6}$$

That is, if Condition (2) with transshipment quantities $\hat{T}_{ijk}$ (k = 1,..., n) holds, then we transship all potential transshipment quantities (of all items); otherwise, the transshipment is not profitable and we do not transship any unit of any item.

THEOREM 1: *Given any replenishment quantities $\vec{\boldsymbol{Q}}$, the multi-item constrained complete pooling policy defined by (6) is an optimal transshipment policy.*

PROOF: We have already claimed that it is profitable to transship units from i to j if and only if Condition (2) holds. In the first case of (6), the condition with transshipment quantities $\hat{T}_{ijk}$ holds. Moreover, if we transship less than $\hat{T}_{ijk}$ units of item k, we lose a positive gain of a_{ijk} to the system profit. By the assumptions on the parameters, every extra unit beyond $\hat{T}_{ijk}$ is also not profitable. In the second case of (6), Condition (2) with transshipment quantities $\hat{T}_{ijk}$ does not hold and increasing the transshipment quantities beyond this value will decrease the system profit even further. □

Notice that when $\sum_{k=1}^{n} a_{ijk}\, \hat{T}_{ijk} - A_{ij} = 0$, we are indifferent about whether to implement the transshipment or not. Accordingly, the optimal transshipment policy, in general, is not unique. Noting that under the optimal transshipment policy, the transshipment quantity T_{ijk} is a random variable that depends on the replenishment quantities $\vec{\boldsymbol{Q}}$, we can denote

$$\pi(\vec{\boldsymbol{Q}}, \vec{\boldsymbol{T}}) \equiv \pi(\vec{\boldsymbol{Q}}).$$

Finally note that when $A_{ij} = 0$ (i.e., there is no fixed transshipment cost), Condition (2) is always satisfied for every $T_{ijk} \geq 0$, and therefore, the optimal policy according to (6) is $T_{ijk} = \hat{T}_{ijk}$ for all k. Namely, the multi-item constrained complete pooling policy reduces to the complete pooling policy for each item.

Our next step is to determine the replenishment policies. The replenishment policy in the multi-item problem relies a great deal on that of the single-item problem, which by itself has not been characterized previously in the literature. Therefore, we start in Section 4 with a detailed analysis of the replenishment policy in the single-item case and discuss the multi-item case in Section 5.

4. THE REPLENISHMENT POLICY FOR THE SINGLE-ITEM PROBLEM

In this section, we analyze the replenishment policy for the single-item (i.e., when n = 1) problem with fixed transshipment costs. We use the notation presented in the previous sections but omit the index k that represents item identity. Note that A_{ij} remains unchanged. Our approach is to represent the expected profit as a function of the replenishment

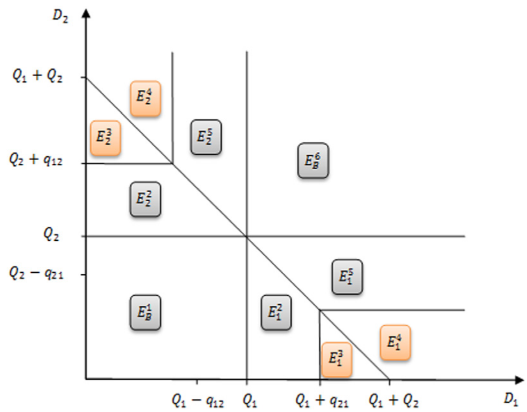


Figure 1. Graphical illustration of events (single item). [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

quantities using optimal transshipment quantities. We then derive an optimal replenishment policy based on first-order conditions. In Section 4.1, we provide a detailed characteristics and properties of the expected profit function and in Section 4.2, we perform a sensitivity analysis of the obtained solution.

Similarly to the definitions of a_{ijk} and q_{ijk} in the multi-item case, we define for the single-item case: $a_{ij} \equiv r_j + p_j - s_i - \tau_{ij} = v_j - s_i - \tau_{ij}$ and $q_{ij} \equiv A_{ij}/a_{ij}$. The optimal transshipment policy reduces in the single-item case to the following: find the complete pooling quantity $\hat{T}_{ij}$ according to (4), then if $\hat{T}_{ij} \geq q_{ij}$ set $T_{ij} = \hat{T}_{ij}$, else $T_{ij} = 0$, for all $i \neq j$.

Our next step is to determine the optimal replenishment quantities. In determining the desired replenishment quantities, our analysis takes into consideration, the transshipments that may be performed after demand realization, according to the constrained complete pooling policy.

To specify the expected profit function for given Q_1 and Q_2, we define all possible events, describing possible outcomes of the demand relative to the replenishment quantity values. Figure 1 graphically illustrates all possible events. It is based on the illustrations presented in Tagaras [20] and in Rudi et al. [18], but adapted to our model. In particular, the vertical line drawn at level $Q_1 + q_{21}$ denotes the demand level at Retailer 1 beyond which it may be worth transshipping units from Retailer 2 to this retailer. In addition, it is required that Retailer 2 should observe a demand of no more than $Q_2 - q_{21}$ (denoted by a horizontal line) to make this transshipment profitable. Together these two conditions define events E_1^3 and E_1^4. Similar events are defined by these conditions and the other vertical and horizontal lines shown in the figure. We use the following notation for the event numbers defined below: E_i^e= event e at retailer i, e = 1,..., 6, and: E_B^e= event e at Both retailers, e = 1,6.

For events E_i^1, E_i^3, and E_i^4 (i = 1,2), we introduce the notation for their respective probabilities, which in some cases depend also on $j \neq i$:

$$\begin{aligned}
\alpha_i(Q_i) &\equiv \Pr(D_i \leq Q_i) = \Pr(E_i^1), 1 - \alpha_i(Q_i) \\
&\equiv \Pr(Q_i < D_i) = 1 - \Pr(E_i^1) \\
\beta_{ji}(Q_i, Q_j) &\equiv \Pr(Q_i + Q_j - D_i < D_j \leq Q_j - q_{ji}) \\
&= \Pr(E_i^4), \quad j = 1, 2, j \neq i \\
\gamma_{ji}(Q_i, Q_j) &\equiv \Pr(Q_i + q_{ji} < D_i \leq Q_i + Q_j - D_j) \\
&= \Pr(E_i^3), \quad j = 1, 2, j \neq i
\end{aligned}$$

Note that the rest of the events in the analysis below can be represented by the above notation.

For each retailer i = 1,2 and $j \neq i$, Table 1 summarizes all possible events and specifies optimal transshipment quantities for each event. Notice that the events presented in Table 1 cover all of the sample space (the sum of the event probabilities equals one) and that the probability of event E_i^1, $\alpha_i(Q_i)$, includes the symmetric events that refer to retailer j: E_j^2, E_j^3, E_j^4, E_j^5.

Through the event diagram and the associated optimal transshipment policy, it is possible to characterize for each event the marginal contribution to the profit function that results from increasing slightly (e.g., by one unit) the replenishment quantity at retailer i, Q_i. The marginal profit contribution for each possible event as a result of increasing Q_i (omitting the unit purchase cost c_i, common to all events) is described in Table 2.

By combining all possible events and their probabilities, and collecting terms, we obtain the partial derivative of the expected profit function with respect to Q_i:

$$\begin{aligned}
\frac{\partial \pi(Q_i, Q_j)}{\partial Q_i} &= v_i[\Pr(E_i^2) + \Pr(E_i^4) + \Pr(E_i^5) + \Pr(E_B^6)] \\
&\quad + (\tau_{ji} + s_j)[\Pr(E_i^3)] + s_i[\Pr(E_B^1) + \Pr(E_j^2) \\
&\quad + \Pr(E_j^3) + \Pr(E_j^5)] + (-\tau_{ij} + v_j)[\Pr(E_j^4)] \\
&\quad - c_i \quad i, j = 1, 2\, i \neq j \qquad (7)
\end{aligned}$$

Using the notation introduced in Table 1 and making the derivative (7) equal to zero, we obtain:

$$\begin{aligned}
\frac{\partial \pi(Q_i, Q_j)}{\partial Q_i} &= v_i[1 - \alpha_i(Q_i) - \gamma_{ji}(Q_i, Q_j)] \\
&\quad + (\tau_{ji} + s_j)\gamma_{ji}(Q_i, Q_j) \\
&\quad + s_i[\alpha_i(Q_i) - \beta_{ij}(Q_i, Q_j)] \\
&\quad + (-\tau_{ij} + v_j)\beta_{ij}(Q_i, Q_j) - c_i = 0 \\
&\quad i, j = 1, 2\, i \neq j \qquad (8)
\end{aligned}$$

Table 1. Events description and transshipment quantities.

Event	Description	Event probability	Optimal transshipment quantities
E_i^1	$D_i \le Q_i$	$\alpha_i(Q_i)$	$T_{ji} = 0$
E_i^2	$Q_i < D_i \le Q_i + q_{ji}$ and $D_j \le Q_i + Q_j - D_i$		$T_{ji} = 0, T_{ij} = 0$
E_i^3	$Q_i + q_{ji} < D_i \le Q_i + Q_j - D_j$	$\gamma_{ji}(Q_i, Q_j)$	$T_{ji} = D_i - Q_i, T_{ij} = 0$
E_i^4	$Q_i + Q_j - D_i < D_j \le Q_j - q_{ji}$	$\beta_{ji}(Q_i, Q_j)$	$T_{ji} = Q_j - D_j, T_{ij} = 0$
E_i^5	$Q_i + Q_j - D_j < D_i$ and $Q_j - q_{ji} < D_j \le Q_j$		$T_{ji} = 0, T_{ij} = 0$
E_B^6	$Q_i < D_i$ and $Q_j < D_j$		$T_{ji} = 0, T_{ij} = 0$

Table 2. The marginal addition to the profit function at each event.

Events	Use of an additional unit at retailer i	Marginal addition
$E_i^2, E_i^4, E_i^5, E_B^6$	The additional unit is sold at retailer i	v_i
E_i^3	The transshipment quantity from j to i is decreased by one, and an additional unit is salvaged at retailer j	$\tau_{ji} + s_j$
$E_B^1, E_j^2, E_j^3, E_j^5$	The additional unit is salvaged at retailer i	s_i
E_j^4	The additional unit is transshipped from i to j and sold at retailer j	$-\tau_{ij} + v_j$

Rearranging (8), the first order conditions are:

$$\alpha_i(Q_i) - \beta_{ij}(Q_i, Q_j)\frac{v_j - s_i - \tau_{ij}}{v_i - s_i} + \gamma_{ji}(Q_i, Q_j)\frac{v_i - \tau_{ji} - s_j}{v_i - s_i} = \frac{v_i - c_i}{v_i - s_i}\ i, j = 1, 2\ i \neq j$$

We note that the first-order conditions have the exact same structure as in Rudi et al. [18] except that the probabilities $\beta_{ij}(Q_i,Q_j)$ and $\gamma_{ji}(Q_i,Q_j)$ (denoted by Rudi et al. as $\beta_i(Q_i,Q_j)$ and $\gamma_i(Q_i,Q_j)$, respectively) are affected in our case by the minimal transshipment quantities q_{ij} and q_{ji} and are, therefore, more general. Indeed, if we set the fixed transshipment costs to zero, then q_{ij} and q_{ji} are also equal to zero, and we obtain the exact same expressions and first-order conditions.

We also notice that if the fixed transshipment costs are very high (A_{ij},$A_{ji} \to \infty$) then q_{ij},$q_{ji} \to \infty$ and thus, $\beta_{ij}(Q_i,Q_j) \to 0$, $\gamma_{ji}(Q_i,Q_j) \to 0\ \forall\, i,j \neq i$. In this case, the first-order conditions reduce to: $\alpha_i(Q_i) = \Pr(D_i \le Q_i) = \frac{v_i - c_i}{v_i - s_i}\ \forall i$ which is the solution for two independent newsvendor problems.

4.1. Characteristics and Properties of the Expected Profit Function

After deriving the first-order conditions, we need to understand whether the replenishment quantities which satisfy them are optimal. Toward that, we investigate the characteristics and properties of the expected profit function $\pi(Q_i,Q_j)$. Recall that for the problem without fixed transshipment costs, Robinson [16] and Herer et al. [10] showed that the equivalent expected cost function is convex, and therefore, the first-order conditions are sufficient for optimality. However, when non-negligible fixed transshipment costs exist, it is unknown whether this property continues to hold.

In this section, we investigate the properties of the expected profit function, and specifically provide sufficient conditions for concavity. We show that the expected profit function is not concave in general and characterize the range of replenishment quantities in which the concavity conditions hold. Since the optimal replenishment quantities are within the domain of the demand distribution functions, we limit ourselves in this article to the expected profit function within this domain only. All proofs of the results stated here appear in Appendix A.

The first lemma we present provides sufficient conditions for the concavity of the expected profit function over a limited range of the (Q_i,Q_j) domain.

LEMMA 1: *For D_i with pdf $f_i(d_i)$ and D_j with pdf $f_j(d_j)$, if $f_i(d_i)$ is concave within the range (a_i,b_i), $a_i < b_i$ and $f_j(d_j)$ is concave within the range (a_j,b_j), $a_j < b_j$, then the expected profit function is jointly concave within the range: $a_i + q_{ij} \le Q_i \le b_i - q_{ji}$ and $a_j + q_{ji} \le Q_j \le b_j - q_{ij}$.*

Since it is well known that the normal density function is concave within the range $(\mu - \sigma,\ \mu + \sigma)$, it is immediately apparent from Lemma 1 that when the demand follows a normal distribution, the expected profit function is concave within the range $\mu_i - \sigma_i + q_{ij} \le Q_i \le \mu_i + \sigma_i - q_{ji}$ and $\mu_j - \sigma_j + q_{ji} \le Q_j \le \mu_j + \sigma_j - q_{ij}$. Note that since these are only sufficient conditions, it is likely that the concavity range of the expected profit function is actually larger. Additional sufficient conditions are presented later in this section.

LEMMA 2: *For D_i with pdf $f_i(d_i)$ and D_j with pdf $f_j(d_j)$, if $f_i(d_i)$ and $f_j(d_j)$ are concave functions over their entire domain, then the expected profit function is jointly concave over its entire domain.*

Since the uniform density function is linear (hence concave) over its entire domain, we have the following interesting corollary:

COROLLARY 1: *When D_i and D_j are independent and uniformly distributed, that is, $f_i(d_i) \sim U[a_i, b_i], 0 \le a_i < b_i$ and $f_j(d_j) \sim U[a_j, b_j], 0 \le a_j < b_j$, the expected profit function is jointly concave in (Q_i, Q_j) over the entire domain.*

The next lemma presents our most general sufficient condition for concavity. It can be used to check whether joint concavity is satisfied for certain (Q_i, Q_j) values, and in particular for (Q_i, Q_j) values that satisfy the first-order conditions. Moreover, it is the basis for the rest of our analysis concerning more specific conditions. To simplify some expressions, we use the following definitions:

$$\theta_{ji}^- \equiv F_j(Q_j - q_{ij}), \quad \theta_{ji}^+ \equiv F_j(Q_j + q_{ij}) \quad \forall i, j \ne i.$$

Note that $0 \le \theta_{ji}^- \le \theta_{ji}^+ \le 1$

LEMMA 3: *For D_i with pdf $f_i(d_i)$ and D_j with pdf $f_j(d_j)$, the expected profit function is jointly concave over a range of Q_i, Q_j that satisfies*:

$$\begin{aligned}
&\frac{a_{ij} f_i(Q_i - q_{ij}) - (v_i - s_i) f_i(Q_i)}{a_{ij} f_i(Q_i - q_{ij}) - a_{ji} f_i(Q_i + q_{ji})} > \theta_{ji}^- \quad \text{if } a_{ij} f_i(Q_i - q_{ij}) - a_{ji} f_i(Q_i + q_{ji}) < 0 \quad \forall i, j \ne i \\
&\frac{a_{ij} f_i(Q_i - q_{ij}) - (v_i - s_i) f_i(Q_i)}{a_{ij} f_i(Q_i - q_{ij}) - a_{ji} f_i(Q_i + q_{ji})} \le \theta_{ji}^+ \quad \text{if } a_{ij} f_i(Q_i - q_{ij}) - a_{ji} f_i(Q_i + q_{ji}) > 0 \quad \forall i, j \ne i
\end{aligned} \tag{9}$$

Note that (9) refers to both cases of the condition. In the first case, the denominator of the left expression is negative; in the second, it is positive.

To obtain an insight on the meaning of Condition (9) and a clear characterization of the (Q_i, Q_j) values that satisfy it, we consider in the rest of this section, the case when demand at both retailers is normally distributed, that is, $f_i(x_i) = \frac{e^{-(x_i-\mu_i)^2/2\sigma_i^2}}{\sqrt{2\pi}\sigma_i}, f_j(x_j) = \frac{e^{-(x_j-\mu_j)^2/2\sigma_j^2}}{\sqrt{2\pi}\sigma_j}$. To simplify the notation and analysis, we assume from this point on that the cost parameters at both retailers are identical, referred to as the symmetric case. Note that the distribution parameters μ_i and σ_i are still retailer-dependent. A similar analysis can be derived for the nonsymmetric case. However, since the expressions obtained are more complex and additional cases need to be considered, they are not presented here.

