The Art and Science of Holography

A Tribute to Emmett Leith and Yuri Denisyuk

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Optical Transforms in Digital Holography

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The birth of digital holography was declared in 1971, the year that the Nobel Prize was awarded to Dennis Gabor for holography’s invention. Just as optical holography is a technique for optical recording, processing and reconstruction of wavefields, digital holography is the analysis, synthesis and simulation of wavefields in digital computers and processors. One of the most fundamental problems of digital holography is that of adequate and accurate representation of optical signals and transforms in digital computers. In this chapter, we review basic principles of such a representation applied to discrete representation of optical transforms such as convolution, Fourier and Fresnel integral transforms.

4.1 Digital Holography: The Revival after Hibernation (a Second Wind)

Denis Gabor invented holography in 1948. In his Nobel Lecture about the development of holography he said,

“Around 1955, holography went into a long hibernation. The revival came suddenly and explosively in 1963, with the publication of first successful laser hologram by Emmett N. Leith and Juris Upatnieks of the University of Michigan, Ann Arbor. Their success was due not only to the laser, but to the long theoretical preparation of Emmett Leith (in the field of the “side-looking radar”) which started in 1955 ... Another important development in holography [happened] in 1962, just before the “holography explosion.” Soviet physicist Yuri N. Denisyuk published an important paper in which he combined holography with the ingenious method of photography in natural colors, for which Gabriel Lippman received the Nobel Prize in 1908.”

Gabor received his Nobel Prize in 1971. That same year, T. Huang published a paper entitled “Digital holography” in the Proceedings of the IEEE. This paper marked the next step in the development of holography: the use of digital computers for recording, generating and simulating wavefields. Lohmann, Huang and Goodman pioneered digital holography in the mid 1960s and prompted a burst
of publications in the early and mid-1970s. At that time, most of the main ideas of digital holography were suggested and tested.12-14,16 Numerous potential applications of digital holography, such as the fabrication of sophisticated computer-generated optical elements and spatial filters for optical information processing, 3D holographic displays and holographic television and holographic computer vision, stimulated a great enthusiasm among researchers. However, the implementation of these ideas was hampered by the limited speed and memory capacity of computers available at that time and the absence of electronic means and media for sensing and recording optical holograms. In the 1980s, digital holography went into a sort of hibernation similar to what happened to holography in the 1950s and 1960s.

With an advent in the end of 1990s of a new generation of high-speed microprocessors, high-resolution electronic optical sensors and liquid crystal displays, as well as advances in microlens and mirror-array fabrication technology, digital holography is gaining a second wind. Tasks that once required hours and days of computer time in the 1970s can now be solved in almost "real" time in a fraction of a second. Optical holograms can now be directly sensed by high-resolution photonic sensors and fed into computers in "real" time with no need for any wet photochemical processing. Microlens and mirror arrays promise a breakthrough in the recording of computer-generated holograms and the creation of holographic displays. The recent flow of publications in digital holographic metrology and microscopy indicate revival of digital holography from its hibernation.*

Development of optical holography, one of the most remarkable inventions of the twentieth century, was driven by a clear understanding of the information nature of optics and holography.1-4,6,15 This nature is especially seen in digital holography. Wavefields recorded in the form of a hologram in optical holography are represented in digital holography by a digital signal that carries the wavefield information. With the combination of digital holography and digital computers into optical information systems, information optics has reached its maturity.

The most substantial advantage of digital computers over analog electronic and optical information processing devices is the lack of modifications necessary to tackle different tasks. With the same hardware, one can build an arbitrary problem solver by simply selecting or designing an appropriate code for the computer. This feature also makes digital computers ideal vehicles for processing optical signals adaptively since, with the help of computers, they can adapt rapidly and easily to varying signals, tasks and end-user requirements. In addition, acquiring and processing quantitative data contained in optical signals, and connecting optical systems to other informational systems and networks is most natural when data are represented and handled in a digital form. In the same way that currencies are a common denominator in economics, digital signals are a common denominator in information handling. Thanks to its universal nature, the digital signal is an ideal means for integrating different informational systems.

