Chapter 5

Image Recovery from Sparse Samples, Discrete Sampling Theorem, and Sharply Bounded Band-Limited Discrete Signals

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1. INTRODUCTION: SPARSE SAMPLING

Image sampling is a special case of image discretization, the very first step in digital image processing, storage, and transmission. Generally, discretization is the representation of continuous images by sets of numbers. Mathematically, it can be treated as computing image representation coefficients by means of projecting images on discretization basis functions, assuming that continuous images can be restored from the set of the representation coefficients by means of weighted summation of the reconstruction basis functions with signal representation coefficients as weights. Equations (1.1) describe these discretization and reconstruction processes:

\[
\alpha_k = \int_{X} a(x) \varphi_k^{(d)}(x) dx \quad (1.1a)
\]

\[
a(x) \approx \tilde{a}(x) = \sum_{k=0}^{N-1} \alpha_k \varphi_k^{(r)}(x), \quad (1.1b)
\]

where \(a(x)\) is a continuous image signal defined as a function of coordinate variable \(x\) in a domain of definition \(X\), \(\tilde{a}(x)\) is its approximation achieved by the reconstruction, \{\(\alpha_k\)\} are the representation coefficients, \{\(\varphi_k^{(d)}(x)\)\} and \{\(\varphi_k^{(r)}(x)\)\} are sets of discretization and reconstruction basis functions, and \(N\) is the number of representation coefficients and, correspondingly, reconstruction basis functions, involved in signal reconstruction.

Discretization basis functions and their reciprocal reconstruction basis functions are physically implemented as point spread functions of discretization and reconstruction (display) devices. They should, in principle, be selected in a manner that minimizes the number of representation coefficients sufficient for image reconstruction with the desired accuracy. However, in reality technological tradition and implementation issues frequently dictate the selection of discretization and reconstruction basis functions.

Almost overwhelmingly, image discretization and display devices implement the principle of image sampling by means of shift, or sampling, basis functions \{\(\varphi^{(s)}(x - k\Delta x)\)\}, \{\(\varphi^{(r)}(x - k\Delta x)\)\}; image discrete representation of images is obtained in the form of samples \{\(a_k\)\} of images after they are prefiltered by the sampling function:

\[
a_k = \int_{-\infty}^{\infty} a(x) \varphi^{(s)}(x - k\Delta x) dx. \quad (1.2)
\]

The samples \{\(a_k\)\} are taken at nodes \(k\Delta x\) of a regular sampling grid with interval \(\Delta x\) (sampling interval).
The theoretical foundation of this approach to discretization originates from the sampling theorem (Kotelnikov, 1933; Shannon, 1948). The sampling theorem in its classical formulation states that signals with a band-limited Fourier spectrum can be precisely reconstructed from their samples taken on a uniform sampling grid with a sampling interval inversely proportional to the bandwidth of the signal Fourier spectrum.

In reality, no band-limited signals exist, and the sampling theorem must be reformulated in terms of the accuracy of reconstruction of continuous signals from their samples. In such formulation, the sampling theorem states that

- The least square error approximation \( \tilde{a}(x) \) of signal \( a(x) \) from its samples \( \{a_k\} \) taken on a uniform sampling grid with sampling interval \( \Delta x \) is

\[
\tilde{a}(x) = \sum_{k=-\infty}^{\infty} a_k \text{sinc} \left( \frac{2\pi (x - k\Delta x)}{\Delta x} \right),
\]  

provided that signal samples \( \{a_k\} \) are obtained as values of a low-pass-filtered signal at sampling points:

\[
a_k = \frac{1}{\Delta x} \int_{-\infty}^{\infty} a(x) \text{sinc} \left( \frac{2\pi (x - k\Delta x)}{\Delta x} \right) dx.
\]

- The approximation mean square error (MSE) is minimal in this case and is equal to the signal energy outside the frequency interval \([ -1/2\Delta x, 1/2\Delta x ]\):

\[
\int_{-\infty}^{\infty} |a(x) - \tilde{a}(x)|^2 dx = \int_{-1/\Delta x}^{1/\Delta x} |\alpha(f)|^2 df + \int_{1/\Delta x}^{\infty} |\alpha(f)|^2 df = 2 \int_{1/\Delta x}^{\infty} |\alpha(f)|^2 df,
\]

where

\[
\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx) dx
\]

is the signal Fourier spectrum and \( f \) is frequency.

The sinc function \( \text{sinc}(x) = \sin x / x \) in Eqs. (1.3) and (1.4) has a uniform spectrum within the frequency interval \([ -1/2\Delta x, 1/2\Delta x ]\) and is a point
spread function of the ideal low-pass filter

\[ \text{sinc}(2\pi x/\Delta x) = \Delta x \int_{-1/\Delta x}^{1/\Delta x} \exp(-i2\pi fx)df. \quad (1.7) \]

Fourier spectra of images usually decay quite rapidly with frequency \( f \). However, high-frequency spectral components carry highly important information for image analysis, object detection, and recognition that cannot be neglected despite the fact that their contribution to signal energy \( \int_{-\infty}^{\infty} |a(x)|^2 dx = \int_{-\infty}^{\infty} |a(f)|^2 df \) is relatively small. For this reason, the sampling interval \( \Delta x \) must be sufficiently small to preserve image-essential high frequencies. As a consequence, image representation by samples is frequently quite redundant because samples are highly correlated. This means, that, in principle, far fewer samples would be sufficient for image reconstruction if the reconstruction could be done in a more sophisticated manner than conventional weighted summation according to Eq. (1.1b).

Apart from the general desire to reduce the number of image samples required for image storage and transmission, there are many real applications, where, contrary to the common practice of uniform sampling, sampled data are collected in an irregular fashion. The following are some typical instances:

- Samples are taken not where the regular sampling grid dictates them to be taken but where it is feasible because of technical or other limitations.
- The pattern of sample disposition is dictated by physical principles of the work of the measuring device (e.g., in interferometry or moiré technique, where samples are taken along level lines).
- The sampling device and sample positioning are jittery as a result of camera or object vibrations or other irregularities, such as imaging through a turbulent medium.
- Some samples of the regular sampling grid are lost or unavailable due to losses in communication channels.

Because display devices, sound synthesizers, and other devices for reconstruction of continuous signals from their samples, as well as computer software for processing sampling data, assume the customary use of a regular uniform sampling grid, in all these cases it is necessary to convert irregularly sampled images to regularly sampled ones. Generally, the corresponding regular sampling grid must contain more samples than are available, because the coordinates of the positions of available samples might be known with subpixel accuracy—that is, with accuracy (in units of image size) better than \( 1/K \), where \( K \) is the number of available pixels.