For the normal distribution: $af_i(Q_i - q) - af_i(Q_i + q) < 0$ when $Q_i < \mu_i$ and $af_i(Q_i - q) - af_i(Q_i + q) > 0$ when $Q_i > \mu_i$. Therefore, for the symmetric case, when the demand is normally distributed, Condition (9) reduces to:

$$\frac{af_i(Q_i - q) - (a + \tau) f_i(Q_i)}{af_i(Q_i - q) - af_i(Q_i + q)} > \theta_{ji}^- \text{ if } Q_i < \mu_i \ \forall i, j\ i \ne j$$

$$\frac{af_i(Q_i - q) - (a + \tau) f_i(Q_i)}{af_i(Q_i - q) - af_i(Q_i + q)} \le \theta_{ji}^+ \text{ if } Q_i > \mu_i \ \forall i, j\ i \ne j \tag{10}$$

where we recall that a_{ij} was defined as the marginal profit per unit transshipped from retailer *i* to retailer *j*, which in the symmetric case is denoted by a and $a + \tau = v - s$ since $a = v - s - \tau$.

Lemma 4 presents equivalent conditions to those of (10), using Z_i and Z_j, the standardized replenishment quantities for retailers *i* and *j*, respectively (i.e., $Z_i = \frac{Q_i - \mu_i}{\sigma_i}$, $Z_j = \frac{Q_j - \mu_j}{\sigma_j}$). These equivalent conditions present more directly the relationship between the two standardized replenishment quantities that satisfy Condition (10). Lemma 4 is based on a detailed investigation of the left-hand side of (10), a function which we denote as $g_i(Q_i)$, that is, $g_i(Q_i) \equiv \frac{af_i(Q_i-q)-(a+\tau)f_i(Q_i)}{af_i(Q_i-q)-af_i(Q_i+q)}$. The detailed investigation of $g_i(Q_i)$ is provided in Appendix A.

LEMMA 4: *When D_i and D_j are independent and normally distributed, the cost parameters of both retailers are symmetric, and Z_i and Z_j are the standardized replenishment quantities, the expected profit function is jointly concave over a range of Z_i, Z_j which satisfy*:

$$-\infty \le Z_j \le \infty \quad \text{when} \quad -\frac{q}{2\sigma_i} + \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a + \tau}\right) \le Z_i \le \frac{q}{2\sigma_i} - \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a + \tau}\right) \forall i, j \ne i \tag{11}$$

$$Z_j < \Phi^{-1}(g_i(\mu_i + Z_i\sigma_i)) + \frac{q}{\sigma_j} \quad \text{when} \quad Z_i < -\frac{q}{2\sigma_i} + \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a + \tau}\right) \quad \forall i, j \ne i \tag{12}$$

$$Z_j \ge \Phi^{-1}(g_i(\mu_i + Z_i\sigma_i)) - \frac{q}{\sigma_j} \quad \text{when} \quad Z_i > \frac{q}{2\sigma_i} - \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a + \tau}\right) \quad \forall i, j \ne i \tag{13}$$

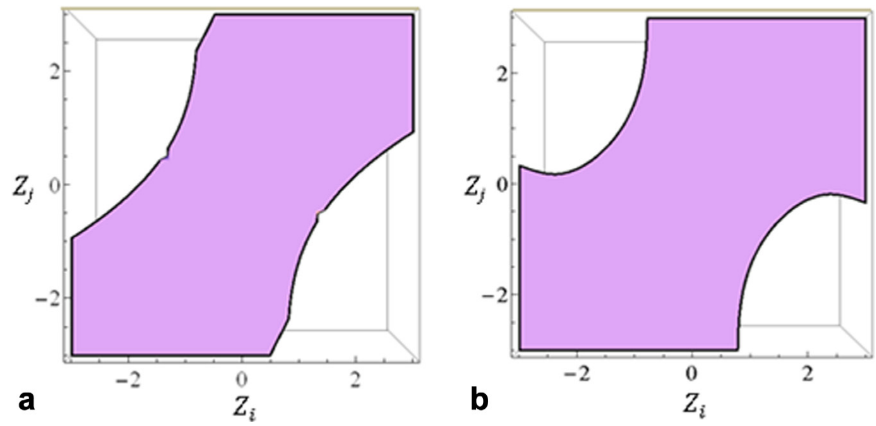


Figure 2. (a) Numerical example of the range where Condition (10) is satisfied, (b) Numerical example of the range where the expected profit function is concave. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Consider (11)–(13) when applied, for example, to retailer i, and note that these conditions cover all possible values of Z_i. We observe from (11) that for intermediate values of Z_i, the sufficient condition is satisfied for any Z_j. Note that since this condition needs to be satisfied for all i and $j \neq i$, it is satisfied for intermediate values of Z_i and Z_j. In each of the ranges (12) and (13), $g_i(\mu_i + Z_i\sigma_i)$ and $\Phi^{-1}(g_i(\mu_i + Z_i\sigma_i))$ increase in Z_i. Therefore, when Z_i in the range of (12) decreases, the upper bound on Z_j becomes tighter, so that the largest Z_j that satisfies the condition decreases. Similarly, when Z_i in the range of (13) increases, the lower bound on Z_j becomes tighter, so that the smallest Z_j that satisfies the condition increases. Considering these two conditions for all i and $j \neq i$, a similar behavior is obtained, although the combined restrictions may be tighter. From these observations, we conclude that in order for the sufficient conditions to be satisfied, Z_i and Z_j must increase or decrease simultaneously, so that they cannot obtain extreme opposite values. Based on this observation, we conjecture that in the symmetric case, since $Z_i = Z_j$ (as claimed more generally for the multi-item case in Section 5), concavity is always satisfied. While we could not prove this conjecture analytically, the numerical study presented at the end of this section supports it.

The range in which the conditions of Lemma 4 [or alternatively, Condition (10)] are satisfied, has the general structure demonstrated in Fig. 2a, drawn for values (Z_i, Z_j) within the range $-3 \leq Z_i \leq 3$ (the horizontal axis) and $-3 \leq Z_j \leq 3$ (the vertical axis), which is the relevant solution range. In this specific example: $c_i = 2, r_i = 7, p_i = 0, s_i = 0.8\ \forall i; \tau_{ij} = 1, A_{ij} = A = 200\ \forall i, \forall j \neq i$; $D_i \sim N(150, 30)$, $D_j \sim N(200, 60)$. The dark area indicates the range in which the above conditions are satisfied. Note that this range refers to sufficient conditions only and the range where the expected profit function is concave may be larger, but appears to have a similar shape. The actual range is numerically derived by examining the grid of (Z_i, Z_j) with high resolution, and testing for each combination whether the Hessian matrix is negative definite. The results, for the same example, are presented in Fig. 2b. A more detailed derivation is depicted in Fig. A2 (in Appendix A).

We see both analytically and numerically two "problematic" areas in which the expected profit function may not be concave. These areas are in the left upper and right bottom parts of the relevant solution range. This demonstrates our observation that the expected profit function is concave as long as Z_i and Z_j are not extreme opposite values. To illustrate the nonconcave areas of the expected profit function, we present in Fig. 3a, a three-dimensional demonstration of the expected profit function for the example described above. The general view appears to agree with the findings stated above. Since it is difficult to see in this figure the nonconcavity areas (at some opposite values of Z_i and Z_j), we present in Fig. 3b, a cut along the (nonoptimal) value $Z_j = 2.5$. The nonconcavity area can be clearly observed for small values of Z_i, which correspond to the upper left white area in Fig. 2.

The sensitivity of the shape of the concavity area with respect to the fixed transshipment cost is demonstrated in Fig. 4 for the same example described above. In this figure, the dark area indicates the concavity area for varying values of the fixed transshipment cost, A. When $A = 0$ and when A is very large, the expected profit function is concave over the entire domain. This is as expected because in these cases the expected profit function is known theoretically to be concave for any demand distribution. We further observe that for intermediate values of A, the nonconcave area is the largest and gradually decreases with smaller or larger values of A. The dark dot in these figures indicates the solution obtained by solving the first-order conditions and is observed to be included within the concavity range in all these cases.

To check whether the solution obtained by the first-order conditions is within the concavity area for a wider range of instances, we conducted an extensive numerical experiment.

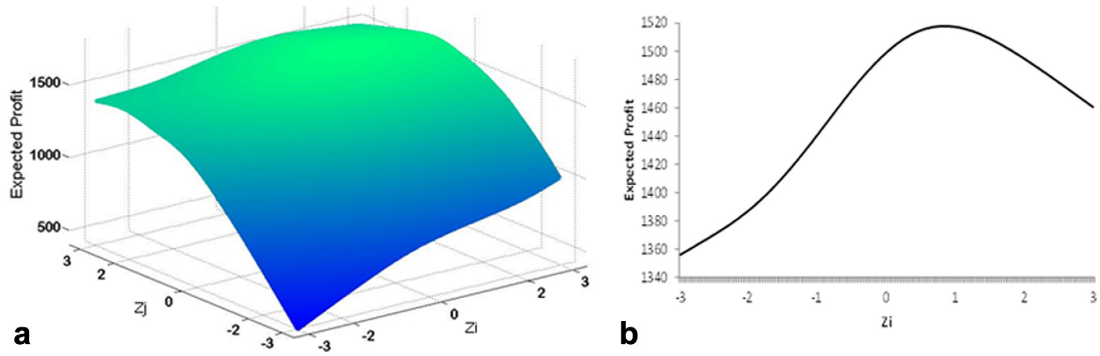


Figure 3. (a) A three dimensional demonstration of the expected profit function, (b) Demonstration of the expected profit function: cut along $Z_j = 2.5$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

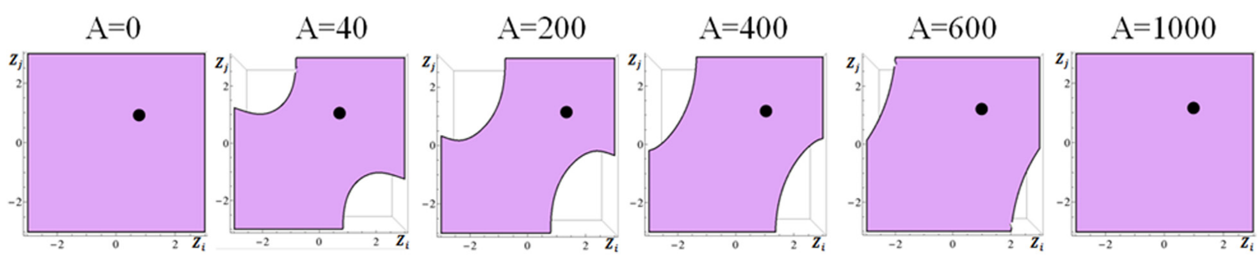


Figure 4. Concavity area for varying values of the fixed transshipment cost A. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

For each instance in this experiment, we solved the first order conditions and examined whether the solution obtained satisfies the most general sufficient concavity conditions for the nonsymmetric case, that is, Condition (9). In particular, we generated 19,086 instances of the problem by considering all combinations of the following parameters (excluding instances that do not satisfy the basic parameter assumptions, specified in Section 2):

$$\frac{v_i - c_i}{v_i - s_i} = 0.5, 0.7, 0.9, 0.95, v_i - s_i = 15, 25, 35,$$
$$a_{ij} = 5, 15, \ldots v_i - s_i, \mu_i = 100, \sigma_i = 10, 20, 30,$$
$$q_{ij} = 5, 15, \ldots q \leq \min(3\sigma_i, 3\sigma_j)\, \forall i, j \neq i$$

The above parameters are those that affect the first-order conditions as well as the concavity conditions. Note that other parameters that are not specified here are implied by the above parameters, for example, A_{ij} is implied by a_{ij} and q_{ij}. It can be observed that the range of values chosen for these parameters represent realistic settings. For example, note that $(v_i - c_i)/(v_i - s_i)$ represent the critical ratio of retailer i and σ_i is varied from a low value up to 30% of the expected demand (to avoid negative demand values). For all 19,086 generated instances, the solution that was obtained satisfied Condition (9). Therefore, in all of the above examined instances, representing a wide variety of parameters, the solution obtained was within the concavity range.

We conclude that although the expected profit function is not concave over its entire domain when normal distribution functions are used, we expect the solution obtained by the first-order condition to be optimal in most, if not all cases. In case of other demand distributions, concavity must be verified similarly using Condition (9), or by checking whether the Hessian matrix is numerically negative definite.

4.2. Sensitivity Analysis

To obtain insights on the solution's behavior, we present in this section, the results of additional experiments. Specifically, we examine the sensitivity of the solution's characteristics to changes in the fixed transshipment cost (the parameter which is the focus of this article) as well as to changes in the standard deviation (s.d.) of the demand. The demand at each location is assumed to be independent and normally distributed. For each experiment, we solved the first-order conditions and checked whether the solution satisfied Condition (9). In all the experiments performed in this section, the points obtained from solving the first-order conditions satisfied Condition (9), that is, they were within the concavity area. Thus, in all these experiments, we refer to the solution obtained from solving the first-order conditions as the optimal solution.

Table 3. Experiments 1–2 parameters.

D_1	D_2	c_1	c_2	r_1	r_2	p_1	p_2	s_1	s_2	τ_{12}	τ_{21}
$\sim N(150, 30)$	$\sim N(200, 60)$	2	2	7	7	0	0	0.8	1	1	1

Table 4. Experiment 1: nonidentical retailers, varying both fixed transshipment costs

$A_{12} = A_{21}$	q_{12}	q_{21}	Q_1^*	Q_2^*	$Q_1^* + Q_2^*$	Expected profit	Expected cost	Transshipment probability From 1 to 2 (%)	Transshipment probability From 2 to 1 (%)	Total transshipment probability (%)
0	0.0	0	165.00 (0.69)	250.86 (0.80)	415.86	1,639.10	110.90	14	25	38
40	7.7	8	172.46 (0.77)	249.05 (0.79)	421.51	1,627.00	123.00	12	12	24
80	15.4	16	174.97 (0.80)	251.20 (0.80)	426.17	1,619.40	130.60	8	6	15
120	23.1	24	175.94 (0.81)	253.50 (0.81)	429.44	1,615.10	134.90	5	3	9
160	30.8	32	176.26 (0.81)	255.26 (0.82)	431.53	1,611.60	138.40	3	2	5
200	38.5	40	176.30 (0.81)	256.45 (0.83)	432.74	1,611.30	138.70	2	1	3
240	46.2	48	176.22 (0.81)	257.18 (0.83)	433.40	1,608.60	141.40	1	0	1
280	53.8	56	176.18 (0.81)	257.42 (0.83)	433.60	1,609.20	140.80	1	0	1
$\mathbf{A} \to \infty$	∞		175.94 (0.81)	258.04 (0.83)	433.99	1,609.00	141.00	0	0	0

In Experiments 1 and 2, we vary the fixed transshipment costs, where the rest of the parameters are specified in Table 3. Note that the parameters of both retailers differ only in their demand distribution and salvage value. We first vary the fixed transshipment costs simultaneously in both transshipment directions (Experiment 1) and then we vary this cost in only one direction (from Retailer 1 to 2, Experiment 2). The results are reported in Tables 4 and 5, respectively.

For each case, we calculate the optimal replenishment quantities (Q_1^*, Q_2^*), and their associated expected profits and costs. We include a comparison of the total expected costs, since the expected profits may be largely affected by a fixed term that equals the sum (over both retailers) of the marginal profit per unit multiplied by the expected demand. Moreover, as in the newsvendor problem, in this transshipment problem too, maximizing the expected profit is equivalent to minimizing the expected costs (see Bonshtain-Noham, [3]). As for the (Q_1^*, Q_2^*) values, in addition to the absolute quantities, we indicate (in parenthesis) their cumulative probability distribution $F_i(Q_i^*)$, for comparison purposes. Finally, in the three right columns of each table, we present the transshipment probabilities in each direction separately as well as the total probability (the latter may deviate slightly from the sum of the separate probabilities, due to rounding).

As can be seen from Table 4, increasing the fixed transshipment costs simultaneously in both directions decreases (increases) the expected profit (cost) and increases the total replenishment quantities, as expected. (Minor deviations may be observed due to numerical errors.) The expected profit (cost) decreases (increases) at a larger rate when the value of A is small since this is where transshipments are used and contribute most. The contribution of transshipments is clearly related to the total transshipment probability in the system, specified in the rightmost column of the table. As expected, this probability decreases with the fixed transshipment costs. This risk pooling effect is also reflected in the total units replenished $Q_1^* + Q_2^*$, a quantity that increases as the fixed transshipment cost increases and the total transshipment probability decreases. Retailer 2 holds relatively more inventory than Retailer 1 (its $F(Q_2^*)$-value is larger) due to a larger salvage value and a higher demand s.d. When no fixed transshipment cost exists, the transshipment probability from Retailer 2 to 1 is higher; otherwise the opposite trend is observed, probably due to the lower q_{12} value (the threshold value for transshipment which is also affected by the salvage value), compared to q_{21}.