* Curiously enough, the initial meaning of digital holography is becoming forgotten and, in many recent publications, digital holography is associated only with numerical reconstruction of holograms.
One of the most fundamental problems of digital holography—and, more generally, of integrating digital computers and analog optics—is that of adequate representation of the optical signal and transformations in digital computers. Solving this problem is not simply a matter of replacing integrals by integral sums. It requires consistent accounting for the computer-to-optics interface and for computational complexity issues.

4.2 Physical Reality vs. the Computer World: Representing Optical Signals and Transforms in Computers

Provided that quantum effects are not considered, physical reality is a continuum. By contrast, computers have only a finite number of states. How, then, can one image the physical reality of optical signals and transforms in computers? Two principles lie at the foundation of the digital representation of continuous signal transformations: the consistency principle of digital signal representation and the mutual correspondence principle between continuous and discrete transformations.

The consistency principle requires that the digital representation of signal transformations parallels that of signals. The mutual correspondence principle is said to hold if both continuous and digital transformations act to transform identical input signals into identical output signals. According to this principle, digital processors incorporated into optical information systems should be regarded and treated along with signal digitization and signal reconstruction devices as integrated analogous units (Fig. 4.1).

Signal digitization is carried out in two successive steps: discretization and element-wise quantization. In discretization, analog signals are converted into a set of numbers. Element-wise quantization rounds off these numbers into integers. In computer processing, quantization effects do not usually play a substantial role, so we will discuss only the role of signal discretization in signal and transform representation in computers.

In image scanners and hologram sensors, discretization is a process of measuring coefficients of signal expansion into a series over a set of functions called discretization basis functions. These coefficients represent signals in computers. It

Figure 4.1 The mutual correspondence principle between continuous and digital signal transformations.
is assumed that original continuous signals can be reconstructed with a certain accuracy by the summation of functions called reconstruction basis functions, with weights equal to the corresponding coefficients of a discrete signal representation. The reconstruction is carried out in signal reconstruction devices such as image displays and computer-generated hologram recorders. In digital holography and image processing, discretization and reconstruction basis functions most frequently belong to a family of “shift” basis functions. All functions from this family are obtained from one “mother” function by its spatial shifts by multiples of an elementary shift called a discretization interval. Discrete signal representation coefficients obtained for such functions are called signal samples.

The mathematical formulation of a signal discretization and reconstruction is as follows. Let \( a(x) \) be a continuous signal as a function of a spatial co-ordinate \( x \), let \( \Delta x \) be the discretization interval, \( k \) be the signal sample index, and

\[
\phi^{(d)}_k(x) = \phi^{(d)}(x - k\Delta x) \tag{4.1}
\]

and

\[
\phi^{(r)}_k(x) = \phi^{(r)}(x - k\Delta x) \tag{4.2}
\]

be the discretization and reconstruction basis functions, respectively. At the discretization, signal samples \( a_k \) are computed as

\[
a_k = \int a(x) \phi^{(d)}(x - k\Delta x) dx. \tag{4.3}
\]

Signal reconstruction from the set of samples \( \{a_k\} \) is described as

\[
\tilde{a}(x) = \sum_k a_k \phi^{(r)}(x - k\Delta x). \tag{4.4}
\]

The result \( \tilde{a}(x) \) of the signal reconstruction from the discrete representation obtained by Eq. (4.3) is not, in general, exactly identical to initial signal \( a(x) \). Reconstruction errors are called aliasing errors. According to the sampling theorem, perfect reconstruction is possible only for signals with a Fourier spectrum bounded by interval \( 1/\Delta x \) and with the use of discretization and reconstruction basis functions

\[
\phi^{(d)}_{id} = \frac{1}{\Delta x} \text{sinc} \left[ \pi (x - k\Delta x)/\Delta x \right] \tag{4.5}
\]

and

\[
\phi^{(r)}_{id} = \text{sinc} \left[ \pi (x - k\Delta x)/\Delta x \right], \tag{4.6}
\]
where \( \text{sinc}x = \frac{\sin x}{x} \) and is called a sinc-function (or cardinal function, in Gabor's terminology). According to the consistency principle, Eqs. (4.3) and (4.4) form the base for adequate discrete representation of signal transformations. In what follows, we describe how this signal representation generates corresponding discrete representations of such importance in optics and holography transforms as the convolution integral and Fourier and Fresnel transforms.