The task of converting an irregular sampling grid into a regular one is obviously a special case of the image resampling task, which can be solved
by various methods of numerical interpolation of sampled data. Several approaches can be used to solve this task. One approach is purely empirical and is based on simplistic numerical interpolation procedures, such as interpolation by means of a weighted summation of known samples in close vicinity to the sought samples with weights inversely proportional to the distance between them. A review of these methods can be found in (Lodha & Franke, 1997). Although such an approach meets some practical needs, it is lacking in signal restoration accuracy and optimality.

A more substantiated approach is based on generalizations of the sampling theory to nonuniform sampling. In this approach, it is assumed that the sampled continuous signals belong to a certain approximation subspace $M$ (e.g., subspaces of band-limited signals, splines subspaces) of the parent Hilbert space (usually, $L^2$ Hilbert space of finite energy functions) with the requisite that the interpolation procedure must determine a continuous signal that satisfies two constraints: (1) the interpolated signal must belong to the subspace $M$ and (2) its available samples must be preserved. Conditions for the existence and uniqueness of the solution depend on the signal model (underlying approximation subspace) and the set of given samples. For the band-limited case, Landau (1967) proved that a necessary and sufficient condition for the unique reconstruction of a continuous band-limited one-dimensional (1D) signal with bandwidth $W$ from its irregularly spaced samples is that the density of its samples should exceed the Nyquist rate $1/Bandwidth$. It was also shown that this condition is necessary for $D$-dimensional signals with band-limited Fourier spectra. A comprehensive exposition of this approach can be found in Marvasti (2001).

An alternative approximation model is associated with spline subspaces (Unser, 1999). However, due to their localized nature, their use in the recovery of large gaps in data is limited. A practical numerical algorithm for the interpolation and approximation of two-dimensional (2D) signals, based on multilevel B-splines, is suggested by Lee, Wolberg, and Shin (1997). A similar spline-based algorithm, which uses nonuniform splines for interpolation, was suggested by Margolis and Eldar (2004).

All the aforementioned methods are oriented toward the approximation of continuous signals, specified by their sparse samples. Some publications also consider discrete models. However, those publications treat only various special cases. Ferreira (2001) considers discrete signal recovery from sparse data with the assumption of signal band limitation in the discrete Fourier transform (DFT) domain. Hasan and Marvasti (2001) suggested error detection coding as a method to recover discrete signals with missing data during data transmission. For signal recovery, they suggested using the discrete cosine transform (DCT) domain band-limitation assumption.

In this chapter, we suggest a general framework for the recovery, from a given set of their arbitrary taken samples, of discrete signals
that originate from continuous signals. We treat this problem as an approximation task in the following assumptions:

- Continuous signals are to be represented in computers by their samples taken at some of, say, \( N \) nodes of a regular uniform sampling grid.
- It is assumed that if all \( N \) samples were known, they would be sufficient to represent the continuous signal.
- \( K < N \) samples of signals are available.
- The goal of the processing is generating, from this incomplete set of \( K \) samples, a complete set of \( N \) signal samples in such a way as to secure the most accurate (in terms of MSE) approximation of the discrete signal that would be obtained if the continuous signal it is intended to represent were densely sampled at all \( N \) positions.

The mathematical foundation of the framework is provided by the discrete sampling theorem for “band-limited” discrete signals that have only a few nonzero coefficients in their representation over a certain orthogonal basis. This theorem is introduced in Section 2. The rest of the chapter is organized as follows. Section 3 describes algorithms for signal minimum MSE recovery from sparse sampled data. In Section 4, properties of transforms that are specifically relevant for signal recovery from sparse data are analyzed, and experimental illustrations of precise reconstruction of band-limited signals from sparse data are provided. Section 5 is a discussion of the energy compaction capability of transforms—that is, their ability to compress image energy in a small number of transform coefficients, and illustrates it for such widely used transforms as the DFT, DCT, Walsh transform, and Haar transform. Section 6 addresses application issues and illustrates the discrete sampling theorem–based methodology of discrete signal recovery on examples of image super-resolution from multiple frames and image recovery from sparse projection data. Finally, Section 7 formulates the discrete uncertainty principle and demonstrates the existence of discrete signals sharply limited both in their extent and in their bandwidth in the domain of a transform.

2. DISCRETE SAMPLING THEOREM

Let \( A_N \) be a vector of \( N \) samples \( \{a_k\}_{k=0,...,N-1} \) of a discrete signal, \( \Phi_N \) be an \( N \times N \) orthogonal transform matrix,

\[
\Phi_N = \{\varphi_r(k)\}_{r=0,1,...,N-1} \quad (2.1)
\]
composed of basis functions \( \varphi_r(k) \), and \( \Gamma_N \) be a vector of signal transform coefficients \( \{y_r\}_{r=0,...,N-1} \) such that

\[
A_N = \Phi_N \Gamma_N = \left\{ \sum_{r=0}^{N-1} y_r \varphi_r(k) \right\}_{k=0,1,...,N-1} \quad (2.2)
\]

Assume now that only \( K < N \) signal samples \( \{a_k\}_{k \in \tilde{K}} \) are available, where \( \tilde{K} \) is a \( K \)-size non-empty subset of indices \( \{0, 1, \ldots, N - 1\} \). These available \( K \) signal samples define a system of \( K \) equations:

\[
\begin{cases}
  a_k = \sum_{r=0}^{N-1} y_r \varphi_r(k) \\
  k \in \tilde{K}
\end{cases} \quad (2.3)
\]

for \( K \) signal transform coefficients \( \{y_r\} \) of certain \( K \) indices \( r \).

Select a subset \( \tilde{R} \) of \( K \) transform coefficients indices \( \{\tilde{r} \in \tilde{R}\} \) and define a "Kof N"-band-limited approximation \( \hat{A}_N^{BL} \) to the signal \( A_N \) as

\[
\hat{A}_N^{BL} = \{\hat{a}_k = \sum_{\tilde{r} \in \tilde{R}} y_{\tilde{r}} \varphi_{\tilde{r}}(k)\} \quad (2.4)
\]

Rewrite this equation in a more general form:

\[
\hat{A}_N^{BL} = \left\{ \hat{a}_k = \sum_{r=0}^{N-1} y_{\tilde{r}} \varphi_{\tilde{r}}(k) \right\}, \quad (2.5)
\]

assuming that all transform coefficients with indices \( r \notin \tilde{R} \) are set to zero:

\[
\tilde{y}_r = \begin{cases} 
  y_r, & r \in \tilde{R} \\
  0, & \text{otherwise.}
\end{cases} \quad (2.6)
\]