Table 5 shows similar results, although increasing the transshipment cost only from Retailer 1 to 2 makes it relatively more desirable to keep units at Retailer 2, so that Q_2^* increases as in the previous experiment, while Q_1^* decreases. This means that we rely less (more) on transshipments in the direction that is more (less) costly, as can also be observed from the transshipment probabilities. Moreover, the reduction in the total transshipment probability becomes moderate. The overall number of units is still increasing and the other effects are similar to those observed in Table 4. Figure 5a presents the effect of the fixed transshipment cost on the total transshipment probability for Experiments 1 and 2. As can be seen, the transshipment probability decreases at a faster rate when both fixed costs increase.

In Experiments 3 and 4, we examine the effect of demand variability. We set the fixed transshipment cost equal to 40 in both directions and vary the demand's s.d., first for both retailers (Experiment 3) and then for Retailer 1 only (Experiment 4). The rest of the parameters are the same as those of Experiments 1 and 2 (presented in Table 3) and the results are presented in Tables 6 and 7, respectively. As expected, it can be seen in both tables that as the demand uncertainty

Table 5. Experiment 2: nonidentical retailers, varying the fixed transshipment cost from 1 to 2

A_{12}	A_{21}	q_{12}	q_{21}	Q_1^*	Q_2^*	$Q_1^* + Q_2^*$	Expected profit	Expected cost	Transshipment probability From 1 to 2 (%)	Transshipment probability From 2 to 1 (%)	Total transshipment probability (%)
0	40	0.0	8	179.90 (0.84)	237.47 (0.73)	417.37	1631.10	118.90	22	7	29
40	40	7.7	8	172.46 (0.77)	249.05 (0.79)	421.51	1626.70	123.30	12	12	24
80	40	15.4	8	168.23 (0.73)	255.36 (0.82)	423.59	1622.90	127.10	6	15	21
120	40	23.1	8	165.77 (0.70)	258.82 (0.84)	424.59	1621.30	128.70	3	17	21
160	40	30.8	8	164.32 (0.68)	260.73 (0.84)	425.05	1620.30	129.70	2	18	20
200	40	38.5	8	163.48 (0.67)	261.78 (0.85)	425.26	1620.00	130.00	1	19	20
240	40	53.8	8	162.77 (0.66)	262.61 (0.85)	425.37	1619.70	130.30	0	20	20
280	40	53.9	8	162.77 (0.66)	262.61 (0.85)	425.37	1619.40	130.60	0	20	20

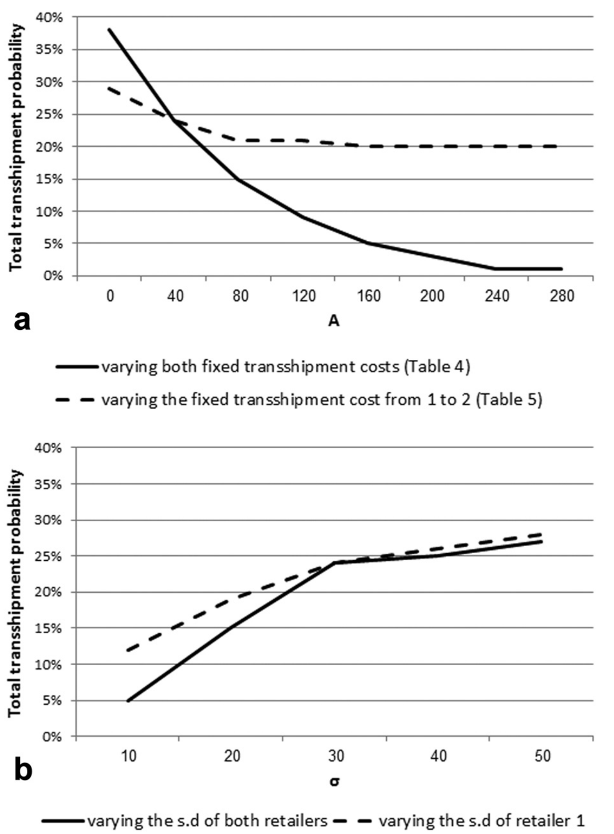


Figure 5. (a) The effect of fixed transshipment cost on the total transshipment probability, (b) the effect of s.d on the total transshipment probability.

increases, the sum of the replenishment quantities increase, the expected system profit (cost) decreases (increases), and the total transshipment probability increases. When we keep the s.d. of the demand at Retailer 2 constant and increase the s.d. of the demand at Retailer 1 (Table 7), Retailer 1's replenishment quantity increases and that of Retailer 2 decreases because the latter can now rely more on transshipments from Retailer 1. However, note that the transshipment probabilities increase in both directions, since in spite of the large replenishment quantity, the increased variability at Retailer 1 may create a significant shortage for this retailer.

Figure 5b presents the total transshipment probability when one or both s.d. values are varied (the varying values are presented on the x-axis). Generally, as expected, the total transshipment probability increases as the s.d. increases and the general shape of the increase appears (visually) to be concave. In particular, the increase in the transshipment probability appears to increase very slowly beyond a certain s.d. value, which is reasonable. This behavior is also apparent when the s.d. of only one retailer increases; as explained above, it increases the transshipment probability in both directions. Finally, transshipments are more likely to occur when both retailers have a medium level of variability (e.g., 30 for both) than when one retailer has a high variability and the second has a low variability (e.g., 60 and 10).

To conclude the numerical results of this section, several interesting observations can be made. As in the case with no fixed transshipment costs, the decision of each retailer depends on the parameters of both retailers, but in our case, the dependency on each direction's fixed transshipment cost is particularly strong. When the fixed transshipment costs are relatively low, the replenishment decisions and the system performance are more sensitive to changes in the parameters because then transshipments are used more. When demand variability increases, even for only one retailer, transshipments are used more in both directions. This mitigates on the system's increased costs, similar to what occurs in the case of no fixed transshipment costs.

5. THE REPLENISHMENT POLICY IN THE MULTI-ITEM PROBLEM

Now we seek to determine the first-order conditions for the multi-item case. In particular, we consider an inventory system that consists of two retailers facing a stochastic demand for n items as described in Section 2. As in the single-item case, we take into consideration the transshipments that may be performed following demand realization, according to the

Table 6. Experiment 3: normal distribution, nonidentical retailers, varying both SD

σ_1	σ_2	Q_1^*	Q_2^*	$Q_1^* + Q_2^*$	Expected profit	Expected cost	Transshipment probability From 1 to 2 (%)	Transshipment probability From 2 to 1 (%)	Total transshipment probability (%)
10	10	158.25 (0.80)	209.38 (0.83)	367.63	1718.90	31.10	2	3	5
20	20	164.80 (0.77)	217.74 (0.81)	382.53	1691.50	58.50	6	9	15
30	30	167.30 (0.72)	219.35 (0.74)	386.65	1663.90	86.10	12	13	24
40	40	176.29 (0.74)	234.24 (0.80)	410.53	1639.90	110.10	10	14	25
50	50	181.74 (0.74)	242.66 (0.80)	424.40	1615.10	134.90	11	16	27

Table 7. Experiment 4: normal distribution, nonidentical retailers, varying the SD from 1 to 2

σ_1	σ_2	Q_1^*	Q_2^*	$Q_1^* + Q_2^*$	Expected profit	Expected cost	Transshipment probability From 1 to 2 (%)	Transshipment probability From 2 to 1 (%)	Total transshipment probability (%)
10	60	160.15 (0.84)	254.47 (0.82)	414.62	1646.50	103.50	9	3	12
20	60	167.29 (0.81)	250.48 (0.80)	417.76	1637.20	112.80	11	8	19
30	60	172.46 (0.77)	249.05 (0.79)	421.51	1627.40	122.60	12	12	24
40	60	177.22 (0.75)	249.04 (0.79)	426.26	1615.90	134.10	12	14	26
50	60	182.03 (0.74)	249.88 (0.8)	431.91	1604.30	145.70	12	16	28

multi-item constrained complete pooling policy. With some adaptations, we use a possible event diagram as defined in the single-item case, for each item k separately. Figure 6 presents the graphical illustration of events for any item k, where we recall that q_{ijk}, defined in (3), represents the minimum transshipment quantity of item k from i to j, *if transshipped alone*. As can be seen, each item has 10 possible events. When we look at the whole system, there are 10^n possible events, representing all combinations of possible events for each of the items. Similar to the single-item case, we use the following notation for the event numbers defined below: E_{ik}^e= event e for item k at retailer i, e = 1,..., 6, and: E_{Bk}^e= event e for item k at Both retailers, e = 1,6. Table 8 describes the events with respect to a specific retailer (retailer i).

Due to the fact that in some of the events the decision of whether to transship a certain item depends on the demand realization of other items, these events are presented in the figure as a union of events. The union includes one subevent in which transshipment occurs and one in which it does not. For example, in event $E_{1k}^2 = E_{1k}^{2a} \cup E_{1k}^{2b}$, the shortage at Retailer 1 from item k is by itself insufficient for transshipment to this retailer. But, in subevent E_{1k}^{2a}, the potential transshipment quantities of other items from Retailer 2 to Retailer 1 do not make the transshipment in this direction profitable while in subevent E_{1k}^{2b}, they do.

For events E_{ik}^1, E_{ik}^{2b}, E_{ik}^3, E_{ik}^4, and E_{ik}^{5b} $\forall$ i = 1, 2, we introduce notations for their respective probabilities that are similar to those presented in the single-item model with the necessary adaptations to the multi-item case:

$$\alpha_{ik}(Q_{ik}) \equiv \Pr(E_{ik}^1), 1 - \alpha_{ik}(Q_{ik}) \equiv 1 - \Pr(E_{ik}^1)$$
$$\beta_{jik}(\boldsymbol{Q_k}) \equiv \Pr(E_{ik}^4)$$

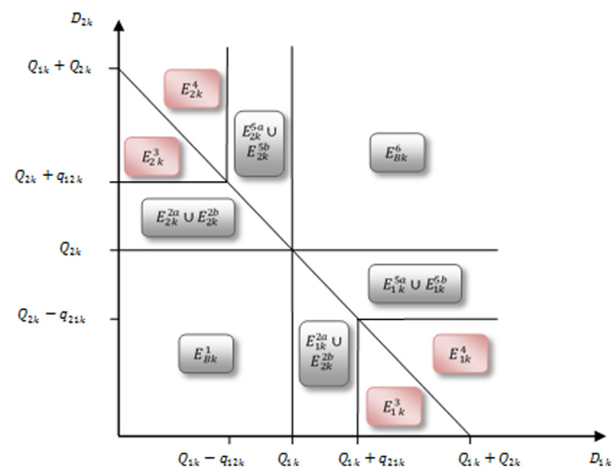


Figure 6. Graphical illustration of events (multi-items). [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

$$\gamma_{jik}(\boldsymbol{Q_k}) \equiv \Pr(E_{ik}^3)$$
$$\delta_{jik}(\vec{\boldsymbol{Q}}) \equiv \Pr(E_{ik}^{2b})$$
$$\varphi_{jik}(\vec{\boldsymbol{Q}}) \equiv \Pr(E_{ik}^{5b})$$

Table 8 summarizes all possible events for retailer i ($i \neq j$) and item k, and specifies optimal transshipment quantities for each event. Note that the probabilities of all events described in Table 8 sum up to one. Note also that the probability $\alpha_{ik}(Q_{ik})$ is a function of the replenishment quantity of item k at retailer i only (it includes the respective events that refer to retailer j: E_{jk}^{2a}, E_{jk}^{2b}, E_{jk}^3, E_{jk}^4, E_{jk}^{5a}, E_{jk}^{5b}). The probabilities $\beta_{jik}(\boldsymbol{Q_k})$ and $\gamma_{jik}(\boldsymbol{Q_k})$ are functions of the

Table 8. Events description and transshipment quantities—multi-item model

Event	Description	Probability	Transshipment
E_{ik}^{1}	$D_{ik} \leq Q_{ik}$	$\alpha_{ik}(Q_{ik})$	$T_{jik} = 0$
E_{ik}^{2a}	$Q_{ik} < D_{ik} \leq Q_{ik} + q_{jik}$ and $D_{jk} \leq Q_{ik} + Q_{jk} - D_{ik}$ and $\sum_{m=1}^{n} a_{jim}\hat{T}_{jim} - A_{ji} < 0$		$T_{jik} = 0, T_{ijk} = 0$
E_{ik}^{2b}	$Q_{ik} < D_{ik} \leq Q_{ik} + q_{jik}$ and $D_{jk} \leq Q_{ik} + Q_{jk} - D_{ik}$ and $\sum_{m=1}^{n} a_{jim}\hat{T}_{jim} - A_{ji} \geq 0$	$\delta_{jik}(\vec{\boldsymbol{Q}})$	$T_{jik} = D_{ik} - Q_{ik}, T_{ijk} = 0$
E_{ik}^{3}	$Q_{ik} + q_{jik} < D_{ik} \leq Q_{ik} + Q_{jk} - D_{jk}$	$\gamma_{jik}(\boldsymbol{Q_k})$	$T_{jik} = D_{ik} - Q_{ik}, T_{ijk} = 0$
E_{ik}^{4}	$Q_{ik} + Q_{jk} - D_{ik} < D_{jk} \leq Q_{jk} - q_{jik}$	$\beta_{jik}(\boldsymbol{Q_k})$	$T_{jik} = Q_{jk} - D_{jk}, T_{ijk} = 0$
E_{ik}^{5a}	$Q_{ik} + Q_{jk} - D_{jk} < D_{ik}$ and $Q_{jk} - q_{jik} < D_{jk} \leq Q_{jk}$ and $\sum_{m=1}^{n} a_{jim}\hat{T}_{jim} - A_{ji} < 0$		$T_{jik} = 0, T_{ijk} = 0$
E_{ik}^{5b}	$Q_{ik} + Q_{jk} - D_{jk} < D_{ik}$ and $Q_{jk} - q_{jik} < D_{jk} \leq Q_{jk}$ and $\sum_{m=1}^{n} a_{jim}\hat{T}_{jim} - A_{ji} \geq 0$	$\varphi_{jik}(\vec{\boldsymbol{Q}})$	$T_{jik} = Q_{jk} - D_{jk}, T_{ijk} = 0$
E_{Bk}^{6}	$Q_{ik} < D_{ik}$ and $Q_{jk} < D_{jk}$		$T_{ijk} = 0$

replenishment quantities of item k at both retailers while probabilities $\delta_{jik}(\vec{\boldsymbol{Q}})$ and $\varphi_{jik}(\vec{\boldsymbol{Q}})$ are functions of all the replenishment quantities from all items at both retailers.

Overall we have n diagrams, one for each item, similar to the one presented in Fig. 6. Through the event diagrams and the associated optimal transshipment policy, it is possible to characterize the marginal contribution to the profit function that results from slightly increasing (e.g., by one unit) the replenishment quantity of item k at retailer i, Q_{ik}. Based on the description of events in Table 8, the marginal profit contribution for each possible event as a result of increasing Q_{ik} (omitting the unit purchase cost c_i, common to all events) is described in Table 9.