### 4.3 The Convolution Integral and Digital Filtering in the Signal Domain

The convolution integral models shift-invariant imaging systems. For a signal \( a(x) \) with a shift-invariant kernel \( h(x) \) (a convolution kernel or system point spread function) the convolution integral is defined as

\[
b(x) = \int_{-\infty}^{\infty} a(\xi)h(x - \xi)d\xi.
\]

Replacing the signal \( a(\xi) \) by Eq. (4.4) through its samples \( l_a^m \) and assuming that samples \( \{b_k\} \) of the convolution result \( b(x) \) are measured as it is described by Eq. (4.3), one can obtain that samples \( \{b_k\} \) of the convolution result \( b(x) \) can be computed as

\[
b_k = \sum_n a_nh_{k-n},
\]

where the summation is carried out over all available signal samples and

\[
h_m = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - \xi, -m\Delta x)\varphi^{(r)}(\xi)\varphi^{(d)}(x)d\xi dx,
\]

are samples of the point spread function. For band-limited signals and sinc-functions as discretization and reconstruction basis functions, representation of Eq. (4.8) of the convolution integral is perfect in the same sense as signal reconstruction from their samples is perfect. Since the spatial extent of the convolution kernel \( h(x) \) is usually much less then that of signals and, consequently, the number of samples \( \{h_m\} \) of the kernel is much less than that of signals, Eq. (4.8) is usually replaced by its alternative form

\[
b_k = \sum_{n=0}^{N_h-1} h_na_{k-n},
\]

in which \( N_h \) is the number of samples of \( \{h_m\} \). When the number of signal samples is infinite, in accordance with infinite limits in the convolution integral, these
two forms are identical. Signal transformation defined by Eq. (4.10) is commonly referred to as signal domain digital filtering by a digital filter defined by its discrete point spread function \( \{h_n\} \).

Equation (4.9) defines how a discrete point spread function \( \{h_m\} \) of a digital filter can be found that corresponds to a given system point spread function. On the other hand, given the discrete point spread function \( \{h_n\} \) of a digital filter, one can, in accordance to the correspondence principle, find the point spread function of an equivalent continuous filter:

\[
h_{eq}(x, \xi) = \sum_{k=0}^{N_h-1} \sum_{m=0}^{N_h-1} h_m \varphi^{(r)}(x - n \Delta x) \varphi^{(d)}[\xi - (k - n) \Delta x], \quad (4.11)
\]

where \( N_h \) is the number of samples \( \{b_k\} \) involved in the reconstruction of signal \( b(x) \).

It is more convenient to characterize the equivalent continuous filter by its frequency response as a Fourier transform of its impulse response:

\[
H_{eq}(f, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{eq}(x, \xi) \exp[i2\pi(fx - p\xi)] dx d\xi. \quad (4.12)
\]

By substitution of Eq. (4.11) into Eq. (4.12), one obtains

\[
H_{eq}(f, p) = \left[ \sum_{m=0}^{N_h-1} h_m \exp(i2\pi pm \Delta x) \right] \left[ \sum_{k=0}^{N_h-1} \exp[i2\pi(f - p)k \Delta x] \right]
\times \int_{-\infty}^{\infty} \varphi^{(r)}(x) \exp(i2\pi fx) dx \int_{-\infty}^{\infty} \varphi^{(d)}(\xi) \exp(-i2\pi p\xi) d\xi. \quad (4.13)
\]

