THE THEORY OF OPTIMAL METHODS FOR LOCALIZATION OF OBJECTS IN PICTURES

BY

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§ 1. Introduction

One of the major goals of picture processing is to provide information about the relative location of objects in space. In many applications, detection and localization (i.e., measurement of coordinates) of objects is of extreme practical importance. Almost all tasks in picture processing and interpretation, especially those of object recognition, can be reduced to this problem.

A copious literature exists on localization and detection of objects in pictures, but the variety of ideas used to solve this problem is not too rich. Essentially, detection and localization of objects is reduced in all methods to some kind of correlation of the given object with the observed picture and subsequent comparison of the result with a threshold. This approach is usually substantiated by an additive observation model, which treats the observed picture signal as an additive mixture of the desired object signal and signal-independent noise.

There are practical reasons for the general adherence to a correlator. The correlation detector-estimator is essentially a version of the so-called linear detector-estimator. Decisions about the presence and coordinates of the desired object are made point-wise on the basis of the signal level in each point of the field at the output of a linear filter acting upon the observed picture. The function of the linear filter in such devices is to transform the signal space in order to enable decision making on the basis of individual signal coordinates of the transformed space rather than using the entire signal. Due to separation into independent linear and point-wise non-linear units, data analysis and implementation of such devices is much simplified.

The implementation issue is of particular importance in picture processing because of the enormous number of degrees of freedom in optical and picture signals, which results in computational complexity in data processing. Fortunately, a very elegant optical feature of the correlator exists which allows image correlation to be performed with the speed of light. It was proposed by Van der Lugt [1964] almost 30 years ago. This development began an "era" of coherent optical correlators in optical information processing. Even now in almost each issue of Applied Optics, Optics Communications
and other optical journals one can find a paper on optical correlators and related problems. This tells about the topicality of the problem. At the same time, this means that the problem is still open despite the passage of 30 years. What is the reason for that? In the present author's opinion, there are two reasons. The first is technical: optical correlators remain too imperfect to be able to compete with digital electronic processors in real applications. The second reason is that, until recently, the theoretical basis was not sufficiently developed. The present paper is an attempt to review the current state of the theory of optimal correlators, and, in this way, to contribute to the solution of the problem.

The performance characteristics of localization devices can be assessed quantitatively in terms of their accuracy and reliability (Woodward and Davies [1950]). Localization accuracy and reliability are limited by the presence of some random noise in the observed picture signal due to the image sensor's noise (e.g., graininess of the photomaterial or intrinsic noise of the video camera, etc.) as well as by signal components from outside background objects. This paper presents an analysis of these factors, starting with the simplest case of additive, signal-independent noise and progressing to a treatment of the most general situation of pictures with a cluttered background.

Section 7 is devoted to a general analysis of the potential accuracy and the reliability of object localization in the presence of additive Gaussian noise. In §2.1, the concepts of accuracy and reliability as well as normal and anomalous errors of localization are introduced. The optimality of the conventional correlator (matched filter) is shown for the case of picture observation with additive white Gaussian noise. In §2.2, the potential accuracy of localization is characterized in terms of the variance of measurement errors along the coordinates. In §2.3, these results are extended to the case of mismatched filter and non-white noise, and in §2.4 they are extended to object localization in color or, generally, in multi-component pictures. The reliability of localization in the presence of additive white Gaussian noise is investigated in §2.5. The probability of anomalous errors is estimated, its threshold behavior is indicated, and the lower bound of the object signal energy per bit of measurement information is found. In §2.6 these results are revisited from the more general perspective of the theory of random processes. Finally, in §2.7 the probability of anomalous localization errors in the presence of additive Gaussian noise and multiple outside objects is estimated. Moreover, the problem of designing devices for localization with minimal probability of anomalous errors in the presence of cluttered background is formulated.

### III. 2.
The Accuracy and Reliability of the Localization of Two-dimensional Objects on a Plane

#### 2.1. Localization of a Single Object in the Presence of Additive White Gaussian Noise: Optimal Localization Device and Two Types of Localization Errors

Let us first consider the simplest discrete observation model when samples \( (b_n) \) of the observed signal can be regarded as a sum of samples \((a_n (x_0, y_0))\) of the signal from a given object having unknown coordinates \((x_0, y_0)\) and samples \((n_n)\) of the interference of noise.

\[
b_n = a_n (x_0, y_0) + n_n.  \tag{2.1}\n\]

Now assume that noise samples are statistically independent of the signal \(a_n (x_0, y_0)\), are non-correlated, and have a Gaussian probability distribution with zero mean and variance \(\sigma^2\). This model describes the simplest situation where the only disturbance that interferes with object localization is the noise of the signal sensor. Thermal noise serves as an example, and can usually be regarded as additive, Gaussian, signal-independent and non-correlated. A typical practical task to which such a model corresponds is, e.g., localization of constellations in stellar navigation.