By combining all possible events and their probabilities, and collecting terms, we obtain the partial derivative of the expected profit function with respect to Q_{ik}:

$$\begin{aligned}\frac{\partial \pi(\vec{\boldsymbol{Q}})}{\partial Q_{ik}} &= v_{ik}[\Pr(E_{ik}^{2a}) + \Pr(E_{ik}^{4}) + \Pr(E_{ik}^{5a} \cup E_{ik}^{5b}) \\ &\quad + \Pr(E_{k}^{6})] + (\tau_{jik} + s_{jk})[\Pr(E_{ik}^{2b})] + \Pr(E_{ik}^{3})] \\ &\quad + s_{ik}[\Pr(E_{Bk}^{1}) + \Pr(E_{jk}^{2a} \cup E_{jk}^{2b}) + \Pr(E_{jk}^{3}) \\ &\quad + \Pr(E_{jk}^{5a})] + (-\tau_{ijk} + v_{jk})[\Pr(E_{jk}^{4}) \\ &\quad + \Pr(E_{jk}^{5b})] - c_{ik} \; i, j = 1, 2 \; i \neq j, k = 1..n\end{aligned} \tag{14}$$

Using the notation introduced in Table 8 and making the derivative (14) equal to zero, we obtain (using the identity: $E_{Bk}^{1} = E_{ik}^{1} - (E_{jk}^{2a} \cup E_{jk}^{2b} \cup E_{jk}^{3} \cup E_{jk}^{4} \cup E_{jk}^{5a} \cup E_{jk}^{5b})$

$$\begin{aligned}\frac{\partial \pi(\vec{\boldsymbol{Q}})}{\partial Q_{ik}} &= v_{ik}[1 - \alpha_{ik}(Q_{ik}) - \delta_{jik}(\vec{\boldsymbol{Q}}) - \gamma_{jik}(\boldsymbol{Q_k})] \\ &\quad + (\tau_{jik} + s_{jk})[\delta_{jik}(\vec{\boldsymbol{Q}}) + \gamma_{jik}(\boldsymbol{Q_k})] \\ &\quad + s_{ik}[\alpha_{ik}(Q_{ik}) - \beta_{ijk}(\boldsymbol{Q_k}) - \varphi_{ijk}(\vec{\boldsymbol{Q}}) + (-\tau_{ijk} + v_{jk}) \\ &\quad \times [\beta_{ijk}(\boldsymbol{Q_k}) + \varphi_{ijk}(\vec{\boldsymbol{Q}})] - c_{ik} = 0 \\ &\quad i, j = 1, 2 \; i \neq j, k = 1..n\end{aligned} \tag{15}$$

Rearranging (15), the first-order conditions are:

$$\begin{aligned}&\alpha_{ik}(Q_{ik}) - [\beta_{ijk}(\boldsymbol{Q_k}) + \varphi_{ijk}(\vec{\boldsymbol{Q}})]\frac{v_{jk} - s_{ik} - \tau_{ijk}}{v_{ik} - s_{ik}} \\ &\quad + [\gamma_{jik}(\boldsymbol{Q_k}) + \delta_{jik}(\vec{\boldsymbol{Q}})]\frac{v_{ik} - \tau_{jik} - s_{jk}}{v_{ik} - s_{ik}} = \frac{v_{ik} - c_{ik}}{v_{ik} - s_{ik}} \\ &\quad i, j = 1, 2 \quad i \neq j, k = 1..n\end{aligned} \tag{16}$$

We can see that the structure of the first-order conditions for each item is very similar to that of the single item. The difference is in the additional probabilities $\delta_{jik}(\vec{\boldsymbol{Q}})$ and $\varphi_{jik}(\vec{\boldsymbol{Q}})$, that are associated with those cases in which the transshipment of an item occurs due to the transshipment of another item (or items) in the same direction. Note also that the probability expressions $\varphi_{jik}(\vec{\boldsymbol{Q}})$ and $\delta_{jik}(\vec{\boldsymbol{Q}})$ in (16) are defined with respect to the demand realization of each of the n items at each of the two retailers, and thus, are complicated expressions to evaluate. This is addressed in the next section by presenting a heuristic for obtaining replenishment quantities.

As in the single-item model, we notice that if the fixed transshipment costs are very high ($A_{ij} \to \infty$ and $A_{ji} \to \infty$) then Condition (2) never holds and thus: $\beta_{ijk}(\boldsymbol{Q_k}) \to 0$, $\gamma_{jik}(\boldsymbol{Q_k}) \to 0, \delta_{jik}(\vec{\boldsymbol{Q}}) \to 0, \varphi_{ijk}(\vec{\boldsymbol{Q}}) \to 0, \forall i, j \neq i, \forall k = 1 \ldots n$. In this case, the first-order conditions reduce again to $\alpha_{ik}(Q_{ik}) = \Pr(D_{ik} \leq Q_{ik}) = \frac{v_{ik} - c_{ik}}{v_{ik} - s_{ik}} \forall i, \forall k = 1 \ldots n$ which is the solution for $2n$ independent newsvendor problems.

Although we could not prove it analytically, we conjecture that due to the similar structure of the first-order conditions, the expected system profit function in the multi-item case has characteristics that are similar to those found in the single-item case. For two items with uniform demand distribution functions, for which we could solve the first-order conditions directly, we conducted an extensive search on the replenishment quantities to verify that this solution achieved the highest expected profit. The following parameters were used: for item 1: $D_{i1} \sim U[0, 500]$, $c_{i1} = 10, r_{i1} = 30, p_{i1} = 5, s_{i1} = 4, \forall i$, for item 2: $D_{i2}, \sim U[0, 400]$, $c_{i2} = 12$, $r_{i2} = 32$, $p_{i2} = 5$, $s_{i2} = 4$, $\forall i$ and $\tau_{ijk} = 1, \forall i, \forall j \neq i, \forall k$;

Table 9. The marginal addition to the profit function at each event

Events	Use of an additional unit of item k at retailer i	Marginal addition
$E_{ik}^{2a}, E_{ik}^{4}, E_{ik}^{5a}, E_{ik}^{5b}, E_{k}^{6}$	An additional unit is sold at retailer i	v_{ik}
E_{ik}^{2b}, E_{ik}^{3}	The transshipment quantity from j to i is decreased by one, and an additional unit is salvaged at retailer j	$\tau_{jik} + s_{jk}$
$E_{k}^{1}, E_{jk}^{2a}, E_{jk}^{2b}, E_{jk}^{3}, E_{jk}^{5a}$	An additional unit is salvaged at retailer i	s_{ik}
E_{jk}^{4}, E_{jk}^{5b}	An additional unit is transshipped from i to j and sold at retailer j	$-\tau_{ijk} + v_{jk}$

the parameters $A_{ij} = A_{ji}$ were varied between 1000, 1400, 1800,..., 3800. In all the cases examined, the replenishment quantities obtained from solving the first-order conditions achieved the highest expected profit and thus our conjecture was confirmed.

Since extending our concavity conditions to the multi-item problem was analytically intractable, we conclude this section with a numerical insight for the multi-item, normal distribution case, with identical cost parameters and demand distribution for all items at both retailers. We note that in this case, a solution with identical replenishment quantities, that is, $Q_{ik} = Q \ \forall\ i, \forall\ k$, which satisfies one of the $2n$ first-order optimality conditions, satisfies all other first-order optimality conditions, since the conditions are symmetric. Therefore, we focus on identical replenishment quantities for all retailers and items, obtaining a single-variable expected profit function. In Fig. 7, the expected profit function is calculated through simulation for one example, for 2, 3, 4, and 10 items, for a replenishment quantity between the range of the expected demand minus/plus three s.ds. In this example, the cost parameters are those of item 1 in the above example, where $A_{ij} = 1000 \ \forall\ i, \forall\ j \neq i$ and $D_{ik} \sim N(200,60)\ \forall\ i, \forall\ k$. Although this does not qualify as a proof and the graph refers to one example only, the functions appear (visually) to be concave, with a relatively steep (flat) slope for replenishment quantities that are smaller (larger) than the maximum point. This leaves the door open for future research that may attempt to find conditions for concavity in the multi-item case and possibly demonstrate that they are satisfied for a wide range of parameters.

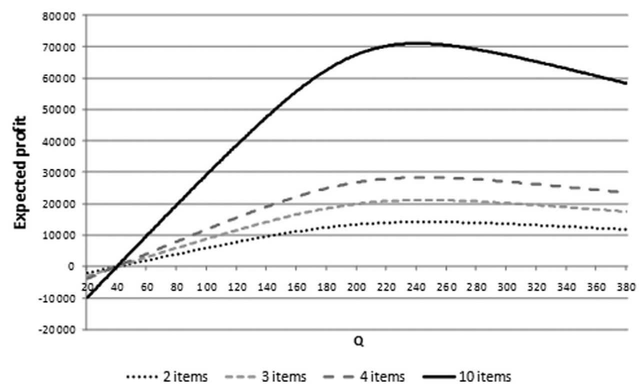


Figure 7. The expected profit function, multiple items, and identical retailers.

5.1. Heuristic for Finding Replenishment Quantities in the Multi-Item Problem

Solving the first-order conditions involves solving equations with multidimensional ($2n$) integrals. For systems with normally distributed demands and two or more items, this is computationally intractable, even when using state-of-the-art mathematical software. Using such software, we were able to solve only problems with two items and uniform demand distributions. Therefore, we suggest a simple heuristic to determine the replenishment quantities while still applying the optimal transshipment policy. Since the suggested heuristic relies on our ability to solve the single-item problem, it decomposes the multi-item problem into subproblems, one for each item. To overcome the fact that in the single-item problems the "cooperation" between the items is not considered, we use modified (lower) fixed transshipment costs in each single-item problem, where this modified cost depends on the number of items.

The heuristic is described as follows:

(i) Decompose the multi-item problem into n single-item problems, one for each item.
(ii) Solve each single-item problem separately. For each item k, use the modified fixed transshipment cost $\hat{A}_{ijk}$, where:

$$\hat{A}_{ijk} = \frac{f_{ijk} \cdot A_{ij}}{n}, \quad \forall i, j \neq i \tag{17}$$

f_{ijk} is a constant factor within the range: $1 \leq f_{ijk} \leq n$.

Note that according to (17), as the number of items increases, the allocated fixed cost per item decreases. This is because incurring the fixed transshipment cost may be beneficial even when each separate item transships according to only a smaller part of the original fixed transshipment cost. Yet, since not all items may necessarily transship together, we believe that the factor f_{ijk} should be larger than one so that even a subset of items can "cover" the fixed transshipment costs.

The impact of the factor f_{ijk}, together with the overall evaluation of the heuristic's performance, was examined through a series of numerical experiments, described in Section 6. Based on these experiments, and aspiring for simplicity, we recommend on a single value of f for all instances. Our results indicate that setting $f = 1.5$ provides the best overall results. We note, however, that slightly higher values of f also provide good results.

5.2. Upper Bounds on the Optimal System Profit

One way of evaluating our heuristic is to compare its expected profit to an upper bound on the optimal expected profit. We first show, in Lemma 5, that when using optimal transshipment quantities, the expected system profit can be expressed as the sum of three components. Those components are: the expected profit when no transshipments are performed (the newsvendor expected profit, denoted for item k by $E[P_k^{\mathrm{NV}}(\boldsymbol{Q_k})]$); the expected profit gained from transshipments, denoted by $E\,[P^{TR}\,(\vec{\boldsymbol{Q}})]$; and the expected fixed transshipment cost (with a minus sign), denoted by $E\,[F^{TR}\,(\vec{\boldsymbol{Q}})]$. These terms satisfy the following identities:

$$E[P_k^{\mathrm{NV}}(\boldsymbol{Q}_k)] = \sum_{i=1}^{2}[E(r_{ik}\min(D_{ik}, Q_{ik}) - p_{ik}(D_{ik} - Q_{ik})^+ + s_{ik}(Q_{ik} - D_{ik})^+) - c_{ik}Q_{ik}] \tag{18}$$

$$E[P^{\mathrm{TR}}(\vec{\boldsymbol{Q}})] = E\left[\sum_{k=1}^{n}(a_{jik}T_{jik} + a_{ijk}T_{ijk})\right] \tag{19}$$

$$E[F^{\mathrm{TR}}(\vec{\boldsymbol{Q}})] = A_{ji}E(X_{ji}) + A_{ij}E(X_{ij}) \tag{20}$$

And as stated above, we have the following result:

LEMMA 5: *The expected system profit $\pi(\vec{\boldsymbol{Q}})$ under the optimal transshipment policy can be written as the following sum*:

$$\pi(\vec{\boldsymbol{Q}}) = \sum_{k=1}^{n} E[P_k^{\mathrm{NV}}(\boldsymbol{Q_k})] + E[P^{\mathrm{TR}}(\vec{\boldsymbol{Q}})] - E[F^{\mathrm{TR}}(\vec{\boldsymbol{Q}})] \tag{21}$$

The proofs of all claims and lemmas presented in this section appear in Appendix B.

In addition, we use the following notation which is used in the lemmas below:

- $\vec{\boldsymbol{Q}}^{*(A=0)} \equiv$ Optimal replenishment quantities when $A_{ij} = A_{ji} = 0$
- $\vec{\boldsymbol{Q}}^{*(A>0)} \equiv$ Optimal replenishment quantities with strictly positive fixed transshipment costs A_{ij} and A_{ji}
- $\pi(\vec{\boldsymbol{Q}}, A = 0) \equiv$ Expected system profit with replenishment quantities $\vec{\boldsymbol{Q}}$ and fixed transshipment costs equal to zero. In a similar way $\pi(\vec{\boldsymbol{Q}}, A > 0)$ denotes the expected system profit with replenishment quantities $\vec{\boldsymbol{Q}}$ and strictly positive fixed transshipment costs A_{ij} and A_{ji}.
- $E[P^{TR}(\vec{\boldsymbol{Q}}, A = 0)] \equiv$ Expected profit from transshipments with replenishment quantities $\vec{\boldsymbol{Q}}$ and fixed transshipment costs equal to zero (when following the optimal transshipment policy). In a similar way $E[P^{TR}(\vec{\boldsymbol{Q}}, A > 0)]$ denotes the expected profit from transshipments with replenishment quantities $\vec{\boldsymbol{Q}}$ and strictly positive fixed transshipment costs A_{ij} and A_{ji}.

Claim 1 is used to prove Claim 2, which is used to prove Lemma 6 below.

CLAIM1 1: Using the same replenishment quantities ($\vec{\boldsymbol{Q}}$), the expected profit from transshipments (weakly) decreases in A and in particular:

$$E[P^{\mathrm{TR}}(\vec{\boldsymbol{Q}}^{*(A>0)}, A = 0)] \geq E[P^{\mathrm{TR}}(\vec{\boldsymbol{Q}}^{*(A>0)}, A > 0)] \tag{22}$$

CLAIM2 2: The expected system profit from using replenishment quantities $\vec{\boldsymbol{Q}}^{*(A>0)}$ in a system with $A = 0$ is (weakly) higher than the optimal expected profit of the same system with A_{ij} and $A_{ji} > 0$, that is,

$$\pi(\vec{\boldsymbol{Q}}^{*(A>0)}, A = 0) \geq \pi(\vec{\boldsymbol{Q}}^{*(A=0)}, A > 0) \tag{23}$$

We are now ready to define and prove our first upper bound.

LEMMA 6: *The system's optimal expected profit with $A_{ij} = A_{ji} = 0$ is an upper bound on the optimal expected profit of the same system with A_{ij}* and *$A_{ji} > 0$. We denote it as UB1, that is,*

$$\mathrm{UB1} \equiv \pi(\vec{\boldsymbol{Q}}^{*(A=0)}, A = 0) \geq \pi(\vec{\boldsymbol{Q}}^{*(A>0)}, A > 0)$$

UB1 is a trivial upper bound. In the following lemma, we present a tighter upper bound, denoted as UB2.

LEMMA 7: *Subtracting the optimal expected fixed transshipment costs in a system with strictly positive fixed transshipment costs A_{ij} and A_{ji} (denoted as $E[F^{TR}(\vec{\boldsymbol{Q}}^{*(A>0)}, A > 0)]$) from UB1 is an upper bound on the optimal expected profit. We denote it as UB2, that is,*

$$\mathrm{UB2} \equiv \mathrm{UB1} - E[F^{\mathrm{TR}}(\vec{\boldsymbol{Q}}^{*(A>0)}, A > 0)]$$

$$= \pi(\vec{Q}^{*(A=0)}, A = 0) - E[F^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A > 0)]$$
$$\geq \pi(\vec{Q}^{*(A>0)}, A > 0) \quad (24)$$

To use UB2, we need to calculate $E[F^{TR}(\vec{Q}^{*(A>0)}, A > 0)]$ and for this, we need to find the probabilities for executing transshipments in each direction. For two or more items, it is difficult to compute these probabilities (except for two items in the uniform case). Furthermore, they are based on the optimal replenishment quantities, which are also hard to compute (otherwise, we would not need an upper bound). To overcome this problem, we suggest the following alternative that leads to an additional bound: estimate the transshipment probabilities by simulation, using the replenishment quantities received from the heuristic solution. While using this estimated value does not necessarily result in an upper bound, we believe it would be close to UB2, and therefore, we refer to it as an approximate upper bound, denoted as $\widetilde{\mathrm{UB}}2$. A numerical study, in which $\widetilde{\mathrm{UB}}2$ is compared to the heuristic solution when $n \geq 2$, is presented in the next section.

We conclude this section with a theorem that demonstrates the advantage of considering items together, while performing transshipments in a multi-item system with fixed transshipment costs. For this purpose, we examine the average (per item) expected profit (i.e., $\pi(\vec{Q})/n$) in a system with identical items and retailers. While it is obvious that adding an item to a system increases the expected system profit, the effect on the profit per item is not so obvious. This is because as the number of items increases, we transship more, which increases the expected profit from transshipments ($E[P^{TR}(\vec{Q})]$), but also increases the expected fixed cost paid for transshipments ($E[F^{TR}(\vec{Q})]$). As we prove in the next theorem, the profit per item is (weakly) increasing with the number of items.

THEOREM 2: *The optimal average (per item) expected profit in a multi-item system with identical items and retailers is (weakly) increasing with the number of items, n. That is, if the optimal expected profit with n items is denoted by* $\pi_n(Q_n^*)$, *then:* $\frac{\pi_{n+1}(Q_{n+1}^*)}{n+1} \geq \frac{\pi_n(Q_n^*)}{n}$ *where* Q_n^* *is the optimal replenishment quantity for each item k and retailer i, in a system with n identical items and two identical retailers.* $(Q_{ik}^* = Q_n^* \, \forall i, \forall k)$.