This expression contains four terms:

\[
H_{eq}(f, p) = DFR(p) \cdot \Phi^{(r)}(f) \cdot \Phi^{(r)}(-p) \cdot SV(f, p), \quad (4.14)
\]

where

\[
DFR(p) = \sum_{n=0}^{N_h-1} h_n \exp(i2\pi pn \Delta x); \quad (4.15)
\]

\[
\Phi^{(r)}(f) = \int_{-\infty}^{\infty} \varphi^{(r)}(x) \exp(i2\pi fx) dx; \quad (4.16)
\]

\[
\Phi^{(d)}(p) = \int_{-\infty}^{\infty} \varphi^{(d)}(x) \exp(i2\pi px) dx; \quad (4.17)
\]
and

\[ SV(f, p) = \frac{\sin[\pi(f - p)N_b \Delta x]}{\sin[\pi(f - p) \Delta x]} \exp\left[i\pi(f - p)(N_b - 1)\Delta x\right]. \]  

(4.18)

The term \( DFR(p) \) is referred to as discrete frequency response of the digital filter. It is a periodical function in the frequency domain. Terms \( \Phi^{(r)}(f) \) and \( \Phi^{(d)}(-p) \) are frequency responses of the signal reconstruction and discretization devices (Fourier transforms of their point spread functions), respectively. If signal reconstruction and discretization bases are, as required by the sampling theorem, sinc-functions, then discretization and reconstruction devices act as ideal low-pass filters:

\[ \Phi^{(r)}(f) = \Delta x \Phi^{(d)}(f) = \text{rect}(1 + 2f \Delta x) = \begin{cases} 1, & -1/2 < f < 1/2 \Delta x \\ 0, & \text{otherwise} \end{cases} \]  

(4.19)

that remove all but one period of the discrete frequency response. This is not the case in reality, and therefore one should anticipate aliasing effects in the convolution results similar to those in signal reconstruction.

The term \( SV(f, p) \) reflects the fact that the digital filter defined by Eq. (4.10) and obtained as a discrete representation of the convolution integral is nevertheless not shift-invariant. The reason is that the number \( N_b \) of samples \( \{b_k\} \) of the filter output signal involved in the reconstructed continuous output signal \( b(x) \) is finite. When \( N_b \) increases, the contribution of boundary effects into the reconstructed signal diminishes and the quality of the digital filter approximation to the convolution integral improves. In the limit, when \( N_b \to \infty \),

\[
\lim_{N_b \to \infty} SV(f, p) = \lim_{N_b \to \infty} \frac{N_b}{N_b} \frac{\sin[\pi(f - p)N_b \Delta x]}{\sin[\pi(f - p) \Delta x]} \exp\left[i\pi(f - p)(N_b - 1)\Delta x\right] = \delta(f - p),
\]

(4.20)

where \( \delta \) represents Dirac’s delta function, and the filter is spatially homogeneous. Boundary effects caused by scantness of signal samples require special treatment of signal borders in digital filtering.

### 4.4 The Fourier Integral and Discrete Fourier Transforms

The integral Fourier transform of a signal \( a(x) \) is defined as

\[
\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx)dx.
\]

(4.21)

The transformation kernel of the integral Fourier transform \( \exp(i2\pi fx) \) is shift-variant. Therefore, in the derivation of its discrete representation one should, in
distinction to the case of the convolution integral, account for possible arbitrary shifts of signal and its spectrum sample positions with respect to signal and its spectrum coordinate systems such as those defined by, for instance, the system's optical axis. This principle is illustrated in Fig. 4.2 where \( u \) and \( v \) are shifts of the signal and its Fourier spectrum samples with respect to the corresponding coordinate systems.

Let the signal discretization and reconstruction bases be, respectively, \( \{ \varphi^{(d)}_k(x) = \varphi^{(d)}[x - (k + u)\Delta x] \} \) and \( \{ \varphi^{(r)}_k(x) = \varphi^{(r)}[x - (k + u)\Delta x] \} \), and the signal spectrum discretization and reconstruction bases be, respectively, \( \{ \phi^{(d)}_r(x) = \phi^{(d)}[f - (r + v)\Delta f] \} \) and \( \{ \phi^{(r)}_r(x) = \phi^{(r)}[f - (r + v)\Delta f] \} \) where \( k \) and \( r \) are integer indices of the signal and its Fourier spectrum samples \( \{ a^{(u)}_k \} \) and \( \{ \alpha^{(u,v)}_r \} \).