Then the vector \( \tilde{A}_K \) of available signal samples \( \{a_{\tilde{k}}\} \) can be expressed in terms of the basis functions \( \{\varphi_{\tilde{r}}(k)\} \) of transform \( \Phi_N \) as

\[
\tilde{A}_K = \text{KofN}_\Phi \cdot \tilde{\Gamma}_K = \left\{ a_{\tilde{k}} = \sum_{\tilde{r} \in \tilde{R}} y_{\tilde{r}} \varphi_{\tilde{r}}(k) \right\}, \quad (2.7)
\]

where the \( K \times K \) sub-transform matrix \( \text{KofN}_\Phi \) is composed of samples \( \varphi_{\tilde{r}}(k) \) of the basis functions with indices \( \{\tilde{r} \in \tilde{R}\} \) for signal sample indices \( \tilde{k} \in \tilde{K} \), and \( \tilde{\Gamma}_K \) is a vector composed of the corresponding subset \( \{y_{\tilde{r}}\} \) of
signal nonzero transform coefficients \{\gamma_r\}. This subset of the coefficients can be found as

\[ \tilde{\Gamma}_K = \{\tilde{\gamma}_r\} = KofN_\phi^{-1} \cdot \tilde{A}_K \quad (2.8) \]

provided the matrix \(KofN_\phi^{-1}\) inverse to the matrix \(KofN_\phi\) exists, which, in general, is conditioned, for a specific transform, by the positions \(\tilde{k} \in \tilde{K}\) of available signal samples and by the selection of the subset \(\tilde{R}\) of transform basis functions.

By virtue of the Parseval relationship for orthonormal transforms, the band-limited signal \(\hat{A}^{BL}_N\) approximates the complete signal \(A_N\) with MSE as follows:

\[ MSE = \left\| A_N - \hat{A}^{BL}_N \right\|^2 = \sum_{k=0}^{N-1} |a_k - \hat{a}_k|^2 = \sum_{r \notin \tilde{R}} |\gamma_r|^2. \quad (2.9) \]

This error can be minimized by an appropriate selection of the \(K\) basis functions of the sub-transform \(KofN_\phi\). In order to do so, the energy compaction ordering of basis functions of the transform \(\Phi_N\) must be known. If, in addition, it is known that, for a class of signals, a certain transform features the best energy compaction in the smallest number of transform coefficients, then selecting this transform can secure the best minimum MSE band-limited approximation of the signal \(\{a_k\}\) for the given subset \(\{\hat{a}_k\}\) of its samples.

In this manner, we arrive at the following discrete sampling theorem that can be formulated in these two statements:

**Statement 1** For any discrete signal of \(N\) samples defined by its \(K \leq N\) sparse and not necessarily regularly arranged samples, its band-limited, in terms of certain transform \(\Phi_N\), approximation defined by Eq. (2.5) can be obtained with the MSE defined by Eq. (2.9) provided the positions of the samples secure the existence of the matrix \(KofN_\phi^{-1}\) inverse to the sub-transform matrix \(KofN_\phi\) that corresponds to the band limitation. The approximation error can be further minimized by using a transform with the best energy compaction property.

**Statement 2** Any signal of \(N\) samples that is known to have only \(K \leq N\) nonzero transform coefficients for certain transform \(\Phi_N\) (\(\Phi_N\)-transform “band-limited” signal) can be precisely recovered from exactly \(K\) of its samples provided the positions of the samples secure the existence of the matrix \(KofN_\phi^{-1}\) inverse to the sub-transform matrix \(KofN_\phi\) that corresponds to the band limitation.

In this formulation, the discrete sampling theorem is applicable to signals of any dimensionality. It also does not require any assumption
regarding the compactness of nonzero signal spectral coefficients in the transform domain. The signal dimensionality affects only the formulation of the signal band limitedness. For 2D images and transforms such as discrete Fourier, discrete cosine, and Walsh transforms, the most natural is compact "low-pass" band-limitedness by a rectangle or circle sector.

3. ALGORITHMS FOR SIGNAL RECOVERY FROM SPARSE SAMPLED DATA

For optimal signal/image band-limited approximation from sampled data, the first step is choosing the following:

- the transform, which promises the best approximation;
- the type of band limitation (e.g., low-pass, band-pass, high-pass, the shape of the figure in the transform domain that is supposed to contain nonzero transform coefficients); and
- the number $N$ of samples to be recovered.

The choice of the transform can be based on the transform energy compaction capability. As for the type of band limitation, it is an issue of a priori knowledge of the class of images at hand. If for a particular image or set of images the best transform and the type of band limitation are not certain, several different transforms and types of band limitations can be selected and then, from the obtained approximation results, the one with the highest energy can be chosen. The number $N$ of samples to be recovered is also a matter of a priori assumption of how many samples of a regular uniform sampling grid would be enough to represent the images.

Implementation of signal recovery/approximation from sparse non-uniformly sampled data according to Eq. (2.8) requires matrix inversion, which is usually a very computationally demanding procedure. However, for some transforms (DFT, DCT, Walsh, Haar, and others that feature fast Fourier transform [FFT]-type algorithms), pruned versions of these algorithms may be used (Yaroslavsky, 2004). For applications in which signal reconstruction with certain limited accuracy is satisfactory, one can use a simple iterative reconstruction algorithm of the Gerchberg–Papoulis type for the reconstruction (Gerchberg & Saxton, 1972; Papoulis, 1975). The flow diagram of the algorithm is shown in Figure 1. In this algorithm, the initial guess is generated from available sparse signal samples on a dense sampling grid of $N$ samples, supplemented with a guess of the rest of the samples, for which, for instance, zeros, signal mean value, or random numbers can be used. Then, at each iteration, the signal is subjected to the selected transform, the obtained transform coefficients are zeroed
initial guess: available signal samples on a dense sampling grid defined by the accuracy of measuring sample coordinates, supplemented with a guess of the rest of the samples, for which zeros, signal mean value or random numbers can be used

**FIGURE 1** Flow diagram of the iterative signal recovery procedure.

according to the band-limitation assumption and inverse transformed, after which the next iteration of the restored signal is generated by restoring available signal samples. We used this algorithm in the experiments described herein. A review of other iterative and non-iterative algorithmic options can be found in Ferreira (2001).

4. ANALYSIS OF TRANSFORMS

The applicability of a particular transform for band-limited image approximation depends first on whether the \(K_{\text{of N}_{\text{DFT}}}^{\text{LP}}\) matrix for this transform is invertible—that is, whether the available signal samples are compatible with the band-limitation type selected for this transform. In this section, we address the invertibility conditions for DFT, DCT, Fresnel, Walsh, and wavelet transforms most widely used in applications.