Because of random noise, the problem of optimal localization should be treated statistically. We shall seek a way to obtain the statistically best
estimation of the coordinates \((x_0, y_0)\) of the object given samples of the observed signal \(b_1\). The statistically best estimation is known to be the estimation by the maximum a posteriori probability of \((x_0, y_0)\), or its equivalent, \((a_k(x_0, y_0))\) (see, e.g., Woldenstief and Jacobs [1965]). The a posteriori probability of \((a_k(x_0, y_0))\) given \(b_1\) may be found from the Bayes rule in probability theory as

\[
P(a_k(x_0, y_0)|b_1) = \frac{P(a_k(x_0, y_0)) P(b_1|a_k(x_0, y_0))}{P(b_1)},
\]

where \(P(\cdot | \cdot)\) are the corresponding a priori probabilities and \(P(\cdot | \cdot)\) are the corresponding conditional probabilities.

It is evident that for the model of eq. (2.1)

\[
P(b_1|a_k(x_0, y_0)) = P(b_1 = b_1 - a_k(x_0, y_0)),
\]

and \(P(b_1)\) does not depend on \((x_0, y_0)\). Therefore, the optimal estimation of the object coordinates will be

\[
(x_0, y_0) = \arg \max_{(x_0,y_0)} P(a_k(x_0, y_0)) P(b_1 = b_1 - a_k(x_0, y_0)).
\]

Such an estimation is called a maximum a posteriori probability (MAP) estimation. It requires a knowledge of the a priori probabilities of the object coordinates. If the estimation is made without regard to an a priori distribution, or, equivalently, made using the assumption of a uniform a priori distribution, it is called a maximum likelihood (ML) estimation.

Let us now write an explicit expression for \(P(a_k = b_1 - a_k(x_0, y_0))\). Since by our assumption the samples \(a_k\) are non-correlated Gaussian numbers with variance \(\sigma^2\),

\[
P(a_k = b_1 - a_k(x_0, y_0)) = C \prod_{k=0}^{K-1} \exp \left\{ -\frac{1}{2\sigma^2} [a_k(x_0, y_0) - b_1]^2 \right\},
\]

where \(C\) is an unimportant normalization factor and \(K\) is the total number of signal and noise samples. Substituting eq. (2.5) into eq. (2.4) and taking into account that \(\exp(\cdot)\) is a monotonic function and that \(\sum_{k=0}^{K-1} |a_k|^2\) and \(\sum_{k=0}^{K-1} |a_k(x_0, y_0)|^2\) do not depend on the coordinates \((x_0, y_0)\), we obtain

\[
(x_0, y_0) = \arg \max_{(x_0,y_0)} \left\{ \sum_{k=0}^{K-1} b_k a_k(x_0, y_0) + 2\sigma^2 \ln P(x_0, y_0) \right\}.
\]

According to the theory of discrete representation of signal transforms (see, e.g., Yaroslavsky [1985])

\[
\sum_{k=0}^{K-1} b_k a_k(x_0, y_0) = 2\pi F \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(x, y) a(x - x_0, y - y_0) \, dx \, dy.
\]

where \(b(x, y)\) and \(a(x - x_0, y - y_0)\) are the continuous signals that correspond to the set of samples \(b_k\) and \(a_k(x_0, y_0)\), and \(2\pi F\) are the frequency bandwidths which correspond to the chosen sampling rate for the discrete model of eq. (2.1). Then for continuous signals we obtain for MAP-estimation

\[
(x_0, y_0) = \arg \max_{(x_0,y_0)} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(x, y) a(x - x_0, y - y_0) \, dx \, dy \right. \]

\[
+ 2\pi F \ln P(x_0, y_0) \right\},
\]

where \(N_0 = 2\pi F\) is the spectral density of the noise, and for ML-estimation we obtain

\[
(x_0, y_0) = \arg \max_{(x_0,y_0)} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(x, y) a(x - x_0, y - y_0) \, dx \, dy \right. \]

\[
+ 2\pi F \ln P(x_0, y_0) \right\}.
\]

Thus, the optimal ML-estimator should calculate the mutual correlation function between the object signal \(a(x - x_0, y - y_0)\) and the observed signal \(b(x, y)\) and take the coordinates of the maximum of the correlation pattern as a coordinate estimation. The optimal MAP-estimator also consists of a correlator and a decision making device that locates the maximum in the correlation pattern (fig. 1). The only difference is that the correlation pattern should be biased by the appropriately normalized pattern of the a priori probability distribution of the coordinates of the object.