6. THE HEURISTIC PERFORMANCE

In this section, we present numerical experiments related to our suggested heuristic. These experiments had several goals. First, we examined the sensitivity of the heuristic's performance to the choice of the f_{ijk} parameter. Second, the performance of the heuristic was evaluated, and finally, we examined the sensitivity of the heuristic performance to changes in the fixed transshipment costs and the number of items.

The impact of the factor f_{ijk} was examined through a series of numerical experiments in which we fixed $f_{ijk} = f \; \forall \; i, j \neq i, k$, for simplicity. We then varied its value, calculated the heuristic profit and compared it to the "best" profit found by an extensive search of the replenishment values. In particular, we tested the values f = 1, 1.5, 2,...,n for n = 4, 6, 8, and 10, for A = 1000, 2000, and 3000 for both symmetric and nonsymmetric instances. The exact setting of this experiment and the detailed results are presented in Appendix C.

Three important observations were obtained from the results. (1) In the symmetric case, all values of f in the above range provided profits that are extremely close to the best profit, where the largest deviation was 0.15%. Thus, any f within the above interval would provide close to optimal results. (2) In the nonsymmetric case, while the deviations of the expected profit of the heuristic solutions from those of the best solutions are higher than in the symmetric case, they are still relatively low, with an average deviation of 0.52% and a maximal deviation of 2.55%. (3) In the nonsymmetric case, setting f = 1.5 provided the overall best results, with a deviation of only 0.28% on average. Based on these observations, we recommend for the sake of simplicity setting f = 1.5 in all cases, even though this value did not achieve the best results in every instance. While it may be possible to slightly improve the performance of the heuristic by setting f values that depend on some of the problem's parameters (e.g., A and/or n), this requires a further extensive investigation, which we believe cannot improve the performance significantly. Finally, while recommending to set f = 1.5 in all cases, we note that slightly higher values of f also provide good results.

The above experiment also demonstrated that the heuristic with f = 1.5 performs very well, with an average gap from the best solution of 0.04% in the symmetric case and 0.28% in the nonsymmetric case. We note that performing such an extensive search is not practical; it is performed here only for comparison purposes.

The next experiment provides more information about the performance of the heuristic, as well as on the performance of the upper bound. Table 10 presents a comparison between the approximate upper bound $\widetilde{\mathrm{UB}}2$, the best expected profit and the heuristic expected profit for systems with identical items and retailers. The cost parameters are equal to those of Experiment C1 in Appendix C. $\hat{A}$ is the modified, fixed transshipment cost used in the heuristic, computed according to (17) with f = 1.5 and $\hat{q}$ is the minimal transshipment quantity for the single-item model when $A = \hat{A}$. $Q(H)$ and $\pi(Q(H))$ are the replenishment quantity and expected profit obtained from the heuristic, respectively, and in the next column, the one directional transshipment probability is presented. We examine systems with 2, 3, 4, and 10 items. The

Table 10. Comparison between $\widetilde{\text{UB}}2$, the best solution and the heuristic solution

	Best solution			Heuristic solution				Heuristic Gap	
	A	Q^*	$\pi(Q^*)$	$\hat{A}$	$Q(H)$	$\pi(Q(H))$	One direction transshipment probability (%)*	Δ from $\widetilde{\text{UB}}2$ (%)	Δ from best solution (%)
$n = 2$	0	238 (0.74)	14534.0	0	237.1 (0.73)	14534.0	36	0.46	0.00
	1000	247 (0.78)	14102.4	750	245.5 (0.78)	14101.8	11	2.40	0.00
	2000	249 (0.79)	13985.8	1500	249.9 (0.80)	13984.7	3	3.55	0.01
	3000	252 (0.81)	13962.9	2250	251.4 (0.81)	13963.1	1	3.97	0.00
	$A \to \infty$	252 (0.81)	13957.5	$\hat{A} \to \infty$	251.9 (0.81)	13958.1	0	3.97	0.00
$n = 3$	0	237 (0.73)	21803.4	0	237.1 (0.73)	21804.3	48	0.00	0.00
	1000	244 (0.77)	21183.5	500	243.1 (0.76)	21180.8	18	1.28	0.01
	2000	250 (0.80)	20993.5	1000	247.4 (0.79)	20987.4	5	2.95	0.03
	3000	250 (0.80)	20948.2	1500	249.9 (0.80)	20946.2	1	3.67	0.01
	$A \to \infty$	252 (0.81)	20938.0	$\hat{A} \to \infty$	251.9 (0.81)	20937.1	0	3.97	0.00
$n = 4$	0	237 (0.73)	29070.3	0	237.1 (0.73)	29069.4	59	0.00	0.00
	1000	243 (0.76)	28281.2	375	241.7 (0.76)	28275.6	25	1.06	0.02
	2000	249 (0.79)	28010.7	750	245.5 (0.78)	28002.3	8	2.67	0.03
	3000	249 (0.79)	27937.5	1125	248.2 (0.79)	27932.5	2	3.51	0.02
	$A \to \infty$	252 (0.81)	27916.4	$\hat{A} \to \infty$	251.9 (0.81)	27916.1	0	3.97	0.00
$n = 10$	0	237 (0.73)	72677.5	0	237.1 (0.73)	72674.3	89	0.00	0.00
	1000	240 (0.75)	71199.3	150	239.0 (0.74)	71195.5	59	0.43	0.01
	2000	244 (0.77)	70360.4	300	240.8 (0.75)	70336.4	31	1.56	0.03
	3000	249 (0.79)	69996.6	450	242.5 (0.76)	69938.4	14	2.68	0.08
	$A \to \infty$	252 (0.81)	69790.9	$\hat{A} \to \infty$	251.9 (0.81)	69790.3	0	3.97	0.00

* In all cases, we evaluate the transshipment probabilities based on the heuristic replenishment quantities $Q(H)$.

optimal replenishment quantities (Q^*) were found through an extensive search on integer values between the expected demand plus/minus three s.ds. For each value of Q in this interval, we calculated the expected profit through simulation and chose the best solution whose profit is $\pi(Q^*)$. Using symmetric parameters for each item and each retailer enabled us to search for only one value of Q^* instead of a series of $Q^*_{ik}{}'s$ (because, as claimed in Section 5, the replenishment quantities are equal for each item and each retailer). Consequently, we were able to keep the search time at a reasonable level. In general, searching for the best solution requires large computational efforts, thus, using a simple heuristic with a good performance is preferable. Using our upper bound, we verify that a certain solution is not far from optimality.

As can be seen from Table 10, the gap between the expected profits of the heuristic solutions and the corresponding best solutions do not exceed 0.08%, with an average gap of 0.01%. These results are similar to those obtained in the experiment presented in Appendix C. The replenishment quantities of both solutions are also very close, where the largest differences are obtained for intermediate values of the fixed transshipment cost. For a given number of items, the gap between the expected profit of the heuristic and $\widetilde{\text{UB}}2$ is relatively small for low values of A but increases in A up to a maximum value of about 3.97% when $A \to \infty$. Note that for $A \to \infty$, $\widetilde{\text{UB}}2$ is equal to the optimal solution of the problem with $A = 0$, since no transshipments occur, so that no expected fixed transshipment cost is reduced from UB1. Thus, this bound is weaker for large values of A. This observation is insensitive to the number of items. In other words, for a given number of items, the deterioration in the quality of $\widetilde{\text{UB}}2$ as A increases can be explained by the decrease in the actual transshipments in the optimal solution (as can be observed by the transshipment probability column), a decrease which is not accounted for in the calculation of $\widetilde{\text{UB}}2$.

The decrease in the transshipment probability and the impact of the number of items on this probability are depicted in Fig. 8a. As we would expect, the figure shows that for every value of fixed transshipment costs, the transshipment probability is higher as the number of items is larger. For relatively small values of fixed transshipment costs and a large number of items, transshipment will occur with a very high probability, since "covering" the fixed transshipment cost can be accomplished by many combinations of item identities and quantities. In Fig. 8b, the transshipment probability is depicted for an example with the same parameters except that $A = 1000$ and the s.d. of the demand distribution functions vary between 20 and 80. Again, the transshipment probability increases with the number of items, as well as with the s.d. value.

In the case of varying fixed transshipment costs, we calculated for each system the average (per item) expected profit (i.e., $\pi(\vec{Q})/n$). The numerical results, presented in Table 11,

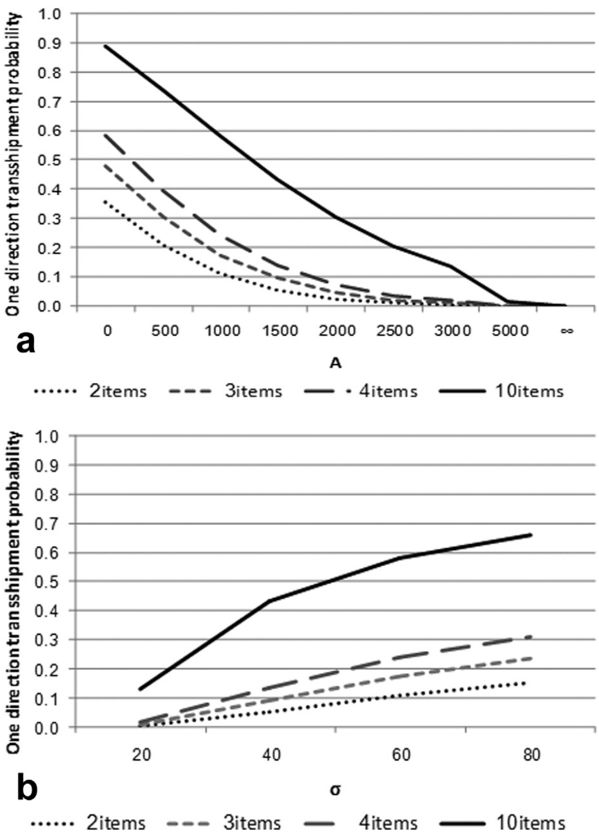


Figure 8. (a) The effect of fixed transshipment cost and number of items on transshipment probability, (b) The effect of standard deviation and number of items on transshipment probability (A = 1000).

Table 11. Profit per item

n	$A = 0$	$A = 1000$	$A = 2000$	$A = 3000$	$A \to \infty$
2	7267.5	7050.9	6992.3	6981.5	6979.0
3	7267.5	7060.3	6995.8	6982.1	6979.0
4	7267.5	7068.9	7000.6	6983.1	6979.0
10	7267.5	7119.5	7033.6	6993.8	6979.0

support the analytical result presented in Theorem 2 and it can be seen (for the same fixed transshipment cost), that as the number of items increases, the average (per item) expected profit increases.

7. CONCLUSIONS

In this work, we studied single and multi-item transshipment problems with fixed transshipment costs, which extends the single-item transshipment problem with negligible fixed transshipment costs studied in the literature. Our work enables transshipment strategy to be implemented to more complex and realistic environments than current methods are capable of coping with.

We proved that a policy referred to as the constrained complete pooling policy is an optimal transshipment policy for the multi-item problem (and for the single-item as a special case). To determine the optimal replenishment policy, we derived the first-order conditions for both the single and multi-item problems. In the single-item case, these conditions are accompanied by a thorough analysis of the characteristics of the expected profit function. Our analysis indicates that the solution obtained from solving the first-order conditions yields an optimal solution in most, if not all cases. For the multi-item case, the first-order conditions are computationally difficult to solve. Consequently, we suggest a heuristic based on the single-item solution. We also developed upper bounds on the optimal solution value. An extensive numerical study provided sensitivity analysis and managerial insights for both the single and multi-item problems. The heuristic performance was examined and shown to be very effective for a wide range of parameters. Finally, we proved that for the special case of identical items and retailers, the average (per item) expected profit is (weakly) increasing in the number of items. This demonstrates the advantage of considering items together in multi-item systems.

Throughout the article, we assumed a single period and independency between the demands. However, based on arguments similar to those presented in Herer et al. [10], we can show that the analysis and results continue to hold for multiperiod settings. As for dependent demands, the optimal transshipment policy and first-order conditions remain unchanged, albeit joint density functions need to be used and consequently computational difficulties may arise. Furthermore, investigation of the properties of the expected profit function and an analysis of the performance of our suggested heuristic in the case of dependent demands are interesting directions for future research.

Another useful extension to the current models is to consider multiple retailers, where, in addition to the decisions regarding transshipment quantities, one has to determine the origin and the destination of every unit transshipped. This may be achieved using mixed integer linear programming that may require substantial computational resources. Moreover, in such cases the optimal transshipment quantities may not be easy to represent in a closed-form formula, and thus, it may not be possible to express the expected profit function as a function of the replenishment quantities only. Another interesting direction is to consider the same problem in a decentralized system. In this case, the effect of the fixed transshipment cost on a desired coordination mechanism and the resulting system profit is not clear. We know that in the absence of fixed transshipment costs, this is already a complex problem that requires a nontrivial coordinating mechanism (Hanany et al., [7]). Accordingly, we believe that fixed transshipment costs would complicate the problem even more.

APPENDIX A

In this appendix, we present the properties of the expected profit function and prove the lemmas and corollaries stated in Section 4.1. We start by presenting the second-order partial derivatives, which are the basis for the analysis. We use Leibniz's rule, and obtain the following Hessian matrix of the expected profit function:

$$\begin{pmatrix} \begin{array}{l} \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i} = (s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i)[-\int_0^{Q_i-q_{ij}} f_i(y_i) f_j(Q_i + \\ Q_j - y_i) dy_i + f_i(Q_i - q_{ij})[1 - F_j(Q_j + q_{ij})]] + \\ (\tau_{ji} + s_j - v_i)[\int_{Q_i+q_{ji}}^{Q_i+Q_j} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i \\ - f_i(Q_i + q_{ji}) F_j(Q_j - q_{ji})] \end{array} & \begin{array}{l} \frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j} = \\ (v_j - \tau_{ij} - s_i)[\int_0^{Q_i-q_{ij}} - f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] \\ +(\tau_{ji} + s_j - v_i)[\int_{Q_i+q_{ji}}^{Q_i+Q_j} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] \end{array} \\ \begin{array}{l} \frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i} = \\ (v_i - \tau_{ji} - s_j)[\int_0^{Q_j-q_{ji}} - f_j(y_j) f_i(Q_j + Q_i - y_j) dy_j] \\ +(\tau_{ij} + s_i - v_j)[\int_{Q_j+q_{ij}}^{Q_j+Q_i} f_j(y_j) f_i(Q_j + Q_i - y_j) dy_j] \end{array} & \begin{array}{l} \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j} = \\ (s_j - v_j) f_j(Q_j) + \\ (v_i - \tau_{ji} - s_j)[-\int_0^{Q_j-q_{ji}} f_j(y_j) f_i(Q_j + Q_i - y_j) dy_j + \\ f_j(Q_j - q_{ji})[1 - F_i(Q_i + q_{ji})]] + \\ (\tau_{ij} + s_i - v_j)[\int_{Q_j+q_{ij}}^{Q_j+Q_i} f_j(y_j) f_i(Q_j + Q_i - y_j) dy_j \\ - f_j(Q_j + q_{ij}) F_i(Q_i - q_{ij})] \end{array} \end{pmatrix}$$

It is well known that a function is concave iff the Hessian matrix is negative definite over its entire domain. For the expected profit function, Conditions (A1) and (A2) need to be satisfied.

$$|H_1| = \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i} < 0 \quad \forall Q_i \tag{A1}$$

and:

$$|H_2| = \begin{vmatrix} \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i} & \frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j} \\ \frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i} & \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j} \end{vmatrix}$$

$$= \left(\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i}\right)\left(\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j}\right) - \left(\frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j}\right)\left(\frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i}\right) > 0 \quad \forall Q_i, Q_j \tag{A2}$$

LEMMA A1: *When Condition (A3) is satisfied then Conditions (A1) and (A2) are satisfied.*

$$(s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1 - F_j(Q_j + q_{ij})) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) F_j(Q_j - q_{ji}) < 0 \quad \forall i, j \neq i \tag{A3}$$

PROOF: We first show that when (A3) is satisfied, then (A1) is satisfied.

$$\begin{aligned} \frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i} &= (s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i)\Big[-\int_0^{Q_i-q_{ij}} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i + f_i(Q_i - q_{ij})[1 - F_j(Q_j + q_{ij})]\Big] \\ &\quad + (\tau_{ji} + s_j - v_i)\Big[\int_{Q_i+q_{ji}}^{Q_i+Q_j} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i - f_i(Q_i + q_{ji}) F_j(Q_j - q_{ji})\Big] \\ &\quad < (s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1 - F_j(Q_j + q_{ij})) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) F_j(Q_j - q_{ji}) < 0 \end{aligned} \tag{A4}$$

The first inequality of (A4) holds because $(v_j - \tau_{ij} - s_i)[-\int_0^{Q_i-q_{ij}} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] < 0$ and $(\tau_{ji} + s_j - v_i)[\int_{Q_i+q_{ji}}^{Q_i+Q_j} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] < 0$ due to the parameters assumptions. The second inequality of (A4) holds due to (A3). Similar to (A4), we obtain:

$$\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j} < (s_j - v_j) f_j(Q_j) + (v_i - \tau_{ji} - s_j) f_j(Q_j - q_{ji}) \times (1 - F_i(Q_i + q_{ji})) + (v_j - \tau_{ij} - s_i) f_j(Q_j + q_{ij}) F_i(Q_i - q_{ij}) < 0 \tag{A5}$$

so that Condition (A1) is satisfied. To show that Condition (A2) is satisfied, notice also that: $\frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j} = (v_j - \tau_{ij} - s_i)[\int_0^{Q_i-q_{ij}} - f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] + (\tau_{ji} + s_j - v_i)[\int_{Q_i+q_{ji}}^{Q_i+Q_j} f_i(y_i) f_j(Q_i + Q_j - y_i) dy_i] < 0$, since due to the parameters assumptions this expressions is the sum of two negative expressions. Similarly, $\frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i} < 0$. In addition: $(\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i}) - (\frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j}) = (s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1 - F_j(Q_j + q_{ij})) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) F_j(Q_j - q_{ji}) < 0 \, \forall i, j \neq i$ where the inequality is due to (A3). Similarly, $(\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j}) - (\frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i}) < 0$, and therefore, when (A3) holds, $|H_2| = (\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_i})(\frac{\partial\pi(Q_i,Q_j)}{\partial^2 Q_j}) - (\frac{\partial\pi(Q_i,Q_j)}{\partial Q_i \partial Q_j})(\frac{\partial\pi(Q_i,Q_j)}{\partial Q_j \partial Q_i}) > 0$, establishing Condition (A2). □

Now we can prove the results stated in Section 4.1. To simplify some notation, we use the following definition introduced in Section 4.1: $\theta_{ji}^- \equiv F_j(Q_j - q_{ij})$, $\theta_{ji}^+ \equiv F_j(Q_j + q_{ij}) \, \forall i, j \neq i$. Note that $0 \leq \theta_{ji}^- \leq \theta_{ji}^+ \leq 1$.