4.1. Discrete Fourier Transform

Consider the \(K_{\text{of N}_{\text{DFT}}}^{\text{LP}}\)-trimmed \(DFT_N\) matrix

\[
K_{\text{of N}_{\text{DFT}}}^{\text{LP}} = \left\{ \exp\left( i2\pi \frac{\tilde{k}\tilde{r}_{\text{LP}}}{N} \right) \right\}
\]  

(4.1)

that corresponds to the DFT \(K_{\text{of N}_{\text{DFT}}}^{\text{LP}}\)-low-pass band-limited signal. Due to the complex conjugate symmetry of DFT or real signals, \(K\) must be an odd number, and the set of frequency domain indices of \(K_{\text{of N}_{\text{DFT}}}^{\text{LP}}\) low-pass
band-limited signals in Eq. (4.1) is defined as

$$\tilde{r}_{LP} \in \tilde{R}_{LP} = \{[0, 1, \ldots, (K - 1)/2, N - (K - 1)/2, \ldots, N - 1]\}. \quad (4.2)$$

For such a case, the following theorems hold:

**Theorem 1** KofN-low-pass DFT band-limited signals of N samples with only K nonzero low-frequency DFT coefficients can be precisely recovered from exactly K of their samples taken in arbitrary positions.

**Proof.** As it follows from Eqs. (2.3)-(2.8), the theorem is proven if matrix KofN_{DFT} is invertible. A matrix is invertible if its determinant is nonzero. To check whether the determinant of the matrix KofN_{DFT} is nonzero, permute the order of columns of the matrix as follows:

$$\tilde{r} \in \tilde{R} = \{[N - (K - 1)/2, (K - 1)/2, \ldots, 0, 1, \ldots, (K - 1)/2]\} \quad (4.3)$$

and obtain the matrix

$$\text{KofN}_{DFT} = \begin{vmatrix} \text{exp} \left[ i2\pi \frac{\tilde{k} \tilde{r}}{N} \right] \\
\end{vmatrix} = \begin{vmatrix} \text{exp} \left[ i2\pi \frac{N - (K - 1)/2}{N} \frac{\tilde{k}}{\tilde{r}} \right] \delta(\tilde{k} - \tilde{r}) \\
\end{vmatrix} \begin{vmatrix} \text{exp} \left[ i2\pi \frac{\tilde{k} \tilde{r}}{N} \right] \\
\end{vmatrix}, \quad (4.4)$$

where

$$\tilde{r} \in \tilde{R} = \{[0, \ldots, K - 1]\}. \quad (4.5)$$

The first matrix, \{exp\[i2\pi \frac{N - (K - 1)/2}{N} \tilde{k} \tilde{r}\]\}, in this product of matrices is a diagonal matrix, which is obviously invertible. The second one, \{exp(i2\pi \tilde{k} \tilde{r}/N)\}, is a version of Vandermonde matrices, which are also known to have nonzero determinants if, as in our case, their ratios for each row are distinct (Horn & Johnson, 1991). Because permutation of the matrix columns does not change the absolute value of its determinant, Eq. (4.4) implies that the determinant of KofN-trimmed DFT_{N} matrix KofN_{LP} of Eq. (4.1) is also nonzero for an arbitrary set \( \tilde{K} = \{\tilde{k}\} \) of positions of K available signal samples.
For DFT Kof N-high-pass band-limited signals, for which

$$K_{of}N_{DFT}^{HP} = \left\{ \exp \left( i2\pi \frac{\tilde{\tau}_{HP}}{N} \right) \right\}, \quad (4.6)$$

where

$$\tilde{\tau}_{HP} \in \tilde{\mathcal{R}}_{HP} = \{[(N - K + 1)/2, (N - K + 3)/2, \ldots, (N + K - 1)/2]\}, \quad (4.7)$$

a similar theorem holds.

**Theorem 2** Kof N-high-pass DFT band-limited signals of N samples with only K nonzero high-frequency DFT coefficients can be precisely recovered from exactly K of their arbitrarily taken samples.

Theorems 1 and 2 can be extended to a more general case of signal DFT band limitation, when indices \{\tilde{\tau}\} of nonzero DFT spectral coefficients form arithmetic progressions with the common difference other than 1 such as, for instance,

$$\tilde{\mathcal{R}}_{mLP} \subset \tilde{\mathcal{R}}_{mLP}$$

$$= \left\{ 0, m, \ldots, m \frac{(K - 1)}{2}, N - m \frac{(K - 1)}{2}, \ldots, N - m \frac{(K - 1)}{2} + \frac{(K + 1)}{2} \right\} \quad (4.8)$$

The plots in Figure 2a and 2b illustrate examples of the exact reconstruction of a DFT "band-limited" signal (solid line) by matrix inversion for two cases: (1) All available signal samples are randomly placed within signal support and (2) available signal samples form a compact group. Note that, as the experiments show, the convergence of the iterative algorithm heavily depends on the realization of sample positions and, for some realizations of positions of available samples, it might be quite slow.

### 4.2. Discrete Cosine Transform

The N-point DCT of a signal \{a_k\}, k = 0, 1, \ldots, N - 1 is equivalent to the 2N-point Shifted Discrete Fourier transform (SDFT) with shift parameters (1/2, 0) of the 2N-samples' signal

$$\tilde{a}_k = \begin{cases} a_k, & k = 0, 1, \ldots, N - 1 \\ a_{2N-1-k}, & k = N, \ldots, 2N - 1 \end{cases} \quad (4.9)$$
FIGURE 2 Restoration of a DFT low-pass band-limited signal by means of matrix inversion for random (a) and compactly placed signal samples (b) and that using the iterative algorithm (c). Plot (d) shows the standard deviation of the signal restoration error as a function of the number of iterations. The experiment was conducted for test signal length of 64 samples and bandwidth of 13 frequency samples (~1/5 of the signal base band). In all images, the original signal is represented with a solid line obtained, for display purposes, by linear interpolation of its samples; available samples are represented by stems, and samples reconstructed by matrix inversion are represented by dots.