The operation of correlation of the mixture of signal and noise with the copy of the signal is often called matched filtering. The filter which carries

![Fig. 1. Block diagram of optimal localization device for localization of objects observed in the presence of additive white Gaussian noise.](image-url)
out this operation is appropriately called a matched filter. The frequency response of the matched filter, according to the properties of Fourier transform, is evidently \( \pi f_0 f_1 \), the complex conjugate of the object signal spectrum. A correlation, or matched filter type of localization device may be implemented easily by optical and holographic means. As was already mentioned, this was recognized at the very early stages of holography (Van der Lugt [1964]).

Let us determine the performance characteristics of the optimal ML coordinate estimator for the situation under consideration. In this analysis, we should distinguish between two essentially different types of possible errors of estimation (Yaroslavsky [1972]):

(i) small errors of measurements due to distortion of the object shape by the noise when coordinate estimations lie in the vicinity of their actual values, and

(ii) large errors due to false localization of the object very far from its actual location due to possible big noise outburst outside the object. These large errors are similar to the so-called "false alarm" errors in signal detection, or false recognition errors in object recognition.

Following the terminology of Kotel'nikov [1956], we shall refer to the first type of errors as normal errors because, as we shall see later, their distribution density is very close to a Gaussian one. Errors of the second type we shall call anomalous errors. Normal errors characterize the accuracy of measurements while anomalous errors characterize the measurement reliability.

2.2. LOCALIZATION OF A SINGLE OBJECT IN THE PRESENCE OF ADDITIVE WHITE GAUSSIAN NOISE: POTENTIAL ACCURACY OF COORDINATE MEASUREMENTS

Statistical characteristics of normal errors were found by Yaroslavsky [1972, 1992] from an analysis of the correlator output signal

\[
R_\lambda(x, y) = R_\lambda(x - x_0, y - y_0) + R_\lambda(x, y),
\]

where

\[
R_\lambda(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(\xi, \eta) a(\xi - x, \eta - y) d\xi d\eta.
\]

\[
R_\lambda(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\xi, \eta) a(\xi - x, \eta - y) d\xi d\eta.
\]

(2.11a)

(2.11b)

\( R_\lambda(x, y) \) is the auto-correlation function of the object signal

\[
R_\lambda(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\xi, \eta) a(\xi - x, \eta - y) d\xi d\eta,
\]

and

\[
R_\lambda(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\xi, \eta) a(\xi - x, \eta - y) d\xi d\eta.
\]

(2.11b)

is a random Gaussian field. It follows from eq. (2.11b) that its correlation function is equal to \( N_\lambda R_\lambda(x, y) \).

In the case of normal errors, the maximum of the signal \( R(x, y) \) at the correlator output occurs in the close vicinity of the coordinates \((x_0, y_0)\) of a maximum of \( R_\lambda(x, y) \). The location of the maximum of \( R(x, y) \) can be found from the following system of equations

\[
\begin{aligned}
\frac{\partial}{\partial x} R(x, y) &= \frac{\partial}{\partial x} R_\lambda(x - x_0, y - y_0) + \frac{\partial}{\partial x} R_\lambda(x, y) = 0, \\
\frac{\partial}{\partial y} R(x, y) &= \frac{\partial}{\partial y} R_\lambda(x - x_0, y - y_0) + \frac{\partial}{\partial y} R_\lambda(x, y) = 0.
\end{aligned}
\]

(2.12)

Let the solution of this system be

\[
\begin{aligned}
x &= x_0 + \eta_1; \\
y &= y_0 + \eta_2;
\end{aligned}
\]

(2.13)

and assume that the errors \( \eta_1 \) and \( \eta_2 \) are small. Then after some computations one can obtain for \( \eta_1 \) and \( \eta_2 \) the following relationships

\[
\begin{aligned}
\eta_1 &= \frac{D_x}{D_x D_y - D_{xy}^2} D_{xy} R_\lambda(x - x_0, y - y_0) \\
\eta_2 &= \frac{D_y}{D_x D_y - D_{xy}^2} D_{xy} R_\lambda(x - x_0, y - y_0)
\end{aligned}
\]