PROOF OF LEMMA 1: Recall that Condition (A3) is sufficient for concavity. Note that $\theta_{ji}^- = F_j(Q_j - q_{ij}) \leq F_j(Q_j + q_{ij}) = \theta_{ji}^+$ and consider the left hand side of (A3):

$$
\begin{aligned}
&(s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1-\theta_{ji}^{+}) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) Q_{ji}^{-} \\
&\quad < -(v_i - s_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1-\theta_{ji}^{+}) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) \theta_{ji}^{+} \\
&\quad \le -(v_i - s_i) f_i(Q_i) + (v_i - s_i) f_i(Q_i - q_{ij})(1-\theta_{ji}^{+}) + (v_i - s_i) f_i(Q_i + q_{ji}) \theta_{ji}^{+} \\
&\quad = (v_i - s_i)((1-\theta_{ji}^{+}) f_i(Q_i - q_{ij}) + \theta_{ji}^{+} f_i(Q_i + q_{ji}) - f_i(Q_i))
\end{aligned} \tag{A6}
$$

The first inequality of (A6) holds because $\theta_{ji}^{-} \le \theta_{ji}^{+}$ and the second inequality of (A6) holds due to the assumptions about the parameters: $v_i - s_i \ge v_j - \tau_{ij} - s_i \ge 0$ and $v_i - s_i \ge v_i - \tau_{ji} - s_j \ge 0$.

Now note that if $f_i(d_i)$ is concave within the range (a_i, b_i), $a_i < b_i$ then by definition of concavity and since $0 \le \theta_{ji}^{+} \le 1$, $(1-\theta_{ji}^{+}) f_i(Q_i - q_{ij}) + \theta_{ji}^{+} f_i(Q_i + q_{ji}) - f_i(Q_i) \le 0$, within the range $a_i + q_{ij} \le Q_i \le b_i - q_{ji}$, this establishes the result that the expression in (A6) is nonpositive. A similar argument holds for $f_j(d_j)$, θ_{ij}^{+} and $a_j + q_{ji} \le Q_j \le b_j - q_{ij}$, and thus, (A3) is satisfied. □

PROOF OF LEMMA 2: Suppose that the domain of $f_i(d_i)$ and $f_j(d_j)$ are (a_i, b_i), $a_i < b_i$ and (a_j, b_j), $a_j < b_j$, respectively. (In the case $a_i = b_i$ and $a_j = b_j$, the function is trivially concave over its entire domain, which consists of one point only.)

CASE A: $a_i + q_{ij} \le Q_i \le b_i - q_{ji}$ and $a_j + q_{ji} \le Q_j \le b_j - q_{ij}$. In this range, Condition (A3) is satisfied by Lemma 1.

CASE B: $a_i \le Q_i \le a_i + q_{ij}$ or $a_j \le Q_j < a_j + q_{ji}$. Consider (A6), which, by the proof of Lemma 1, when nonpositive, is sufficient for Condition (A3) to hold:

$$
\begin{aligned}
&(v_i - s_i)((1-\theta_{ji}^{+}) f_i(Q_i - q_{ij}) + \theta_{ji}^{+} f_i(Q_i + q_{ji}) - f_i(Q_i)) \\
&\quad < (v_i - s_i)((1-\theta_{ji}^{+}) f_i(a_i) + \theta_{ji}^{+} f_i(Q_i + q_{ji}) - f_i(Q_i)) \\
&\quad < 0 \quad \forall i, j \ne i
\end{aligned} \tag{A7}
$$

The first inequality of (A7) holds because in this case $f_i(Q_i - q_{ij}) = 0$. The second inequality of (A7) holds from the concavity of $f_i(d_i)$. Thus, (A3) holds.

CASE C: $b_i - q_{ji} < Q_i \le b_i$ *or* $b_j - q_{ij} < Q_j \le b_j$. Similar to case B, since in this case $f_i(Q_i + q_{ji}) = 0$, we obtain $\forall\, i,j \ne i$:

$$
\begin{aligned}
&(v_i - s_i)((1-\theta_{ji}^{+}) f_i(Q_i - q_{ij}) + \theta_{ji}^{+} f_i(Q_i + q_{ji}) - f_i(Q_i)) \\
&\quad < (v_i - s_i)((1-\theta_{ji}^{+}) f_i(Q_i - q_{ij}) + \theta_{ji}^{+} f_i(b_i) - f_i(Q_i)) < 0
\end{aligned} \tag{A8}
$$

Thus, (A3) holds. □

PROOF OF LEMMA 3: Similar to the proof of Lemma 1, recall that Condition (A3) is sufficient for concavity, and consider the left hand side of (A3):

$$
\begin{aligned}
&(s_i - v_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1-\theta_{ji}^{+}) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) \theta_{ji}^{-} \\
&\quad < -(v_i - s_i) f_i(Q_i) + (v_j - \tau_{ij} - s_i) f_i(Q_i - q_{ij})(1-\theta_{ji}^{-}) + (v_i - \tau_{ji} - s_j) f_i(Q_i + q_{ji}) \theta_{ji}^{-} \\
&\quad = -(v_i - s_i) f_i(Q_i) + a_{ij} f_i(Q_i - q_{ij})(1-\theta_{ji}^{-}) + a_{ji} f_i(Q_i + q_{ji}) \theta_{ji}^{-}
\end{aligned} \tag{A9}
$$

(A9) holds because $\theta_{ji}^{-} \le \theta_{ji}^{+}$ and by the definition of a_{ij} $\forall\, i,j \ne i$.

Rearranging the first part of (9), the following inequality holds:

$$
-(v_i - s_i) f_i(Q_i) + a_{ij} f_i(Q_i - q_{ij})(1-\theta_{ji}^{-}) + a_{ji} f_i(Q_i + q_{ji}) \theta_{ji}^{-} < 0 \tag{A10}
$$

Note that the expression to the left of inequality (A10) is exactly the third line in (A9). The left hand side of (A3) is smaller than the third line in (A9), which is nonpositive by (A10). Consequently, when the first part of (9) is satisfied, (A3) is satisfied. Similarly, rearranging the second part of (9), we obtain that the following inequality holds:

$$
-(v_i - s_i) f_i(Q_i) + a_{ij} f_i(Q_i - q_{ij})(1-\theta_{ji}^{+}) + a_{ji} f_i(Q_i + q_{ji}) \theta_{ji}^{+} \le 0 \tag{A11}
$$

Note that the expression to the left of inequality (A11) is exactly the second line in (A6). The left hand side of (A3) is smaller than the second line in (A6), which is nonpositive by (A11). Consequently, when the second part of (9) is satisfied, (A3) is satisfied. □

Recall that we define in Section 4.1: $g_i(Q_i) \equiv \frac{a f_i(Q_i - q) - (a+\tau) f_i(Q_i)}{a f_i(Q_i - q) - a f_i(Q_i + q)}$.

We provide here a detailed investigation of $g_i(Q_i)$ and analyze for which values of (Q_i, Q_j) it satisfies (10). The results of this analysis are used at the end of Section 4.1.

A Detailed Investigation of $g_i(Q_i)$

LEMMA A2: *If the demand at both retailers is normally distributed, the function $g_i(Q_i)$ has the following characteristics*:

(a) *The function has a vertical asymptote at $Q_i = \mu_i$*

(b) *The function is increasing in the range $Q_i < \mu_i$ and in the range $Q_i > \mu_i$*

(c) $g_i(Q_i) = 1$ *when* $Q_i = \mu_i - \frac{q}{2} + \frac{\sigma_i^2}{q} \cdot \ln(\frac{a}{v-s}) < \mu_i$

(d) $g_i(Q_i) = 0$ *when* $Q_i = \mu_i + \frac{q}{2} - \frac{\sigma_i^2}{q} \cdot \ln(\frac{a}{v-s}) > \mu_i$

(e) *When* $Q_i < \mu_i, g_i(Q_i) > 0$

(f) *When* $Q_i > \mu_i, g_i(Q_i) < 1$

PROOF:

(a) Vertical asymptote: $g_i(Q_i)$ is not defined when $a f_i(Q_i - q) - a f_i(Q_i + q) = 0$, that is, when $f_i(Q_i + q) = f_i(Q_i - q)$, that is, when $Q_i = \mu_i$.

(b) Taking the derivative of $g_i(Q_i)$ (both to the right and to the left of the asymptote) and rearranging it, results in the following expression:

$$\frac{dg_i\left(Q_i\right)}{dQ_i}=\frac{q\cdot \mathrm{e}^{\frac{2q^2+(Q_i-\mu_i)^2}{2\sigma_i{}^2}}\left(-\left(a+\tau\right)\cdot \mathrm{e}^{\frac{(q+Q_i-\mu_i)^2}{2\sigma_i{}^2}}-\left(a+\tau\right)\cdot \mathrm{e}^{\frac{(q-Q_i+\mu_i)^2}{2\sigma_i{}^2}}+2a\cdot \mathrm{e}^{\frac{(Q_i-\mu_i)^2}{2\sigma_i{}^2}}\right)}{-a\sigma_i{}^2\left(\mathrm{e}^{\frac{(q+Q_i-\mu_i)^2}{2\sigma_i{}^2}}-\mathrm{e}^{\frac{(q-Q_i+\mu_i)^2}{2\sigma_i{}^2}}\right)^2}$$

It is easy to see that the denominator of the derivative is negative because: $a > 0$, $\sigma_i^2 > 0$, and $\left(\mathrm{e}^{\frac{(q+Q_i-\mu_i)^2}{2\sigma_i{}^2}}-\mathrm{e}^{\frac{(q-Q_i+\mu_i)^2}{2\sigma_i{}^2}}\right)^2 > 0$

To see that the numerator of the derivative is also negative, we simplify the notation by defining: $\tilde{e}\equiv \mathrm{e}^{\frac{1}{2\sigma_i{}^2}}$ and obtain that the numerator is equal to:

$$\begin{aligned}
&q\cdot \tilde{\mathrm{e}}^{2q^2+(Q_i-\mu_i)^2}\left(-\left(a+\tau\right)\cdot \tilde{\mathrm{e}}^{(q+Q_i-\mu_i)^2}-\left(a+\tau\right)\cdot \tilde{\mathrm{e}}^{(q-Q_i+\mu_i)^2}+2a\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}\right)\\
&=q\cdot \tilde{\mathrm{e}}^{2q^2+(Q_i-\mu_i)^2}[-\left(a+\tau\right)\cdot \tilde{\mathrm{e}}^{q^2}\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}\cdot \tilde{\mathrm{e}}^{2q(Q_i-\mu_i)}-\left(a+\tau\right)\cdot \tilde{\mathrm{e}}^{q^2}\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}\cdot \tilde{\mathrm{e}}^{-2q(Q_i-\mu_i)^2}+2a\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}]\\
&=q\cdot \tilde{\mathrm{e}}^{2q^2+(Q_i-\mu_i)^2}[-\left(a+\tau\right)\tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}\left(\tilde{\mathrm{e}}^{q^2}\left(\tilde{\mathrm{e}}^{2q(Q_i-\mu_i)}+\tilde{\mathrm{e}}^{-2q(Q_i-\mu_i)}\right)\right)+2a\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}]\\
&=q\cdot \tilde{\mathrm{e}}^{2q^2+(Q_i-\mu_i)^2}[-\left(a+\tau\right)\tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}\left(\tilde{\mathrm{e}}^{q^2}\left(\tilde{\mathrm{e}}^{2q(Q_i-\mu_i)}+\tilde{\mathrm{e}}^{-2q(Q_i-\mu_i)}\right)-2\right)-2\tau\cdot \tilde{\mathrm{e}}^{(Q_i-\mu_i)^2}]
\end{aligned}$$

Now observe, as explained below:

$$q\cdot \tilde{\mathrm{e}}^{2q^2+(Q_i-\mu_i)^2} > 0 \tag{A12}$$

$$-\left(a+\tau\right)\tilde{\mathrm{e}}^{(Q_i-\mu_i)^2} < 0 \tag{A13}$$

$$\tilde{\mathrm{e}}^{q^2} \geq 0 \tag{A14}$$

$$\tilde{\mathrm{e}}^{2q(Q_i-\mu_i)}+\tilde{e}^{-2q(Q_i-\mu_i)} \geq 2 \tag{A15}$$

$$-2\tau\tilde{\mathrm{e}}^{(Q_i-\mu_i)^2} < 0 \tag{A16}$$

Inequalities (A12) and (A14) clearly hold. Inequalities (A13) and (A16) hold because $a + \tau > 0$ and $\tau > 0$, respectively. Inequality (A15) holds because $(\mathrm{e}^x + \mathrm{e}^{-x})$ is a convex function in x with a minimum at $x = 0$. In this case, the function value equals 2. We conclude from (A12) to (A16) that the numerator of the derivative is negative. Since both its numerator and the denominator are negative, the derivative of $g_i(Q_i)$ is positive and $g_i(Q_i)$ is increasing in Q_i.

(c) $g_i\left(Q_i\right)=\frac{af_i(Q_i-q)-(a+\tau)f_i(Q_i)}{af_i(Q_i-q)-af_i(Q_i+q)}=1$ when $\left(a+\tau\right)f_i\left(Q_i\right)=af_i\left(Q_i+q\right)$, and therefore, when

$$\frac{f_i\left(Q_i+q\right)}{f_i\left(Q_i\right)}=e^{\frac{(Q_i-\mu_i)^2-(Q_i-\mu_i+q)^2}{2\sigma_i^2}}=\frac{a}{(a+\tau)}$$

This implies that $\frac{2(Q_i-\mu_i)q+q^2}{2\sigma_i{}^2}=\ln\left(\frac{a}{a+\tau}\right)$ and hence $g_i(Q_i) = 1$ when $Q_i=\mu_i-\frac{q}{2}+\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)$. Since $\frac{a}{a+\tau}<1, \ln\left(\frac{a}{a+\tau}\right)<0$, therefore, $\mu_i-\frac{q}{2}+\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)<\mu_i$.

(d) $g_i\left(Q_i\right)=\frac{af_i(Q_i-q)-(a+\tau)f_i(Q_i)}{af_i(Q_i-q)-af_i(Q_i+q)}=0$ when $af_i\left(Q_i-q\right)-\left(a+\tau\right)f_i\left(Q_i\right)=0$, and therefore, when $\frac{f_i(Q_i)}{f_i(Q_i-q)}=e^{\frac{(Q_i-\mu_i-q)^2-(Q_i-\mu_i)^2}{2\sigma_i^2}}=\frac{a}{(a+\tau)}$, implying that $\frac{-2(Q_i-\mu_i)q+q^2}{2\sigma_i^2}=\ln\left(\frac{a}{a+\tau}\right)$ and hence $g_i(Q_i)=0$ when $Q_i=\mu_i+\frac{q}{2}-\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)$.