obtained from the initial one by its mirror reflection (Yaroslavsky, 2004). The \( \text{Kof} \cdot \text{N-trimmed} \) matrix of SDFT \((1/2,0)\)

\[
\text{Kof} \cdot \text{N}_{\text{SDFT}} = \left\{ \exp \left( \frac{i2\pi (\tilde{k} + 1/2)\tilde{r}}{2N} \right) \right\}
\]  \hspace{1cm} (4.10)

can be represented as a product

\[
\text{Kof} \cdot \text{N}_{\text{SDFT}} = \left\{ \exp \left( \frac{i2\pi \tilde{k}\tilde{r}}{2N} \right) \left\{ \exp \left( \frac{i\pi \tilde{r}}{2N} \right) \delta(k - r) \right\} \right\} = \text{Kof} \cdot \text{N}_{\text{DFT}} \left\{ \exp \left( \frac{i\pi \tilde{r}}{2N} \right) \delta(k - r) \right\}
\]  \hspace{1cm} (4.11)

of a 2\( N \)-point DFT matrix and a diagonal matrix \( \{ \exp(i\pi \tilde{r}/2N)\delta(k - r) \} \). The latter matrix is invertible and the invertibility of the \( \text{Kof} \cdot \text{N-trimmed} \) \( \text{DFT}_{2N} \) matrix \( \text{Kof} \cdot \text{N}_{\text{DFT}} \) can be proved, for above described band-limitations, as for the previous DFT case. Therefore, for DCT theorems similar to those for DFTs hold.
These theorems hold also for 2D DFT and DCT transforms provided band-limitation conditions are separable. The case of non-separable band limitation is more involved and requires further study.

We illustrate the above reasoning by some simulation examples. Figures 3 and 4 illustrate the precise restoration from sparse data of images band limited in the DCT domain by a square (separable band limitation) and by a 90° circle sector (a pie piece, inseparable band limitation). The image presented in Figure 3 is a 64 × 64-pixel test image low-pass band limited in the DCT domain by a square of 9 × 9 samples (Figure 3b). It has only 9 × 9 = 81 nonzero DCT spectral components of the 64 × 64 ones. This image was sampled at 82 “random” positions obtained from the standard Matlab pseudo-random number generator. One can see from the figure that the iterative algorithm provides quite accurate restoration of the initial image, though precise restoration may require a large number of iterations. An important peculiarity of the iterative restoration process is that the convergence of iteration is very nonuniform within the image area. Usually, the restoration error rapidly becomes very small almost everywhere in the image, and only in some parts where the sample density happens to be low do the restoration errors remain substantial and decay quite slowly.

Image band limitation by a square is separable over image coordinates and, as was shown earlier, it does not impose any limitation on the positions of sparse samples. It is, however, not isotropic. The situation is quite different in the case of isotropic band limitation in the DCT domain by a circle sector (a pie piece). Experiments show that the speed of convergence of the iterative algorithm substantially drops in this case. Many more iterations are needed to make the overall standard deviation of the restoration error low enough, though again, the restoration error remains substantial only in limited areas of the image. The convergence speed of the iterative algorithm in the case of isotropic circle sector band limitation can be substantially improved if the number of available image samples exceeds the number of nonzero DCT spectral coefficients—that is, if it is redundant from the point of view of the discrete sampling theorem. This is illustrated in Figure 4. The image presented is a 64 × 64-pixel test image low-pass band limited in the DCT domain by a circle sector. It has 73 (of 64 × 64) nonzero DCT spectral components, all located within a circle sector (shown in white in Figure 4b). In contrast to the image in Figure 3, this one was sampled at 93 “random” positions. The redundancy 93/73 = 1.27 in the number of samples with respect to the number of nonzero spectral coefficients is approximately equal to the ratio of the area of a square to the area of the circle sector inscribed into this square. As Figure 4e shows, with such a redundancy, iterative restoration converges quite fast and the overall restoration error after 100,000 iterations is comparable to that for the separable band.
**FIGURE 3** Recovery of an image band limited in the DCT domain by a square. (a) Initial image with 82 "randomly" placed samples in positions shown by white dots. (b) The shape of the image spectrum in the DCT domain. (c) The image restored by the iterative algorithm after 100,000 iterations. (d) Iterative algorithm restoration error (white areas show large errors; black areas show small errors). (e) Restoration error standard deviation versus the number of iterations for the iterative algorithm.
FIGURE 4  Recovery of an image band limited in the DCT domain by a circle sector. (a) Initial image with 93 "randomly" placed samples in positions shown by white dots. (b) The shape of the image spectrum in the DCT domain. (c) Image restored by the iterative algorithm after 100,000 iterations. (d) Iterative algorithm restoration error (white areas show large errors; black areas show small errors). (e) Restoration error standard deviation versus the number of iterations for the iterative algorithm.
limitation by a square illustrated in Figure 3. Once again, one can see that
the convergence of the iterative algorithm is nonuniform over the image
and relatively large restoration errors occur only in a small area of the
image where the density of available samples happens to be low.

In some applications, there is a natural and substantial redundancy in
the number of available image samples with respect to its bandwidth. One
such case is illustrated in Figure 5, which presents an example of image
restoration from its level lines. A 256 × 256-pixel image, shown in the
figure, is band limited in the DCT domain by a circle sector and contains
302 nonzero spectral coefficients. The image was sampled in 6644 samples
on a set of its level lines (8 levels), which resulted in 22-fold redundancy
with respect to the image spectrum 2D bandwidth. As seen in the figure,
such a redundancy substantially accelerated the convergence of the itera-
tive algorithm and enabled restoration with a quite low restoration error
after a few tens of iterations.

4.3. Discrete Fresnel Transforms

Discrete Fresnel transforms are discrete representations of the Fresnel inte-
gral transform. They are used for numerical reconstruction of holograms
digitally recorded in the near diffraction zone. There are two types of the
discrete Fresnel transform (Yaroslavsky, 2004) defined for signal \( \{a_k\} \) of \( N \)
samples and its Fresnel spectrum \( \{\alpha_r\} \) as

\[
a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp \left[ -i\pi \frac{(k\mu - r/\mu)^2}{N} \right] \tag{4.12,a}
\]

and

\[
a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp \left[ -i\pi \frac{(k - r)^2}{\mu^2N} \right], \tag{4.13,a}
\]

where \( \mu^2 = N\lambda Z/\lambda F \) (\( \lambda \) is the wavelength of the radiation used to record
the hologram, \( Z \) is the object-to-hologram distance, and \( X \) and \( F \) are the
1D sizes of the object and its hologram). They are applicable (i.e., they
are free of aliasing artifacts) in different regions \([1 \div \infty]\) and \([0 \div 1]\) of the
"distance" parameter (Yaroslavsky, 2004).