(2.14a)

(2.14b)

with

\[
\begin{aligned}
D_x &= \frac{\partial^2}{\partial x^2} R_\lambda(x - x_0, y - y_0) \bigg|_{x=x_0} = -4\pi^2 f_0^2 E_0; \\
D_y &= \frac{\partial^2}{\partial y^2} R_\lambda(x - x_0, y - y_0) \bigg|_{x=x_0} = -4\pi^2 f_0^2 E_0.
\end{aligned}
\]

(2.14c)

(2.14d)
\[ D_{x} = \frac{\partial}{\partial x} R_{x}(x, y) \bigg|_{y=y_{0}} = -4\pi^{2} p_{x}^{2} E_{x}. \]  

where \( E_{x} \) is object signal energy given by

\[ E_{x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathcal{F}(f_{x}, f_{y})|^{2} df_{x} df_{y}. \]  

\( \mathcal{F}(f_{x}, f_{y}) \) is Fourier spectrum of the object signal and is given by

\[ \mathcal{F}(f_{x}, f_{y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{A}(x, y) \exp\left(-j2\pi(f_{x}x + f_{y}y)\right) dx dy. \]  

\( p_{x}, p_{y}, p_{z} \) are the inertial moments of the object signal power spectrum along the corresponding axes,

\[ p_{x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x} \mathcal{F}(f_{x}, f_{y})^{2} df_{x} df_{y}, \]  

\[ p_{y} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{y} \mathcal{F}(f_{x}, f_{y})^{2} df_{x} df_{y}, \]  

and

\[ p_{z} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(f_{x}, f_{y})^{2} df_{x} df_{y}. \]  

These relationships hold because the derivatives of the Gaussian random process \( R_{x}(x, y) \), with correlation function \( N_{0} R_{x}(x, y) \) are also Gaussian random processes with zero mean and with variances \( 4\pi^{2} N_{0}, \mathcal{F}(f_{x}, f_{y})^{2} \) and \( 4\pi^{2} N_{0}, \mathcal{F}(f_{x}, f_{y})^{2} \), respectively.

Equations (2.14a)-(2.14e) imply that the small errors \( n_{x}, n_{y} \) in the determination of the object coordinates by the ML-estimator have a Gaussian distribution with zero mean and variances given by

\[ \sigma_{n_{x}}^{2} = \frac{1}{1 - \mu^{2}} \frac{N_{0}}{4\pi^{2} E_{x} p_{x}^{2}}, \]  

\[ \sigma_{n_{y}}^{2} = \frac{\mu}{1 - \mu^{2}} \frac{N_{0}}{4\pi^{2} E_{x} p_{y}^{2}}. \]  

where

\[ \mu = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x} \mathcal{F}(f_{x}, f_{y})^{2} df_{x} df_{y}}{(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(f_{x}, f_{y})^{2} df_{x} df_{y})^{2}}. \]

This means that variances of the normal errors are determined by the signal-to-noise ratio \( E_{x}/N_{0} \) and inertia moments (eqs. (2.17a)-(2.17d)) of the object signal power spectrum. These last are the only characteristics of the object shape that affect its potential localization accuracy.

If the object signal power spectrum is symmetrical in relation to the coordinate axes, then its diagonal moments given by

\[ \mathcal{F}(f_{x}, f_{y}) = |\mathcal{F}(f_{x}, f_{y})| \]  

and

\[ |\mathcal{F}(f_{x}, f_{y})| = |\mathcal{F}(f_{x}, f_{y})| = |\mathcal{F}(f_{x}, f_{y})|. \]  

eqs. (2.18a, b) take more simple forms:

\[ \sigma_{n_{x}}^{2} = \frac{1}{4\pi^{2} E_{x}} N_{0}, \]  

\[ \sigma_{n_{y}}^{2} = 0. \]  

The last equation means that if the signal power spectrum is axysymmetrical, normal localization errors along the coordinates \( x \) and \( y \) are noncorrelated. This situation arises if the object spectrum \( \mathcal{F}(f_{x}, f_{y}) \), or equivalently, the object signal \( R_{x}(x, y) \), is a separable function of the coordinates. Equation (2.20a) coincides with the classical relationship for one-dimensional signals (e.g., Wittcroft and Jacobs [1965]).