Then, similar to the case of the convolution integral, one finds that the samples \( \alpha^{(u,v)}_r \) can be found from signal samples \( \{ a^{(u)}_k \} \) as

\[
\alpha^{(u,v)}_r = \sum_k a^{(u)}_k \exp[i2\pi(k + u)(r + v)\Delta x\Delta f] \\
\times \int_{-\infty}^{\infty} \Phi^{(r)}(f + (r + v)\Delta f) \varphi^{(d)}(f) \exp[i2\pi f(k + u)\Delta x] df, \quad (4.22)
\]

where \( \Phi^{(r)}(f) \) is the frequency response of the signal reconstruction device [Fourier transform of \( \varphi^{(r)}_k(x) \)]. Superscripts \( (u,v) \) in \( \{ \alpha^{(u,v)}_r \} \) are kept to indicate

---

**Figure 4.2** Geometry of dispositions of the signal and its Fourier spectrum samples in coordinates \( x \) and \( f \).
the displacements $u$ and $v$ of the signal and its Fourier spectrum samples with respect to their corresponding coordinate systems.

For the discrete representation of the Fourier transform, the term

$$
\int_{-\infty}^{\infty} \Phi^{(r)}(f + (r + v) \Delta f) \Phi^{(d)}(f) \exp\left[i2\pi f (k + u) \Delta x\right] df
$$

(4.23)

is usually disregarded as being of second-order importance and only the summation term in Eq. (4.22) is considered.

Let $X$ now be an interval occupied by the signal $a(x)$. Then, assuming that spectrum discretization is carried out according to this interval, with $\Delta f = 1/X$, we establish that

$$\Delta x \Delta f = \frac{\Delta x}{X} = \frac{1}{N},$$

(4.24)

where $N$ is the number of signal samples within the interval $X$. Then, Eq. (4.22) can be rewritten as

$$\alpha_{r}^{(u,v)} \propto \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k + u)(r + v)}{N}\right].$$

(4.25)

This discrete transformation is orthogonal. In order to make it orthonormal, one can introduce a normalizing multiplier $1/\sqrt{N}$ to obtain direct and inverse discrete transforms:

$$\alpha_{r}^{(u,v)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k + u)(r + v)}{N}\right],$$

(4.26)

$$a_{k}^{(u,v)} = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_{r}^{(u,v)} \exp\left[-i2\pi \frac{(k + u)(r + v)}{N}\right].$$

(4.27)

Equations (4.26) and (4.27) contain multipliers $\exp[i2\pi(\mu v/N)]$ and $\exp[-i2\pi(\mu v/N)]$ that do not depend on sample indices and can be discarded. In this way, we arrive at the following transforms:

$$\alpha_{r}^{(u,v)} = \frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{k(r + v)}{N}\right] \right\} \exp\left(i2\pi \frac{ru}{N}\right)$$

(4.28)

and

$$a_{k}^{(u,v)} = \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{N-1} \alpha_{r} \exp\left[-i2\pi \frac{r(k + u)}{N}\right] \right\} \exp\left(-i2\pi \frac{kv}{N}\right)$$

(4.29)
as discrete representations of direct and inverse integral Fourier transforms. A special case of these transforms for the signal and its spectrum sample shifts \( u = 0, \ v = 0 \),

\[
\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi\frac{kr}{N}\right) \tag{4.30}
\]

and

\[
a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp\left(-i2\pi\frac{kr}{N}\right) \tag{4.31}
\]

which are known as direct and inverse Discrete Fourier Transforms. In order to distinguish the general case from this special case, we will refer to transforms defined by Eqs. (4.28) and (4.29) as shifted discrete Fourier transforms SDFT\(_{(u,v)}\).\(^{17}\)

Shift parameters make SDFT more flexible in simulating integral Fourier transforms than conventional DFT. Using them, one can perform a continuous spectrum analysis of signals presented in a discrete form, compute convolution and correlation with sub-sample resolution and flexibly perform signal resampling with almost ideal sinc-interpolation.