The first of these can be expressed via DFT as

\[
a_k = \frac{\exp(-i\pi k^2 \mu^2 / N)}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[ \alpha_r \exp \left( -i\pi \frac{r^2}{\mu^2N} \right) \right] \exp \left( i2\pi \frac{kr}{N} \right) \right\} \tag{4.12,b}
\]
and the second via "\(\mu^2\)-scaled" DFT (Yaroslavsky, 2004) as
\[
a_k = \frac{\exp\left(-i\pi \frac{k^2}{\mu^2N}\right)}{\sqrt{N}} \sum_{r=0}^{N-1} \left[ \alpha_r \exp\left(-i\pi \frac{r^2}{\mu^2N}\right) \right] \exp\left(i2\pi \frac{kr}{\mu^2N}\right).
\] (4.13,b)

Therefore, for the first discrete Fresnel transform above, formulated results for the DFT are applicable. As for the second, the formulation

FIGURE 5  Recovery of an image band limited in the DCT domain by a circle sector from its level lines. (a) Initial image with level lines (shown by white lines). (b) Image restored by the iterative algorithm after 500 iterations. (c) Restoration error standard deviation versus the number of iterations.
of band limitation and requirements to positions of sparse samples are also similar to those for DFT because the scaling does not affect the Vandermonde character of the transform matrix.

4.4. Wavelets and Other Bases

The main peculiarity of wavelet bases is that their basis functions are most naturally ordered in terms of two parameters: scale and position within the scale. Scale index is analogous to the frequency index for DFT. Position index tells only of the shift of the same basis function within the signal extent on each scale. Therefore, band limitation for DFT translates to scale limitation for wavelets. Limitation in terms of position is trivial: It simply means that some parts of the signal are not relevant. Commonly, discrete wavelets are designed for signals whose length is an integer power of 2 ($N = 2^n$). For such signals, there are $s \leq n$ scales and possible band limitations.

The simplest special case of wavelet bases is the Haar basis. Signals with $N = 2^d$ samples and with only a $K$ lower index nonzero Haar transform (the transform coefficients with indices $\{K, \ldots, N - 1\}$ are zero) are $(\delta = (\lfloor \log_2(K - 1) \rfloor + 1))$-band limited, where $\lfloor x \rfloor$ is an integer part of $x$. Such signals are piecewise constant within intervals between basis function zero-crossings. The shortest intervals of the signal constancy contain $2^{n-\delta}$ samples. As can be seen in Figure 5a, where the first eight basis functions of the Haar transform are presented, for any two samples located on the same interval, all Haar basis functions on this and lower scales have the same value. Therefore, having more than one sample per constant interval does not change the rank of the matrix $K/2^N$. Therefore, the condition for perfect reconstruction is to have at least one sample on each of those intervals.

For other wavelets and other bases a general necessary, sufficient, and easily verified condition for the invertibility of the $K_0/2^N$-trimmed transform submatrix has yet to be found. Standard linear algebra procedures for determining matrix rank can be used to test the invertibility of the matrix in each particular case.

For Walsh basis functions, the function index corresponds to the "sequency," or the number of zero crossings of the basis function. The sequency carries a certain analogy to the signal frequency. Ordering basis functions according to their sequency, which is characteristic for the Walsh transform, preserves, for many real-life signals, the property of more or less regular decay of the transform coefficients' energy with their index. Therefore, for the Walsh transform the notion of low-pass band-limited signal approximation, similar to that described for DFT, can be used. On the other hand, as seen in Figure 6b where the first eight Walsh basis
functions are shown, Walsh basis functions, similarly to Haar basis functions, can be characterized by the scale index, which specifies the shortest interval of signal constancy. Signals with $N = 2^n$ samples and band limitation of $K$ Walsh transform coefficients have the shortest intervals of signal constancy of $2^{n-s}$ samples, where $s = (\lfloor \log_2 (K - 1) \rfloor + 1)$. A necessary condition for perfect reconstruction is to have $K$ signal samples taken on different intervals. Unlike the case for Haar transforms, not all the intervals are needed for sampling, only $K$ intervals of the total number of intervals. For a special case of $K$ equal to a power of 2, there are $K$ intervals, each of which must be sampled to secure perfect reconstruction. This is the case when the reconstruction condition for the Walsh transform is identical to that for the Haar transform.

Figure 7 illustrates the case of recovery of a 1D signal, and Figure 8 shows the recovery of an image “band limited” in the Haar transform domain. For image restoration, two examples are shown: an arrangement of sparse samples, for which signal recovery is possible (Figure 8a) and that for which signal is not recoverable from the same number of samples (Figure 8b). Note that when the Haar transform reconstruction is possible, it is reduced to the trivial nearest-neighbor interpolation.

An example of perfect reconstruction of a Walsh transform domain band-limited signal of $N = 512$ and of band limitation $K = 5$ is shown in

![Figure 6](image-url)
FIGURE 7  Top, an illustrative example of the restoration of Haar transform 1D band-limited signal using the iterative algorithm and (bottom) a plot of the approximation error as a function of the number of iterations. The initial signal is shown by the solid line linearly interpolated, for display purposes, from its samples; available sparse signal samples are shown by stems; reconstructed samples are shown by light points on top of the initial signal line.

FIGURE 8  Two cases of sparse sampling of an image band-limited in the Haar transform (a) Non-recoverable case; (b) recoverable case. Sample positions are marked with dots; the image size is 64 × 64 pixels; band limitation is 8 × 8 samples (scale 3).
FIGURE 9  An example of Walsh band-limited signal recovery by means of matrix inversion.

Figure 9. In this example, the resulting $KofN_{Walsh}$ matrix is

$$KofN_{Walsh|K=5} = \begin{bmatrix}
1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1
\end{bmatrix}$$

(4.12)

and its rank equals 5. Note that in this particular example perfect reconstruction in the Haar transform domain is not possible since one of the shortest intervals of the signal constancy contains no samples.

5. SELECTION OF TRANSFORM FOR IMAGE BAND-LIMITED APPROXIMATION

As mentioned previously, the accuracy of signal/image band-limited approximation is determined by the energy compaction capability of the transforms. Theoretically, if the approximation accuracy is evaluated in terms of mean square approximation error over a certain ensemble of images, optimal transforms are the Karhunen–Loève transform (Karhunen, 1947; Loève, 1948) and its discrete version, the Hotelling transform (Hotelling, 1933). However, the practical value of these transforms
is quite limited because of the high computational complexity of generating transform basis functions and computing image representation coefficients. In practice, fast transforms such as the DFT, DCT, Walsh, Haar, and wavelet transforms are preferred for signal representation, restoration, and analysis because they combine quite good energy compaction capability and low computational complexity thanks to the existence of fast transform algorithms. In principle, their efficiency for different classes of signals might differ, and experimental evaluation of transform energy compaction capability is required. Figures 10 through 13 provide illustrative examples of such experimental evaluation.

Figure 10 shows examples of pseudo-random images band-limited in the domain of DCT, Walsh, and Haar transform domains by a circle sector with radius 0.1 size of the base band. They are generated by means of corresponding low-pass filtering arrays of independent pseudo-random numbers.