Sometimes it is more convenient to express the variances of the normal errors in terms of the variance of the noise at the matched filter output rather than in terms of the input noise spectral density \( N_{0} \). Since this variance is evidently equal to

\[ \sigma_{n_{x}}^{2} = N_{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathcal{F}(f_{x}, f_{y})|^{2} df_{x} df_{y} = E_{x} N_{0}. \]
eqs. (2.8a, b) may be rewritten as

\[ \sigma_{\alpha}^2 = \frac{1}{1 - \mu^2} \cdot \frac{\sigma_0^2}{4\pi^2 E_0} \]
\[ \sigma_{\beta}^2 = -\frac{\mu^2}{1 - \mu^2} \cdot \frac{\sigma_0^2}{4\pi^2 E_0} \cdot \frac{\gamma}{(1 + \gamma)^2} \]

(2.22a, 2.22b)

2.3. LOCALIZATION OF A SINGLE OBJECT IN THE PRESENCE OF ADDITIVE GAUSSIAN NOISE: MEASUREMENT ACCURACY FOR NON-OPTIMAL ESTIMATOR; LOCALIZATION IN NON-WHITE NOISE

Implementation of the correlator or matched filter requires an exact knowledge of the shape of the object under study. Of course, in practice the object shape is often not known with high accuracy and/or it must be approximated because of limitations of the implementation. Therefore, it is of interest to estimate losses in localization accuracy due to deviations of a real filter in the localization device from a matched filter. We shall explore these losses for the one-dimensional or separable spectrum case.

Let the frequency response of the filter in the localization device be \( H(f) \) instead of \( \alpha(f) \), the frequency response of the matched filter. The value of the signal at the output of such a filter will be

\[ b(x) = \int_{-\infty}^{\infty} a(f) H(f) \exp[-i2\pi f (x - x_0)] \, df + R_{\alpha}(x) \]

(2.23)

where \( R_{\alpha}(x) \) is the result of filtering the white noise component in the observed signal by the filter \( H(f) \). The location point \( x_0 + n_x \) of the maximum of this signal in the close vicinity of the point \( x_0 \) (\( n_x \) is small) is defined by

\[ \frac{\partial}{\partial x} b(x) \bigg|_{x = x_0 + n_x} = 0 \]

(2.24)

Since \( n_x \) is assumed to be small and \( x_0 \) is the point of the actual location of the object where the filter response is maximal, we obtain

\[ 4\pi^2 \int_{-\infty}^{\infty} f^2 |a(f) H(f) df|^2 \, n_x = \nu_x \]

(2.25)

where as before \( \nu_x = \langle \xi(x) \rangle R_{\alpha}(x) \). The power spectrum of the random process \( \nu_x \) is evidently \( 4\pi^2 \int_{-\infty}^{\infty} f^2 |a(f) H(f)|^2 \). Hence, the variance \( \sigma^2 \) of the small

The third factor in the formula is nothing but a loss factor (LSFR) showing how much the normal error variance for a non-optimal filter exceeds that of an optimal (matched) one,

\[ \sigma^2 = \frac{1}{f^2} \int_{-\infty}^{\infty} f^2 |a(f) H(f)|^2 \, df \]

(2.26)

The third factor in this formula is nothing but a loss factor (LSFR) showing how much the normal error variance for a non-optimal filter exceeds that of an optimal (matched) one,

\[ \text{LSFR} = \frac{\int_{-\infty}^{\infty} f^2 |a(f) H(f)|^2 \, df}{\int_{-\infty}^{\infty} f^2 |a(f) H(f)|^2 \, df} \]

(2.27)

According to the Schwarz inequality \( \text{LSFR} \geq 1 \). LSFR reaches its minimal value of \( 1 \) if the filter is matched to the object signal; i.e., if \( H(f) = \alpha(f) \).

To estimate the order of magnitude of the loss factor, we shall present the results of calculation of the LSFR for an object signal of Gaussian shape with spectrum \( a(f) = \exp[-f^2/2\sigma^2] \), and for the filter pulse response of a similar shape but of different spread, such that \( H(f) = \exp[-f^2/2\sigma'^2] \). For such a case, we obtain

\[ \text{LSFR} = \frac{1 + \sigma^2}{\sigma'^2} \]

(2.28)

A graph of LSFR versus the ratio of the object spread to that of the filter pulse response is shown in fig. 2. Figure 2 demonstrates that the losses become noticeable if this ratio exceeds a value of about 1.5; i.e., when the loss factor is equal approximately to 1.27. This means that if the filter in the localization device is not grossly mismatched to the object under localization, the localization accuracy will remain close to its upper limit.