It is also noteworthy to mention important special cases of the SDFT for signal and spectra that exhibit certain symmetry, such as the discrete cosine transform (DCT):

\[
\alpha_r^{(DCT)} = \text{SDFT}_{1/2,0} \left( \tilde{a}_k = \begin{cases} 
  a_k, & k = 0, 1, \ldots, N-1 \\
  a_{2N-k-1}, & k = N, \ldots, 2N-1 
\end{cases} \right) 
= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left[\pi\frac{(k+1/2)r}{N}\right]; \tag{4.32}
\]

and its complement DcST:

\[
\alpha_r^{(DcST)} = \text{SDFT}_{1/2,0} \left( \tilde{a}_k = \begin{cases} 
  a_k, & k = 0, 1, \ldots, N-1 \\
  -a_{2N-k-1}, & k = N, \ldots, 2N-1 
\end{cases} \right) 
= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi\frac{(k+1/2)r}{N}\right]; \tag{4.33}
\]

the DCT-IV:

\[
\alpha_r^{(DCY-IV)} = \text{SDFT}_{1/2,1/2} \left( \tilde{a}_k = \begin{cases} 
  a_k, & k = 0, 1, \ldots, N-1 \\
  -a_{2N-k-1}, & k = N, \ldots, 2N-1 
\end{cases} \right) 
= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left[\pi\frac{(k+1/2)(r+1/2)}{N}\right]; \tag{4.34}
\]
and DcST-IV:

\[
\alpha_r^{\text{(DcST-IV)}} = \text{SDFT}_{1/2,1/2} \left( \tilde{a}_k = \begin{cases} 
    a_k, & k = 0, 1, \ldots, N - 1 \\
    a_{2N-k-1}, & k = N, \ldots, 2N - 1 
\end{cases} \right)
\]

\[
= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin \left[ \pi \frac{(k + 1/2)(r + 1/2)}{N} \right]; \tag{4.35}
\]

and the discrete sine transform (DST):

\[
\alpha_r^{\text{(DCT)}} = \text{SDFT}_{1,1} \left( \tilde{a}_k = \begin{cases} 
    a_k, & 0 \leq k \leq N - 1 \\
    0, & k = N \\
    -a_k, & N \leq k \leq 2N \\
    0, & k = 2N + 1 
\end{cases} \right)
\]

\[
= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin \left[ \pi \frac{(k + 1)(r + 1)}{N} \right]. \tag{4.36}
\]

These transforms play an important role in digital signal processing and digital holography.

### 4.5 Integral and Discrete Fresnel Transforms

An integral Fresnel transform models “near zone” wave propagation and is defined as

\[
\alpha(f) = \int a(x) \exp[-i\pi(x - f)^2] dx. \tag{4.37}
\]

The transformation kernel of the integral Fresnel transform \( \exp[-i\pi(x - f)^2] \) is shift-invariant. Nevertheless, in the derivation of its discrete representation, it is also useful to account for possible arbitrary shifts of signal, and its Fresnel spectrum samples positions with respect to the signal and spectrum coordinate systems in a similar way as it was done for the discrete representation of Fourier Transform in Section 4.4.

For the same signal reconstructing and spectrum discretization basis functions \( \varphi_k^r(x) = \varphi^r[x - (k + u)\Delta x] \) and \( \varphi_r^d(x) = \varphi^d[f - (r + v)\Delta f] \) that were used for the discrete representation of integral Fourier transform, discrete representation of the Fresnel transform is obtained by

\[
\alpha_r = \sum_k a_k \exp\{-i\pi[(k + u)\Delta x - (r + v)\Delta f]^2\}
\]

\[
\times \int_{-\infty}^{\infty} \varphi_r^d(f) \exp(-i\pi f^2) \tilde{\Phi}(f - s_{rk}) \exp[i2\pi f s_{rk}] df, \tag{4.38}
\]
where $s_{kr} = (k + u)\Delta x - (r + v)\Delta f$ and

$$
\tilde{\Phi}(f) = \int_{-\infty}^{\infty} \varphi(r)(x) \exp(-i\pi x^2) \exp(i2\pi xf) dx
$$