Similarly to the argument in (c), we obtain here that $\mu_i+\frac{q}{2}-\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)>\mu_i$

(e) Consider first the range in which $Q\leq\mu_i-\frac{q}{2}$. In this range: $f_i\left(Q_i-q\right)<f_i\left(Q_i\right)<f_i\left(Q_i+q\right)$ due to the normal distribution shape. Thus: $af_i\left(Q_i-q\right)-\left(a+\tau\right)f_i\left(Q_i\right)<0$ and $af_i\left(Q_i-q\right)-af_i\left(Q_i+q\right)<0$, implying that: $g_i\left(Q_i\right)=\frac{af_i(Q_i-q)-(a+\tau)f_i(Q_i)}{af_i(Q_i-q)-af_i(Q_i+q)}>0$ Then, since $g_i(Q_i)$ increases in Q_i for $Q_i < \mu_i$, $g_i(Q_i) > 0$ also when $\mu_i-\frac{q}{2}<Q_i<\mu_i$.

(f) Similarly to (e), consider the range $\mu_i+\frac{q}{2}\leq Q_i$. In this range: $f_i\left(Q_i+q\right)<f_i\left(Q_i\right)<f_i\left(Q_i-q\right)$ Thus, $af_i\left(Q_i-q\right)>af_i\left(Q_i+q\right)$ and $af_i\left(Q_i+q\right)<\left(a+\tau\right)f_i\left(Q_i\right)$, implying that the denominator of $g_i(Q_i)$ is positive and its numerator is smaller than it. Therefore, $g_i\left(Q_i\right)=\frac{af_i(Q_i-q)-(a+\tau)f_i(Q_i)}{af_i(Q_i-q)-af_i(Q_i+q)}<1$. Then, since $g_i(Q_i)$ increases in Q_i for each $\mu_i < Q_i$, $g_i(Q_i) < 1$ also when $\mu_i<Q_i<\mu_i+\frac{q}{2}$.

□

Figure A1 numerically demonstrates the structure of $g_i(Q_i)$ for different values of q (or A). In these examples, $\mu_i = 150, \sigma_i = 30, v = 7, \tau = 1, s = 0.8 \Rightarrow a = 5.2$. One can see the vertical asymptote at $Q_i = \mu_i = 150$ and that the function increases for each $Q_i < \mu_i$ and $Q_i > \mu_i$. In Figs. A1a and A1b, the points where $g_i(Q_i) = 0$ and $g_i(Q_i) = 1$ are clearly marked with large dots. In Fig. A1c, where $q \to 0$, $g_i(Q_i) > 1$ for all $Q_i > \mu_i$ and $g_i(Q_i) < 0$ for all $Q_i > \mu_i$. This case refers to the special case of negligible fixed transshipment costs, in which the expected profit function is known to be concave over the entire (Q_i, Q_j) range. As we show next, the derivation here provides another proof for that.

From Lemma 2, we obtain the following corollary, which is used in the subsequent proof of Lemma 4 that characterizes the areas of concavity.

COROLLARY A1:

$$0<g_i\left(Q_i\right)<1\,when\;Q_i<\mu_i-\frac{q}{2}+\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)\;Or$$
$$when\;Q_i>\mu_i+\frac{q}{2}-\frac{\sigma_i^2}{q}\cdot\ln\left(\frac{a}{a+\tau}\right)$$

$$g_i\left(Q_i\right) \geq 1 \; when \; \mu_i - \frac{q}{2} + \frac{\sigma_i^2}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) \leq Q_i < \mu_i$$

$$g_i\left(Q_i\right) \leq 0 \; when \; \mu_i < Q_i \leq \mu_i + \frac{q}{2} - \frac{\sigma_i^2}{q} \cdot \ln\left(\frac{a}{a+\tau}\right)$$

PROOF OF LEMMA 4: From corollary A1, when $-\frac{q}{2\sigma_i} + \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) \leq Z_i < 0$, $g_i\left(\mu_i + Z_i\sigma_i\right) \geq 1$, and when $0 < Z_i \leq \frac{q}{2\sigma_i} - \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a+\tau}\right)$, $g_i\left(\mu_i + Z_i\sigma_i\right) \leq 0$. Since $0 \leq \theta_{ji}^-, \theta_{ji}^+ \leq 1$ for any Z_j, when (11) holds, Condition (10) (a sufficient condition for concavity) holds. (Notice that when $Z_i = 0$, Condition (10) holds trivially.) Rearranging the first part of Condition (10) and using the standardized replenishment quantities, we obtain:

$$g_i\left(Q_i\right) = g_i\left(\mu_i + Z_i\sigma_i\right) > \theta_{ji}^- = F_j\left(Q_j - q\right) = \Phi\left(\frac{Q_j - q - \mu_j}{\sigma_j}\right) = \Phi\left(Z_j - \frac{q}{\sigma_j}\right)$$

$\Rightarrow \Phi^{-1}\left(g_i\left(\mu_i + Z_i\sigma_i\right)\right) > Z_j - \frac{q}{\sigma_j}$. Therefore, when (12) holds the first part of Condition (10) holds. Similarly, rearranging the second part of Condition (10), using the standardized replenishment quantities we obtain:

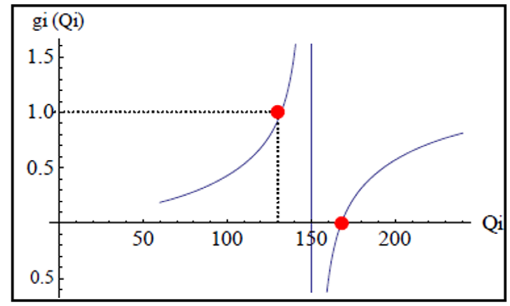
$$g_i\left(Q_i\right) = g_i\left(\mu_i + Z_i\sigma_i\right) \leq \theta_{ji}^+ = F_j\left(Q_j + q\right) = \Phi\left(\frac{Q_j + q - \mu_j}{\sigma_j}\right) = \Phi\left(Z_j + \frac{q}{\sigma_j}\right)$$

$\Rightarrow \Phi^{-1}\left(g_i\left(\mu_i + Z_i\sigma_i\right)\right) \leq Z_j + \frac{q}{\sigma_j}$. Therefore, when (13) holds, the second part of Condition (10) holds. Together, these three conditions cover the entire domain of Z_i, Z_j. □

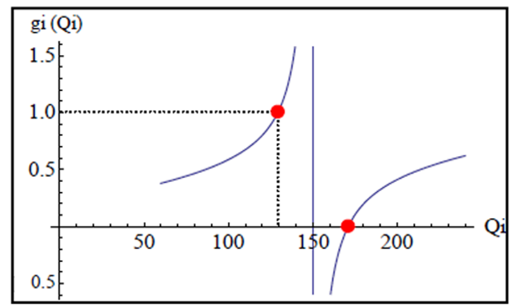
The area in which Condition (10) is satisfied was determined numerically by checking it for each value of (Z_i,Z_j) within the range $-3 \leq Z_i \leq 3$ (the horizontal axis) and $-3 \leq Z_j \leq 3$ (the vertical axis), which is the relevant solution range. The results (the dark area) are demonstrated in Figs. A2a–A2c.

In Fig. A2a, one can observe the ranges discussed above, that is, the range of Z_j obtained for any given Z_i. Note that when i and j are reversed, additional conditions need to be satisfied for (10) to hold since it is defined $\forall\, i,j \neq i$. In Fig. A2b, we observe the range when i and j are reversed, that is, the range of Z_i obtained for any given Z_j. Figure A2c presents the range in which (10) is satisfied $\forall\, i,j \neq i$ (the intersection of A2a and A2b), see further details below. Finally, Fig. A2d presents the range in which the Hessian matrix is numerically negative definite. This range is larger than the one in A2c, because (10) is a sufficient but not a necessary condition. We conclude that within the dark area of Fig. A2d, the expected profit function is concave and we prove it analytically for a subset of it presented in Fig. A2c. Conversely, the function is not concave outside the dark area of Fig. A2d. The parameters used in Fig. A2 are the same as in the example presented in Figs. 2 and A1, with $\mu_j = 200$, $\sigma_j = 60$, and $A = 200$. Therefore, in this example, $q = 15.4$, $-\frac{q}{2\sigma_i} + \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) = -0.6$, $\frac{q}{2\sigma_i} - \frac{\sigma_i}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) = 0.6$, $-\frac{q}{2\sigma_j} + \frac{\sigma_j}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) = -0.81$ and $\frac{q}{2\sigma_j} - \frac{\sigma_j}{q} \cdot \ln\left(\frac{a}{a+\tau}\right) = 0.81$.

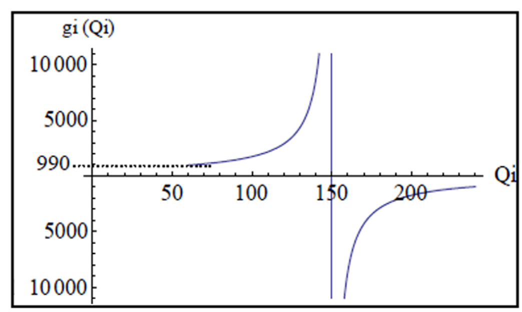
The areas that are labeled A and B in Fig. A2a are those in which $Z_i < -0.6$. While for the values of Z_j in the area labeled A, Condition (12) does not hold; it does hold, however, for the values of Z_j in the area labeled B. Since the area labeled C satisfies $-0.6 \leq Z_i \leq 0.6$ Condition (11) in this area holds for every Z_j. The areas which are labeled D and E are those in which $Z_i > 0.6$. While for the values of Z_j in the area labeled D, Condition (13) holds, for the values of Z_j in the area labeled E Condition (13) does not hold. The same characterization can be done for Fig. A2b.



(a): $q = 20$



(b): $q = 10$



(c): $q \rightarrow 0$

Figure A1. Numerical demonstration of $g_i(Q_i)$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

APPENDIX B

PROOF OF LEMMA 5: The expected system profit is given by:

$$\pi\left(\vec{Q}\right) = \sum_{k=1}^{n}\sum_{i=1}^{2}\left[E\left(r_{ik}R_{ik} - p_{ik}Z_{ik} + s_{ik}U_{ik} - \sum_{\substack{j=1 \\ j\neq i}}^{2}\tau_{ijk}T_{ijk}\right) - c_{ik}Q_{ik}\right] - \sum_{i=1}^{2}\sum_{\substack{j=1 \\ j\neq i}}^{2}E\left(A_{ij}X_{ij}\right)$$

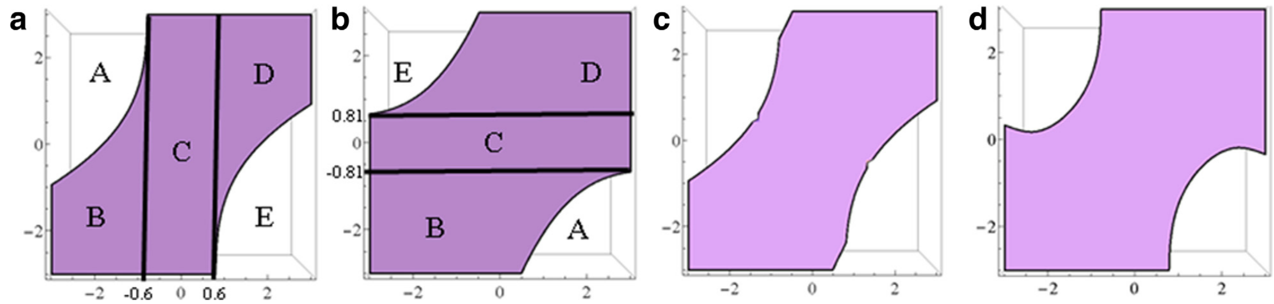


Figure A2. Demonstration of the concavity conditions. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Under the optimal transshipment policy presented in Section 3.2, the random variables R_{ik}, Z_{ik}, and U_{ik} can be written for all i and k as:

$$R_{ik} = \min\left(D_{ik}, Q_{ik} + T_{jik} - T_{ijk}\right) = \min\left(D_{ik}, Q_{ik}\right) + T_{jik} \quad i \neq j$$
$$Z_{ik} = \left(D_{ik} - Q_{ik} - T_{jik} + T_{ijk}\right)^{+} = \left(D_{ik} - Q_{ik}\right)^{+} - T_{jik} \quad i \neq j$$
$$U_{ik} = \left(Q_{ik} - D_{ik} - T_{ijk} + T_{jik}\right)^{+} = \left(Q_{ik} - D_{ik}\right)^{+} - T_{ijk} \quad i \neq j$$

Placing the above expressions in the expected system profit and collecting terms, we obtain:

$$\pi(\vec{Q}) = \sum_{k=1}^{n}\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}[E(r_{ik}\left(\min\left(D_{ik}, Q_{ik}\right) + T_{jik}\right)$$
$$- p_{ik}\left(\left(D_{ik} - Q_{ik}\right)^{+} - T_{jik}\right) +$$
$$s_{ik}\left(\left(Q_{ik} - D_{ik}\right)^{+} - T_{ijk}\right) - \tau_{ijk}T_{ijk}) - c_{ik}Q_{ik}] - \sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2} E\left(A_{ij}X_{ij}\right)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}[E(r_{ik}\min\left(D_{ik}, Q_{ik}\right) - p_{ik}(D_{ik} - Q_{ik})^{+} + s_{ik}(Q_{ik} - D_{ik})^{+}$$
$$+ r_{ik}T_{jik} + p_{ik}T_{jik} - s_{ik}T_{ijk} - \tau_{ijk}T_{ijk}) - c_{ik}Q_{ik}] - \sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2} E\left(A_{ij}X_{ij}\right)$$
$$= \sum_{k=1}^{n}[E(r_{ik}\min\left(D_{ik}, Q_{ik}\right) - p_{ik}(D_{ik} - Q_{ik})^{+} + s_{ik}(Q_{ik} - D_{ik})^{+}$$
$$+ r_{jk}\min\left(D_{jk}, Q_{jk}\right) - p_{jk}\left(D_{jk} - Q_{jk}\right)^{+} + s_{jk}\left(Q_{jk} - D_{jk}\right)^{+})$$
$$- c_{ik}Q_{ik} - c_{jk}Q_{jk}] +$$
$$\sum_{k=1}^{n} E(r_{ik}T_{jik} + p_{ik}T_{jik} - s_{ik}T_{ijk} - \tau_{ijk}T_{ijk} + r_{jk}T_{ijk}$$
$$+ p_{jk}T_{ijk} - s_{jk}T_{jik} - \tau_{jik}T_{jik}) - \sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2} E\left(A_{ij}X_{ij}\right) =$$
$$\sum_{i=1}^{2}\sum_{k=1}^{n}[E(r_{ik}\min\left(D_{ik}, Q_{ik}\right) - p_{ik}(D_{ik} - Q_{ik})^{+} + s_{ik}\left(Q_{ik} - D_{ik}\right)^{+})$$
$$- c_{ik}Q_{ik}] + \sum_{k=1}^{n}[E(r_{ik} + p_{ik} - s_{jk} - \tau_{jik})T_{jik}$$
$$+ \left(r_{jk} + p_{jk} - s_{ik} - \tau_{ijk}\right)T_{ijk}] - A_{ji}E\left(X_{ji}\right) - A_{ij}E\left(X_{ij}\right)$$

We recall that:

1. $E[P_k^{\mathrm{NV}}(\boldsymbol{Q_k})] = \sum_{i=1}^{2}[E(r_{ik}\min(D_{ik}, Q_{ik}) - p_{ik}(D_{ik} - Q_{ik})^{+} + s_{ik}(Q_{ik} - D_{ik})^{+}) - c_{ik}Q_{ik}]$
2. $E[P^{\mathrm{TR}}(\vec{Q})] = \sum_{k=1}^{n} E[(r_{ik} + p_{ik} - s_{jk} - \tau_{jik})T_{jik} + (r_{jk} + p_{jk} - s_{ik} - \tau_{ijk})T_{ijk}] = E[\sum_{k=1}^{n}(a_{jik}T_{jik} + a_{ijk}T_{ijk})]$
3. $E[F^{\mathrm{TR}}(\vec{Q})] = A_{ji}E\left(X_{ji}\right) + A_{ij}E\left(X_{ij}\right)$

Therefore, the expected profit function (under the optimal transshipment policy) can be written as: $\pi(\vec{Q}) = \sum_{k=1}^{n} E[P_k^{NV}(\boldsymbol{Q_k})] + E[P^{TR}(\vec{Q})] - E(F^{\mathrm{TR}}(\vec{Q}))$ □

PROOF OF CLAIM 1: From the condition for executing transshipments (2), one can observe that for each demand realization, the optimal transshipment quantities with A_{ij} and A_{ji} are bigger than or equal to the optimal transshipment quantities with $A'_{ij} > A_{ij}$ and $A'_{ji} > A_{ji}$. Since every unit transshipped contributes a positive profit to the $E[P^{TR}(\vec{Q})]$ term (by definition, see (19)), decreasing both A parameters results in a higher expected profit from transshipments. □

PROOF OF CLAIM 2: According to Lemma 5, we can express the expected profit function as the following sum: $\pi(\vec{Q}) = \sum_{k=1}^{n} E[P_k^{\mathrm{NV}}(\boldsymbol{Q_k})] + E[P^{\mathrm{TR}}(\vec{Q})] - E(F^{\mathrm{TR}}(\vec{Q}))$. Both expressions in (23) have the same replenishment quantities, and therefore, their expected newsvendor profit $\sum_{k=1}^{n} E[P_k^{NV}(\vec{Q}^{*(A>0)})]$ is equal. From Claim 1, the expected transshipment profit $E[P^{TR}(\vec{Q})]$ of the left hand side of (23) is (weakly) larger than $E[P^{TR}(\vec{Q})]$ of the right hand side. In addition, the fixed transshipment cost of the right hand side, $E[F^{TR}(\vec{Q})]$, is positive, whereas the same element in the left hand side equals zero. □

PROOF OF LEMMA 6: Clearly deriving from the optimality of $\vec{Q}^{*(A=0)}$

$$\pi\left(\vec{Q}^{*(A=0)}, A = 0\right) \geq \pi\left(\vec{Q}^{*(A>0)}, A = 0\right) \tag{B1}$$

From (23) and (B1), we obtain: $\pi(\vec{Q}^{*(A=0)}, A = 0) \geq \pi(\vec{Q}^{*(A>0)}, A > 0)$ □

PROOF OF LEMMA 7: From (B1),

$$\pi\left(\vec{Q}^{*(A=0)}, A = 0\right) - E\left[F^{\mathrm{TR}}\left(\vec{Q}^{*(A>0)}, A > 0\right)\right]$$
$$\geq \pi\left(\vec{Q}^{*(A>0)}, A = 0\right) - E\left[F^{\mathrm{TR}}\left(\vec{Q}^{*(A>0)}, A > 0\right)\right]$$

Table 12. Parameters of Experiments C1–C3.