Let us now suppose that the noise is non-white, i.e., its power spectrum is not uniform. Denote the noise power spectrum in this case by \( N(f)H(f,f) \). Evidently, this situation can be reduced to the previous one of white noise if we route the observed signal plus noise mixture through a so-called "whitening filter" with a frequency response of \( 1/H(f,f) \) (Kotelnikov [1956]). At the output of such a whitening filter, the noise power spectrum becomes uniform with a spectral density \( N_0 \), while the object signal spectrum becomes equal to \( a(f) H^2(f,f) \). In this case, an ML-optimal

random error \( n_x \) is

\[ \sigma^2 = \frac{1}{f^2} \int_{-\infty}^{\infty} f^2 |a(f) H(f)|^2 \, df \]

(2.26)
localization device will consist of the whitening filter, followed by a filter matched to the object signal at the output of the whitening filter, and a device for localization of the signal maximum. The whitening and matched filters may be combined into one optimal filter whose frequency response \( H_{o}(f, f_{s}) \) will be (see Van der Lugt [1964])

\[
H_{o}(f, f_{s}) = |H(f, f_{s})| \cdot \frac{1}{|H(f, f_{s})|^{2}}.
\] (2.29)

We thus arrive at the optimal estimator shown in fig. 3.

It is obvious that the potential localization accuracy for such an estimator will be defined by the same formulae (2.18a)–(2.18c) as for localization in

\[
\sigma_{o}^{2} = \frac{1}{1 - \mu_{o}^{2}} \cdot \frac{N_{o}}{4 \pi^{2} E_{w}^{2} E_{o}^{2}}.
\] (2.30a)

\[
\sigma_{o}^{2} = \frac{1}{1 - \mu_{o}^{2}} \cdot \frac{N_{o}}{4 \pi^{2} E_{w}^{2} E_{o}^{2}} \cdot \frac{1}{f_{o}^{2}(f_{0}^{2})^{1/2}}.
\] (2.30b)

\[
\mu_{o} = \frac{f_{o}^{2}(f_{0}^{2})^{1/2}}{f_{o}^{2}(f_{0}^{2})^{1/2}}.
\] (2.30c)

where

\[
E_{w} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(f, f_{s})|^{2}}{|H(f, f_{s})|^{2}} \cdot df_{s} \cdot df_{s}.
\] (2.31a)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(f, f_{s})|^{2}}{|H(f, f_{s})|^{2}} \cdot df_{s} \cdot df_{s}.
\] (2.31b)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(f, f_{s})|^{2}}{|H(f, f_{s})|^{2}} \cdot df_{s} \cdot df_{s}.
\] (2.31c)

2.4. OPTIMAL LOCALIZATION IN COLOR PICTURES

The results discussed above for optimal localization in monochrome pictures were extended by Yaroslavsky [1992b] to the case of localization in color or, more generally, multicomponent pictures observed with additive Gaussian noise.

Let \( a_{m}(x_{n}, y_{n}) \) be the \( m \)th component of the signal of the object to be localized, \( m = 1, 2, \ldots, M \), with \( M \) the number of components (for color pictures, \( M = 3 \) with red, blue and green representing the three components), and \( h_{n,m} \) and \( h_{m} \) be the samples of the corresponding components of the additive Gaussian noise and the observed signal, so that

\[
h_{n,m} = a_{m}(x_{n}, y_{n}) + h_{n,m}.
\] (2.32)

Further assume that the components of the additive noise in each of the \( M \) channels, as well as the samples of the same noise components, are
statistically independent. This assumption implies directly that the a posteriori probability of the coordinates \((x_0, y_0)\), given the observed signal \(b_{0,m}\), is

\[
P(\eta_m = b_0 - a_n(x_0, y_0)) = C \prod_{m=1}^{M} \prod_{n=1}^{K_m} \exp \left\{ -\frac{1}{2\sigma^2_n} (b_{0,m} - a_n(x_0, y_0))^2 \right\},
\]