(4.39)

is the Fourier transform of signal reconstruction basis function $\varphi(r)(x)$ modulated by a chirp function $\exp(-i\pi x^2)$. The function $\varphi(r)(x)$ is concentrated around the point $x = 0$ within an interval of about $\Delta x$, $\exp(-i\pi x^2) \approx 1$ within this interval, and one can regard $\tilde{\Phi}(f)$ as an approximation of a frequency response of a hypothetical signal reconstruction device assumed in the signal discrete representation. Using a similar argument as that made for the discrete representation of the Fourier integral, only the first multiplier in Eq. (4.38) is used for the discrete representation of the Fresnel integral. Completion of the derivation is accomplished by the introduction of the reduced variables

$$
\mu = (\Delta x / \Delta f)^{1/2}
$$

(4.40)

and

$$
w = u\mu - v/\mu.
$$

(4.41)

Then, using the relationship $N = 1/\Delta x \Delta f$ between the number of signal samples and discretization intervals, one can obtain the following for samples of the Fresnel transform spectrum:

$$
\alpha_r^{(\mu,w)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[-i\pi \frac{(k\mu - r/\mu + w)^2}{N}\right]
$$

(4.42)

(where the multiplier $1/\sqrt{N}$ is introduced for normalization purposes). Equation (4.42) can be regarded as a discrete representation of the Fresnel integral transform referred to as the shifted discrete Fresnel transform (SDFrT). Its parameter $\mu$ plays a role of the distance (focal) parameter in the Fresnel approximation of Kirchhoff’s integral, and $w$ is a combined shift parameter of discretization raster shifts in the signal and Fresnel transform domains. It is frequently considered to be equal to zero, and the discrete Fresnel transform (DFrT) is frequently defined as

$$
\alpha_r^{(\mu,w)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[-i\pi \frac{(k\mu - r/\mu)^2}{N}\right].
$$

(4.43)
It can be shown that the Shifted Discrete Fresnel Transform $SDFrT_{(\mu, w)}$ is connected with the shifted DFT by $SDFrT_{(0, -\mu w)}$:

$$\alpha_r^{(\mu, w)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left[ -i\pi \frac{k^2 \mu^2}{N} \right] \exp \left[ i2\pi \frac{k(r - w\mu)}{N} \right] \times \exp \left( -i\pi \frac{(r - w\mu)^2}{N\mu^2} \right).$$

(4.44)

From this relationship, one can conclude that the $SDFrT$ is invertible and that the inverse $SDFrT$ is defined as

$$a_k^{(\mu, w)} = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r^{(\mu, w)} \exp \left[ i\pi \frac{(k\mu - r/\mu + w)^2}{N} \right].$$

(4.45)

Since the shift parameter $w$ is a combination of shifts in the signal and spectral domains, the shift in the signal domain causes a corresponding shift in the transform domain that depends on the focal parameter $\mu$. One can sever this dependence if, in the definition of the discrete representation of integral Fresnel transform, one imposes the symmetry condition

$$\alpha_r^{(\mu, w)} = \alpha_{N-r}^{(\mu, w)}$$

(4.46)

onto the transform

$$\alpha_r^{(\mu, w)} = \exp \left[ -i\pi \frac{(r/\mu - w)^2}{N} \right]$$

(4.47)

of a point source $\delta(k)$. This condition is satisfied when

$$w = \frac{N}{2\mu}$$

(4.48)

and $SDFrT_{(\mu, N/2\mu)}$ for such a shift parameter takes the form

$$\alpha_r^{(\mu, w)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left[ -i\pi \frac{(k\mu - (r - N/2)/\mu)^2}{N} \right].$$

(4.49)