Figure 11 shows four test images and the corresponding plots of a fraction of the image energy contained within a square, which outlines the lowest transform coefficients, as functions of the square size that defines the bandwidth in the fraction of the base band. As the plots show, the DCT and DFT exhibit similar energy compaction capability for these test images and outperform in this respect the Walsh and Haar transforms. Evaluation of image approximation error in terms of error standard deviation is a very convenient, though not fully adequate, evaluation of image quality losses due to the bandwidth limitation. Better insight can be obtained from the images of approximation errors shown in Figure 12 for the same test images and low-pass band limitation by a square of the size of 0.3 of the base band. A common characteristic feature of these images is that they represent edges of objects in the images, components of crucial importance for object detection, localization, and recognition (Yaroslavsky, 2004).
FIGURE 11 Four test images (a1, b1, c1, d1) and plots (a2, b2, c2, d2) of the fraction of image energy contained in their respective DFT, DCT, Walsh transform, and Haar transform band-limited approximations.

6. APPLICATION EXAMPLES

6.1. Image Super-Resolution from Multiple Differently Sampled Video Frames

One potential application of the above signal recovery technique is image super-resolution from multiple video frames with chaotic pixel
FIGURE 12  Low pass (LP) band-limited approximations to the four test images shown in Figure 10 and the corresponding approximation errors for band limitation in a form of a square of 0.3 of the base band in the DCT, Walsh, transform, and Haar transform domains. Error standard deviations (StdErr) are indicated in the image headers in units of image dynamic range [0–255]. BW is bandwidth.
displacements due to atmospheric turbulence, camera instability, or similar random factors (Yaroslavsky, et al., 2007). By means of elastic registration of the sequence of frames of the same scene, the pixel displacements caused by random acquisition factors can be determined for each image frame with subpixel accuracy. Using these data, a synthetic fused image can be generated by placing pixels from all available video frames in their proper positions on the correspondingly denser sampling grid according to their found displacements. In this process, some pixel positions on the denser sampling grid remain unoccupied, especially when a limited number of image frames is fused. These missing pixels can then be restored using the iterative band-limited interpolation algorithm described previously. Computer simulation reported in Yaroslavsky, et al. (2007) showed that application of the iterative interpolation may substantially improve image resolution enhancement results by fusing multiple frames with different local displacements. This is illustrated in Figure 13, which shows one low-resolution frame (Figure 13a), an image fused from 30 frames (13b), and a result of iterative interpolation (13c) achieved after 50 iterations. Image band limitation was set in this experiment twice the base band of raw low-resolution images.

6.2. Image Reconstruction from Sparse Projections in Computed Tomography

Another straightforward application of the sparse data recovery algorithm can be found in tomographic imaging, where it frequently happens that a substantial portion of slices surrounding the body slices’ area frequently is known to be an empty field. This means that slice projections (sinograms) are Radon transform band-limited functions. Therefore,
FIGURE 14  Super-resolution in computed tomography. (a) A set of initial projections supplemented with the same number of presumably lost projections to double the number of projections; initial guesses of the supplemented projections are set to zero. (b) The image reconstructed from initially available projections. (c) The result of iterative restoration of missing projections. (d) The image reconstructed from the restored double set of projections.

for whatever number of projections is available, a certain number of additional projections, commensurate, according to the discrete sampling theorem, with the size of the empty zone of the slice, can be obtained and the corresponding resolution can be increased in the reconstructed images by using the iterative band-limited reconstruction algorithm (Yaroslavsky, Shabat, Salomon, Ideses, & Fishbain, 2009). Figure 14 illustrates such super-resolution by means of recovering the missing half of projections achieved using the fact that by simple segmentation of images shown in Figures 14b and 14d it was found that the outer 55% of the image area is empty.
7. SHARPLY BAND-LIMITED DISCRETE SIGNALS WITH SHARPLY LIMITED SUPPORT

7.1. The Discrete Uncertainty Principle

Discrete signals can, obviously, be sharply limited in their extent. Is this property compatible with the assumption of their band limitedness in the transform domain? This section addresses this issue.

For continuous signals, it is well known that they cannot be both sharply band limited (in terms of their Fourier spectra) and have sharply bounded support. In fact, continuous signals are, strictly speaking, neither band limited nor do they have sharply bounded support. They can only be more or less densely concentrated in signal and spectral domains. This property is mathematically formulated in the form of the "uncertainty principle":

\[ X_{eS} \times F_{eB} > 1, \]  

(7.1)

where \( X_{eS} \) is the interval in the signal domain that contains the \((1 - eS)\)-fraction of its entire energy, \( F_{eB} \) is interval in the signal Fourier spectral domain that contains the \((1 - eB)\)-fraction of signal energy, and both \( eS \) and \( eB \) are assumed to be sufficiently small.

In contrast to continuous signals, sampled signals specified by a finite number of signal samples that represent continuous signals can be sharply bounded both in the signal and spectral domains. This is quite obvious for some signal spectral presentations such as Haar signal spectra. In particular, Haar basis functions are examples of sampled functions sharply bounded in the signal and Haar spectral domains. Another example represents the Radon transform: If an object slice is limited in its extent (such as, for instance, the one shown in Figure 14b), its projections are also limited in the extent (see Figure 14a). Is this property also relevant to discrete transforms, such as DFT and DCT, that originate from the integral Fourier transform, which features the above uncertainty principle?

The answer is "Yes, it is relevant." Such space-frequency sharply bounded signals can be generated using the above-described iterative algorithm that, at each iteration, applies the requested bounds alternatively in the signal and spectral domains. Figure 15 shows examples of such space-frequency sharply bounded images.

The relationship between bounds in the signal and DFT domains is defined by the discrete uncertainty principle. The discrete uncertainty principle can be derived from the continuous one using the same reasoning that is used in signal sampling. Let \( N_{\text{sign}} \) be the number of signal nonzero samples, \( N_{\text{spect}} \) be the number of its nonzero spectral samples, and \( N \) be the number of samples in the signal sampling grid. Then the length of the continuous signal that corresponds to the given sampled signal can be estimated as \( X \approx N_{\text{sign}} \Delta x \), where \( \Delta x \) is the signal sampling
interval, and the length of the interval in the signal Fourier domain occupied by the signal spectrum can be estimated as $F \approx N_{spectr} \Delta f$, where $\Delta f$ is the signal spectrum sampling interval. From the uncertainty principle for continuous signals [Eq. (7.1)] it follows that

$$N_{sign} \Delta x N_{spectr} \Delta f > 1, \quad (7.2)$$

or

$$N_{sign} N_{spectr} > \frac{1}{\Delta x \Delta f} \quad (7.3)$$

Since according to the cardinal sampling relationship, for which DFT represents the integral Fourier transform,

$$\frac{1}{\Delta x \Delta f} = N, \quad (7.4)$$
we obtain finally that

\[ N_{\text{sign}} N_{\text{spectr}} > N, \]  

(7.5)

which formulates the discrete uncertainty principle.