(2.33)

where \(\sigma^2_n\) denotes the variance of the \(n\)-th component of the noise. Therefore, the optimal MAP- and ML-estimations are

\[
\hat{x}_0, \hat{y}_0 = \arg \max_{x_0, y_0} \left\{ \sum_{n=1}^{K} \sum_{n=1}^{K_m} b_{0,m} a_n(x_0, y_0) - \frac{1}{2} \sum_{n=1}^{K_m} \sigma^2_n \ln P(x_0, y_0) \right\}.
\]

(2.34a)

and

\[
\hat{x}_0, \hat{y}_0 = \arg \max_{x_0, y_0} \left\{ \sum_{n=1}^{K} \sum_{n=1}^{K_m} b_{0,m} a_n(x_0, y_0) - \frac{1}{2} \sum_{n=1}^{K_m} \sigma^2_n \ln P(x_0, y_0) \right\}.
\]

(2.34b)

For continuous signals, the MAP-estimation is

\[
\hat{x}_0, \hat{y}_0 = \arg \max_{x_0, y_0} \left\{ \sum_{n=1}^{K} \left\{ \int_{x_0}^{x_0 + K_m/N_0} b_{0,m} a_n(x, y) \times a_n(x-x_0, y-y_0) dx dy \right\} - \frac{1}{2} \sum_{n=1}^{K_m} \sigma^2_n \ln P(x_0, y_0) \right\}.
\]

(2.35)

where \(N_0 = 1/4F_s^2\) is the spectral density of the \(n\)-th component of the noise, and the ML-estimation is

\[
\hat{x}_0, \hat{y}_0 = \arg \max_{x_0, y_0} \left\{ \sum_{n=1}^{K} \left\{ \int_{x_0}^{x_0 + K_m/N_0} b_{0,m} a_n(x, y) \times a_n(x-x_0, y-y_0) dx dy \right\} \right\}.
\]

(2.36)

This means that the optimal localization device should consist of \(M\) parallel correlators (or matched filters) for each component of the signal, an adder for weighted summation of the outputs of the correlators, and a unit for the determination of the coordinates of the signal maximum at the output of the adder (fig. 4).

The variances of the normal errors that characterize the accuracy of

\[
\sigma^2_{x_0, y_0} = \frac{\gamma_0^2 + \gamma_{0,m}^2 - 2\gamma_{0,y}^2}{(1 - \mu^2)^2} \sigma^2_{\eta,x_m},
\]

(2.37a)

\[
\sigma^2_{y_0, y_0} = \frac{\gamma_{0,y}^2 + \gamma_{0,m}^2}{(1 - \mu^2)^2} \sigma^2_{\eta,y_m},
\]

(2.37b)

with

\[
\gamma_{0,y} = \frac{\int_{\eta_0}^{\eta_{0,m}} \eta_{0,y} d\eta_{0,y}}{\int_{\eta_0}^{\eta_{0,m}} d\eta_{0,y}} = \frac{\eta_{0,y}}{\eta_{0,m}},
\]

(2.37c)

\[
\gamma_{0,m} = \frac{\int_{\eta_0}^{\eta_{0,m}} \eta_{0,m} d\eta_{0,m}}{\int_{\eta_0}^{\eta_{0,m}} d\eta_{0,m}} = \frac{\eta_{0,m}}{\eta_{0,m}},
\]

(2.37d)

\[
\gamma_{0,x} = \frac{\int_{\eta_0}^{\eta_{0,m}} \eta_{0,x} d\eta_{0,x}}{\int_{\eta_0}^{\eta_{0,m}} d\eta_{0,x}} = \frac{\eta_{0,x}}{\eta_{0,x}},
\]

(2.37e)

\[
\gamma_{0,y} = \frac{\int_{\eta_0}^{\eta_{0,m}} \eta_{0,y} d\eta_{0,y}}{\int_{\eta_0}^{\eta_{0,m}} d\eta_{0,y}} = \frac{\eta_{0,y}}{\eta_{0,y}},
\]

(2.37f)
\[ s^2 = \frac{\alpha^2 f_2 f_3}{f_1 f_2 f_3} \] (2.37f)

\[ \beta_{\alpha} = \frac{1}{N_{\alpha}} \sum_{f_3} \int_{-\infty}^{\infty} \left| f_{\alpha}(f_1, f_2) \right|^2 df_1 df_2 \] (2.37g)

\[ \beta_{\alpha} = \frac{1}{N_{\alpha}} \sum_{f_3} \int_{-\infty}^{\infty} \left| f_{\alpha}(f_1, f_2) \right|^2 df_1 df_2 \] (2.37h)

\[ \epsilon_{\alpha} = \frac{1}{N_{\alpha}} \sum_{f_3} \int_{-\infty}^{\infty} \left| f_{\alpha}(f_1, f_2) \right|^2 df_1 df_2 \] (2.37i)