The discrete transform defined by Eq. (4.49) allows the position of objects reconstructed from Fresnel holograms to be kept invariant with the focal distance. This version of the discrete Fresnel transform is referred to as the focal plane invariant discrete Fresnel transform.$^{18}$
In many cases of digital reconstruction of a Fresnel hologram, only the magnitude of the reconstructed wavefield (image) is needed. If one removes the exponential phase terms that do not depend on $k$ from Eqs. (4.45) and (4.49), a partial discrete shifted Fresnel transform can be defined as

$$
\hat{\alpha}_r^{(\mu, w)}(u, v) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left( -i \pi \frac{k^2 \mu^2}{N} \right) \exp \left[ i2\pi \frac{k(r - w\mu)}{N} \right], \tag{4.50}
$$

with its inverse transform as

$$
a_k^{(\mu, w)} = \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{N-1} \hat{\alpha}_r^{(\mu, w)}(u, v) \exp \left[ i2\pi \frac{(r - w\mu)k}{N} \right] \right\} \exp \left( i\pi \frac{k^2 \mu^2}{N} \right); \tag{4.51}
$$

and the focal plane invariant partial discrete Fresnel transform can be defined as

$$
\hat{\alpha}_r^{(\mu, w)}(u, v) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left( -i \pi \frac{k^2 \mu^2}{N} \right) \exp \left[ i2\pi \frac{k(r - N/2)}{N} \right]
= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k (-1)^k \exp \left( -i \pi \frac{k^2 \mu^2}{N} \right) \exp \left( i2\pi \frac{kr}{N} \right) \tag{4.52}
$$

with its inverse transform as

$$
a_k = \frac{1}{\sqrt{N}} (-1)^k \exp \left( -i \pi \frac{k^2 \mu^2}{N} \right) \sum_{k=0}^{N-1} \tilde{\alpha}_r^{(\mu, w)}(u, v) \exp \left( -i2\pi \frac{kr}{N} \right). \tag{4.53}
$$

An important role in digital reconstruction of Fresnel holograms is played by the discrete Fourier transform of the discrete chirp-function $\exp(i\pi q k^2 / N)$. It is a periodic function

$$
\text{frincd}(N; q; x) = \frac{1}{N} \sum_{k=0}^{N-1} \exp \left( -i \pi \frac{qk^2}{N} \right) \exp \left( i2\pi \frac{kr}{N} \right). \tag{4.54}
$$

The function $\text{frincd}(N; q; x)$ describes the DFrT of a plane wave to the accuracy of the phase chirp-function $\exp(-i\pi x^2 / q N)$. This can also be regarded as a discrete analog of the function

$$
\text{frinc}(x; F) = \int_{-\infty}^{\infty} \text{rect} \left( \frac{f - F}{2F} \right) \exp \left[ i\pi (x - f)^2 \right] df
= \exp(i\pi x^2) \int_{-F}^{F} \exp(i\pi f^2) \exp(-i2\pi fx) df, \tag{4.55}
$$
the result of an integral Fresnel transform of a rect-function

$$\text{rect}\left(\frac{f - F}{2F}\right) = \begin{cases} 1, & -F \leq f \leq F \\ 0, & \text{otherwise} \end{cases} \quad (4.56)$$

When $q = 0$, function $\text{frincd}(N; q; x)$ coincides with the sincd-function of DFT:

$$\text{frincd}(N; 0, r) = \text{sincd}(N; N; x) = \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(i2\pi \frac{kx}{N}\right), \quad (4.57)$$

which is a discrete analog of a sinc-function for the integral Fourier transform. The corresponding function for the focal-plane-invariant DFrT is defined as

$$\overline{\text{frincd}}(N; q; x) = \frac{1}{N} \sum_{k=0}^{N-1} \exp\left[-i\pi \frac{q(k - N)}{N}\right] \exp\left(i2\pi \frac{kr}{N}\right). \quad (4.58)$$

Figure 4.3 illustrates the behavior of the $\overline{\text{frincd}}$-function for different values of focusing parameter $q$.

Figure 4.3 Absolute values of function $\overline{\text{frincd}}(N; q; x)$ for $N = 256$ and different values of “focusing” parameter $q$. (The function is shown centered around the middle point of the range of its argument.)
References