	$D_{1k} = D_{2k}$	c_{1k}, c_{2k}	r_{1k}, r_{2k}	p_{1k}, p_{2k}	s_{1k}, s_{2k}	τ_{12k}, τ_{21k}
Experiment C1	$\sim N(200, 60)$	10,10	30,30	5,5	4,4	1,1
Experiment C2	$\sim N(200, 60)$	10,12	30,31	5,4	4,5	1,2
Experiment C3	$\sim N(200, 60)$	10,14	30,32	5,5	4,6	1,2

Table 13. Results of Experiment C1.

	$n = 4$			$n = 6$			$n = 8$			$n = 10$			
A f	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	Average gap (%)
1	0.06	0.06	0.08	0.03	0.12	0.13	0.04	0.08	0.15	0.02	0.05	0.14	0.08
1.5	**0.02**	**0.03**	**0.02**	**0.01**	**0.05**	**0.06**	**0.01**	**0.03**	**0.08**	**0.01**	**0.03**	**0.08**	**0.04**
2	0.00	0.01	0.06	0.02	0.05	0.02	0.01	0.03	0.05	0.03	0.02	0.05	0.03
2.5	0.03	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.01	0.02	0.03	0.04	0.02
3	0.02	0.03	0.00	0.01	0.01	0.00	0.02	0.02	0.02	0.02	0.02	0.01	0.02
3.5	0.04	0.02	0.03	0.04	0.00	0.03	0.02	0.00	0.00	0.01	0.01	0.02	0.02
4	0.04	0.00	0.04	0.02	0.01	0.04	0.01	0.00	0.00	0.01	0.02	0.00	0.02
4.5				0.05	0.04	0.02	0.01	0.02	0.01	0.03	0.02	0.01	0.02
5				0.05	0.01	0.02	0.01	0.02	0.01	0.03	0.03	0.03	0.02
5.5				0.09	0.03	0.01	0.02	0.02	0.04	0.04	0.01	0.01	0.03
6				0.08	0.03	0.01	0.05	0.05	0.01	0.03	0.03	0.00	0.03
6.5							0.04	0.06	0.03	0.07	0.06	0.03	0.05
7							0.07	0.05	0.03	0.06	0.06	0.01	0.05
7.5							0.09	0.05	0.04	0.09	0.07	0.04	0.06
8							0.09	0.05	0.05	0.09	0.07	0.05	0.07
8.5										0.09	0.08	0.03	0.07
9										0.10	0.08	0.02	0.07
9.5										0.13	0.09	0.02	0.08
10										0.13	0.08	0.04	0.08

Using Lemma 5, $\pi(\vec{Q}^{*(A>0)}, A = 0) - E[F^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A > 0)] = \sum_{k=1}^{n} E[P_k^{\mathrm{NV}}(Q_k^{*(A>0)})] + E[P^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A = 0)] - E[F^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A > 0)] \geq \sum_{k=1}^{n} E[P_k^{\mathrm{NV}}(Q_k^{*(A>0)})] + E[P^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A > 0)] - E[F^{\mathrm{TR}}(\vec{Q}^{*(A>0)}, A > 0)] = \pi(\vec{Q}^{*(A>0)}, A > 0)$ when the last inequality is due to (22). □

PROOF OF THEOREM 2: In a system with n identical items and identical retailers, the optimal replenishment quantities will be identical as well, that is, $Q_{ik}^* = Q_n^*\ \forall i, \forall k$. We denote: $A_{ij} = A\ \forall i, j \neq i, a_{ijk} = a_{jik} = a\ \forall i, j \neq i, \forall k$,

$$P_k^{\mathrm{NV}}(\boldsymbol{Q_k}) = P^{\mathrm{NV}}(Q_n), \forall k,$$

$$E[P^{\mathrm{TR}}(\vec{Q})] = E[P^{\mathrm{TR}}(Q_n)] \text{ and } E[F^{\mathrm{TR}}(\vec{Q})] = E[F^{\mathrm{TR}}(Q_n)]$$

Thus,

$$\begin{aligned}
\pi_n(Q_n) &= \sum_{k=1}^{n} E[P^{\mathrm{NV}}(Q_n)] + E[P^{\mathrm{TR}}(Q_n)] - E[F^{\mathrm{TR}}(Q_n)] \\
&= \sum_{k=1}^{n} E[P^{\mathrm{NV}}(Q)] + E\left[\sum_{k=1}^{n} \left(a_{jik} T_{jik} + a_{ijk} T_{ijk}\right)\right] \\
&\quad - A_{ji} E\left(X_{ji}\right) - A_{ij} E\left(X_{ij}\right) = nE[P^{\mathrm{NV}}(Q)] \\
&\quad + E\left[\sum_{k=1}^{n} \left(aT_{jik} + aT_{ijk}\right)\right] - \mathrm{AE}\left(X_{ji}\right) - \mathrm{AE}\left(X_{ij}\right) \\
&= nE[P^{\mathrm{NV}}(Q)] + E\left[\sum_{k=1}^{n} (a\hat{T}_{jik}) - A\right]^{+} \\
&\quad + E\left[\sum_{k=1}^{n} (a\hat{T}_{ijk}) - A\right]^{+}
\end{aligned}$$

For the case of identical items and retailers, we define the random variable S_n to be the sum of the n random variables: $a\hat{T}_{ijk}\ k = 1..n$, that is, $S_n \equiv \Sigma_{k=1}^{n}(a\hat{T}_{ijk})$ with the pdf and CDF $f_{s_n}(S)$ and $Fs_n(S)$, respectively. Notice that due to the equal replenishment quantities and identical demand distributions for all items, all random variables $\hat{T}_{ijk}$ are i.i.d. Also notice that $\hat{T}_{ijk}$ depend on the replenishment quantities of item k only, since the other item's replenishment quantities affect the actual but not the potential transshipment quantities.

Using the definition of S_n above: $E[\Sigma_{k=1}^{n}(a\hat{T}_{jik}) - A]^{+} + E[\Sigma_{k=1}^{n}(a\hat{T}_{ijk}) - A]^{+} = 2\int_{S_{n=A}}^{\infty}(s - A) f_{s_n}(s) ds$.

Placing this expression in the expected profit function, we obtain: $\pi_n(Q) = nE[P^{\mathrm{NV}}(Q)] + 2\int_{s_{n=A}}^{\infty}(s - A)\, f_{s_n}(s)\, ds$.

Using the above notation and definitions, we consider the following ratio: the expected system profit in a system with $n + 1$ identical items, calculated using the optimal replenishment quantities of the same system with n items,

Table 14. Results of Experiment C2

	$n=4$			$n=6$			$n=8$			$n=10$			
A f	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	Average gap (%)
1	0.09	0.16	0.09	0.00	0.25	0.22	0.00	0.16	0.40	0.02	0.04	0.35	0.15
1.5	**0.03**	**0.02**	**0.00**	**0.07**	**0.06**	**0.03**	**0.00**	**0.05**	**0.11**	**0.11**	**0.08**	**0.13**	**0.06**
2	0.02	0.02	0.01	0.15	0.03	0.02	0.18	0.02	0.07	0.67	0.11	0.06	0.11
2.5	0.04	0.02	0.00	0.21	0.02	0.01	0.30	0.01	0.02	0.75	0.14	0.03	0.13
3	0.06	0.01	0.01	0.25	0.01	0.00	0.38	0.02	0.04	0.90	0.17	0.00	0.15
3.5	0.08	0.02	0.00	0.27	0.01	0.01	0.47	0.03	0.01	1.02	0.18	0.00	0.17
4	0.09	0.02	0.00	0.31	0.02	0.01	0.52	0.02	0.01	1.04	0.20	0.01	0.19
4.5				0.34	0.03	0.00	0.56	0.04	0.04	1.12	0.21	0.00	0.26
5				0.37	0.03	0.00	0.59	0.04	0.01	1.18	0.23	0.00	0.27
5.5				0.39	0.03	0.00	0.62	0.06	0.02	1.21	0.23	0.01	0.29
6				0.41	0.04	0.01	0.65	0.06	0.03	1.25	0.25	0.01	0.30
6.5							0.67	0.07	0.02	1.29	0.27	0.01	0.39
7							0.68	0.07	0.02	1.30	0.28	0.02	0.40
7.5							0.72	0.08	0.04	1.34	0.27	0.01	0.41
8							0.73	0.07	0.03	1.35	0.29	0.02	0.42
8.5										1.37	0.29	0.01	0.56
9										1.39	0.28	0.01	0.56
9.5										1.42	0.28	0.03	0.58
10										1.43	0.29	0.01	0.58

Table 15. Results of Experiment C3

	$n=4$			$n=6$			$n=8$			$n=10$			
A f	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	1000 (%)	2000 (%)	3000 (%)	Average gap (%)
1	0.38	1.08	0.09	1.71	0.07	0.03	1.13	0.74	0.11	0.00	0.75	0.13	0.52
1.5	**0.40**	**0.04**	**0.03**	**1.46**	**0.02**	**0.01**	**1.65**	**0.73**	**0.02**	**0.92**	**0.81**	**0.03**	**0.51**
2	0.48	0.01	0.06	1.59	0.03	0.03	1.90	0.74	0.01	1.33	0.84	0.01	0.58
2.5	0.50	0.06	0.07	1.69	0.02	0.03	2.05	0.75	0.01	1.40	0.92	0.02	0.63
3	0.55	0.06	0.05	1.74	0.04	0.04	2.15	0.79	0.04	1.58	0.95	0.03	0.67
3.5	0.58	0.08	0.04	1.77	0.06	0.02	2.25	0.81	0.03	1.73	0.96	0.03	0.70
4	0.64	0.07	0.04	1.80	0.08	0.04	2.30	0.83	0.03	1.76	1.00	0.06	0.72
4.5				1.87	0.09	0.04	2.35	0.85	0.06	1.83	1.01	0.05	0.91
5				1.89	0.08	0.03	2.41	0.84	0.04	1.90	1.04	0.05	0.92
5.5				1.92	0.08	0.03	2.44	0.85	0.05	1.92	1.06	0.08	0.94
6				1.93	0.08	0.05	2.46	0.85	0.04	1.99	1.06	0.07	0.95
6.5							2.50	0.86	0.03	2.02	1.07	0.07	1.09
7							2.52	0.87	0.04	2.05	1.09	0.06	1.10
7.5							2.55	0.87	0.04	2.10	1.09	0.08	1.12
8							2.55	0.87	0.04	2.12	1.08	0.05	1.12
8.5										2.13	1.09	0.07	1.09
9										2.16	1.09	0.07	1.11
9.5										2.18	1.09	0.06	1.11
10										2.18	1.08	0.07	1.11

divided by the optimal expected system profit of the same system with n items, that is,

$$\frac{\pi_{n+1}\left(Q_n^*\right)}{\pi_n\left(Q_n^*\right)} = \frac{(n+1)\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{S_{n+1}=A}^{\infty}(s-A)\, f_{s_{n+1}}(s)\, ds}{n\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{S_n=A}^{\infty}(s-A)\, f_{s_n}(s)\, ds} \tag{B2}$$

For $A=0$, the ratio (B2) reduces to:

$$\frac{\pi_{n+1}\left(Q_n^*\right)}{\pi_n\left(Q_n^*\right)} = \frac{(n+1)\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{s_{n+1}=0}^{\infty} s f_{s_{n+1}}(s)\, ds}{n\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{s_n=0}^{\infty} s f_{s_n}(s)\, ds} = \frac{(n+1)\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2E\left(S_{n+1}\right)}{n\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2E\left(S_n\right)}$$

$$= \frac{(n+1)\, E[P^{\mathrm{NV}}\left(Q_n^*\right) + 2\,(n+1)\, E(a\hat{T}_{ij})]}{nE[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2nE(a\hat{T}_{ij})}$$

$$= \frac{(n+1)\, E[P^{\mathrm{NV}}\left(Q_n^*\right) + 2E\left(a\hat{T}_{ij}\right)]}{n[E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2E\left(a\hat{T}_{ij}\right)]} = \frac{n+1}{n}$$

Differentiating $\pi_n(Q)$ with respect to A, we obtain:

$$\frac{d\pi_n\,(Q)}{dA} = \frac{d[n\cdot E[P^{\mathrm{NV}}\,(Q)] + 2\int_{s_n=A}^{\infty}(s-A)\,f_{s_n}\,(s)\,ds]}{dA}$$
$$= -2\int_A^{\infty} f_{s_n}\,(s)\,ds - (A-A) = -2\bar{F}s_n\,(A)$$

Then, differentiating (B2) with respective to A, we obtain:

$$\frac{d}{dA}\left(\frac{\pi_{n+1}\left(Q_n^*\right)}{\pi_n\left(Q_n^*\right)}\right) = \frac{2\bar{F}_{s_n}\,(A)\,[(n+1)\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{s_{n+1}=A}^{\infty}(s-A)\,f_{s_{n+1}}\,(s)\,ds] - 2\bar{F}s_{n+1}\,(A)\,[n\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{s_n=A}^{\infty}(s-A)\,f_{s_n}\,(s)\,ds]}{\left(n\cdot E[P^{\mathrm{NV}}\left(Q_n^*\right)] + 2\int_{s_n=A}^{\infty}(s-A)\,f_{s_n}\,(s)\,ds\right)^2} \tag{B3}$$

Since S_n is the sum of n non-negative random variables:

- $\bar{F}_{s_n}(A) \geq \bar{F}_{s_{n+1}}(A)\ \forall A > 0$
- $\int_{s_{n+1}=A}^{\infty}(s-A)f_{s_{n+1}}(s)ds \geq \int_{s_n=A}^{\infty}(s-A)f_{s_n}(s)ds\ \forall A > 0$

Thus, the numerator of the ratio's derivative (B3) is positive, that is, the ratio (B2) is increasing in A. Since for $A = 0$, we obtain $\frac{\pi_{n+1}(Q_n^*)}{\pi_n(Q_n^*)} = \frac{n+1}{n}$, we conclude: $\frac{\pi_{n+1}(Q_n^*)}{\pi_n(Q_n^*)} \geq \frac{n+1}{n}\ \ \forall A > 0$

$$\frac{\pi_{n+1}\left(Q_n^*\right)}{n+1} \geq \frac{\pi_n\left(Q_n^*\right)}{n} \tag{B4}$$

From the optimality of the solution:

$$\frac{\pi_{n+1}\left(Q_{n+1}^*\right)}{n+1} \geq \frac{\pi_{n+1}\left(Q_n^*\right)}{n+1} \tag{B5}$$

From (B4) and (B5): $\frac{\pi_{n+1}\left(Q_{n+1}^*\right)}{n+1} \geq \frac{\pi_n\left(Q_n^*\right)}{n}$ □

APPENDIX C

In this appendix, we present the results of numerical experiments that we conducted to examine the sensitivity of the heuristic's performance to the f_{ijk} factor. The values of the *f, A*, and *n* parameters are specified in Section 5.1 and can also be observed in the headings of Tables 13–15. The tables contain the results of three experiments, C1–C3, where the following parameters (specified in Table 12) were used ∀ *k*.

The numbers within the tables refer to the gaps (in percentage) between the expected profit of the best solution found by an extensive search and the expected profit obtained by the heuristic solution. The rightmost column in each table presents the average percentage for each *f* value. The performance of *f* = 1.5, the recommended value, is denoted in bold.

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