### 7.2. Sinc-lets: Sharply Band-Limited Basis Functions with Sharply Limited Support

The existence of sharply space-frequency-bounded signals suggests the existence of a family of correspondingly sharply space-frequency-bounded basis functions that can be used to represent such signals. Although the analytical representation of these functions is yet to be found, they can be generated by the same iterative procedure as for the generation of sharply space-frequency-bounded signals using as seed signals delta functions with different locations within the selected interval in the signal domain. We call these functions *sinc-lets* to reflect the fact that they resemble windowed discrete sinc functions (see the illustrative examples in Figure 16).

The following algorithm was used to generate these functions:

\[
\left[ \text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k) \right]^{(t)} = \delta(k - k_0), \quad \text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k) = \lim_{t \to \infty} \left[ \text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k) \right]^{(t)},
\]

(7.6)

where

\[
\left[ \text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k) \right]^{(t)} = S_{\text{lim}} \cdot IDFT \{ B_{\text{lim}} \cdot DFT \{ [\text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k)]^{(t-1)} \} \}, \quad (7.7)
\]

and \( S_{\text{lim}} \) and \( B_{\text{lim}} \) are operators of signal band limitation and space limitations, respectively, and \( k_0 \) is the index of position of the delta function. This algorithm can be regarded as an algorithmic definition of sinc-lets. In the following text, only low-pass band-limited sinc-lets (LP sinc-lets) are considered. As Figure 16 shows, LP sinc-lets are shift-variant functions: Their shape and height depend on the position.

Figure 17 illustrates the speed of convergence of generated signals to their fixed point

\[
\left[ \text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k) \right] = \left[ S_{\text{lim}} \cdot IDFT \left\{ B_{\text{lim}} \cdot DFT \{ [\text{sinclet}_{S_{\text{lim}}}^{B_{\text{lim}}}(k)] \} \right\} \right]
\]

(7.8)

for cases of the space interval of 103 samples and the spectral interval of 51 and 103 samples of 512 available. On the vertical axes on these plots, a fraction of the signal energy outside the selected bounded interval is shown (in this particular case, the interval of 103 [of 512] samples).
FIGURE 16  Examples of LP sinc-lets in different positions within the interval of 103 samples for a signal of 512 samples (images a–c) and their corresponding DFT spectra (images d–f). LP sinc-lets are shown along with the corresponding discrete sinc functions of the same bandwidth.
Figure 17 shows matrices of mutual correlations of LP sinc-lets in different positions obtained for the space interval of 103 samples and the spectral interval of 51 and 103 samples. These matrices allow hypothesizing that sinc-lets shifted by interval $\Delta N = N / B_{\text{lim}}$ inversely proportional to their bandwidth interval $B_{\text{lim}}$ form a family of orthogonal functions. Band limitation in the DCT domain generates similar sinc-lets (Figure 19).

The shape of 2D LP sinc-lets depends on the shape of their space and spectrum limitation. Obviously, for separable space and spectrum limitation, 2D sinc-lets are products of the corresponding 1D sinc-lets. Examples of 2D sinc-lets limited in space by circles and a square and circularly limited in the DFT and DCT domains are shown in Figure 20.

8. CONCLUSION

In summary, this article makes the following statements:

- If an image on a denser uniform sampling grid needs to be reconstructed with the least MSE from a limited number of irregularly taken samples, this can be done using the described iterative or direct matrix inversion algorithms with the assumption that the reconstructed image is band limited in the domain of a certain orthogonal transform. The transform must be selected based on its energy compaction capability. For many
natural images, the DCT and DFT are the best candidates, although other transforms should also be considered. If the optimal selection of transform is uncertain, the restoration can be performed using several different transforms, and the one with the highest energy should be selected from the restored images. A special case of initial samples sparsely taken on a uniform sampling grid is, obviously, a known task of image subsampling, which is optimally and efficiently solved by means of the fast discrete sinc-interpolation (Yaroslavsky, 2007).

- Among the applications of the described image reconstruction method, particularly attractive is computed tomography, for which it offers super-resolution in terms of the number of required projections.
- In contrast to continuous signals that cannot be simultaneously of finite lengths and have band-limited Fourier spectrum, discrete signals that are sharply limited both in the signal domain and in the domain of DCT and DFT, which are discrete representations of the integral Fourier
Figures 19 (a) Sinc-let and (b) its DCT spectrum generated by band limitation in the DCT domain and (c) a plot of signal residual energy outside the selected interval versus the number of iterations. The grey rectangles indicate selected intervals in the space and frequency domains.

Transforms, do exist. The length of such signals and their DCT/DFT bandwidths are linked through the discrete uncertainty principle, an analog of the classical continuous one.

- Experimental results presented in this chapter support a conjecture of the existence of orthogonal families of basis function "sinc-lets" that are both DCT/DFT band limited and sharply bounded in their extent.

As a concluding remark, note that the described methods for image recovery from sparse samples by means of their band-limited approximation in certain transform domain requires explicit formulation of the desired band limitation in the selected transform domain. This a priori knowledge that must be invested is quite natural to assume. If a transform is selected according to its energy compaction capability, one almost definitely knows how this capability works (i.e., what transform coefficients are expected to be zero or nonzero). If however, this is not
FIGURE 20  (a–d) Examples of 2D LP sinc-lets and (e–h) their DFT and DCT spectra. Size limitation (Slrm) and band limitation (Blim) are given in fractions of the size of the corresponding domain. All images are contrast enhanced for display purposes.
known or not certain *a priori*, image recovery can be achieved with the "compressive sensing" approach (a name that, in fact, is quite confusing) (Donoho, 2006). This approach also assumes obtaining band-limited, in certain selected transform domains, approximation of images but does not require explicit formulation of the band limitation, and from $M < N$ available samples $\{a_{m}\}, m = 0, 1, \ldots, M - 1$ a signal $\{a_k\}$ of $N$ samples is recovered that provides minimum to $L_1$ norm $\|a\|_{L_1} = \sum_{r=0}^{N-1} |a_r|$ of signal transform coefficients $\{a_r\}$ for the selected transform. The price paid for the uncertainty regarding band limitation is that the number of required signal samples $M$ must be (in this case) redundant with respect to the given number $K$ nonzero spectral coefficients: $M/K = O(\log N)$.

**REFERENCES**