In the case of a symmetrical signal, i.e., \( \beta_{\alpha} = 0 \),

\[ \sigma_{\alpha}^2 = \frac{\gamma_\alpha}{4 \pi^2 f_1 f_2 f_3} \epsilon_{\alpha} \] (2.38a)

\[ \sigma_{\alpha}^2 = \frac{\gamma_\alpha}{4 \pi^2 f_1 f_2 f_3} \epsilon_{\alpha} \] (2.38b)

\[ \sigma_{\alpha}^2 = 0 \] (2.38c)

2.5. LOCALIZATION OF AN OBJECT IN THE PRESENCE OF ADDITIVE GAUSSIAN NOISE: RELIABILITY OF COORDINATE MEASUREMENTS

Let us now proceed to the characterization of anomalous localization errors. By our definition, anomalous errors occur when the localization device incorrectly locates the object somewhere outside of the area occupied by the object signal at the matched filter output. This takes place if random noise outstrips in the matched filter output exceed the signal value in the point of actual location of the object. Since noise in the output of the matched filter results from filtering the white Gaussian input noise, it is spatially homogenous. Hence, large noise outbursts and consequent anomalous errors are uniformly distributed over the area where the object is supposed to be located. This means that we can characterize anomalous errors simply by their rate of occurrence (i.e., probability).

The probability of anomalous errors was estimated by Yaroslavsky [1972], who used the following reasoning. Let \( \Delta S \) be the area of correlation of the noise in the output of the matched filter in the sense that it is the area that it takes for the correlation between noise values to become negligibly small. Since the correlation function of the noise in the output of the matched filter coincides with that of the object signal, \( \Delta S \) is of the same order of magnitude as the area occupied by the signal in the output of the matched filter. Therefore, in the area of the search, \( S \), there are of the order of \( Q = S / \Delta S \) non-correlated samples of Gaussian noise with variance \( \sigma^2 = N_e / E_c \). The probability \( P \) of anomalous errors is the complement to the probability that none of the \( (Q - 1) \) non-correlated samples of Gaussian noise outside of the object location exceeds the signal value of the point of actual location of the object, which according to the notations in §1.1 is equal to \( R(0, 0) + R_t \), where \( R_t \) is a Gaussian zero-mean random value with variance \( \sigma^2 \) and

\[ R_t(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f_{\alpha}(f_1, f_2) \right|^2 df_1 df_2 = E_c. \] (2.39a)

In this way, we obtain

\[ P = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}n^2\right) \left\{ 1 - \Phi\left( \frac{E_c}{\sqrt{N_e + n}} \right) \right\} \ dn \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}n^2\right) \left\{ 1 - \Phi\left( \frac{E_c}{\sqrt{N_e + n}} \right) \right\} \ dn, \] (2.39b)

with

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2) \ dn. \] (2.39c)

This formula is evidently valid for multicomponent pictures as well, with denotations of the appropriate values involved which were introduced in §2.4.

In communication theory, eq. (2.39b) is known as Kotelnikov's integral and is used to determine the probability of errors in a communication channel with \( M \) orthogonal signals and additive white Gaussian noise (see, e.g., Kotelnikov [1956], Wozencraft and Jacobs [1965]). It is illustrated in fig. 5.
A remarkable feature of Kotelnikov's integral is its threshold behaviour for large $Q$

$$\lim_{Q \to \infty} P_x = \begin{cases} 1, & \text{if } E_s/N_0 \ll \ln Q, \\ 0, & \text{if } E_s/N_0 \gg \ln Q. \end{cases} \quad (2.40)$$

This feature implies that if the field of search is large enough in comparison to the size of the object under localization, the probability of anomalous errors may become enormously high. To minimize this probability, the signal-to-noise ratio should be increased upon increasing the field of search.

A second important implication of eq. (2.40) is that for a given noise intensity there exists a trade-off between accuracy of localization (defined by the variance of normal errors) and localization reliability (described by the probability of anomalous errors). Increases in accuracy which are achieved by widening the object signal spectrum with the signal energy being fixed are accompanied by an increasing probability of anomalous errors. This result arises because widening the spectrum is equivalent to narrowing the signal and consequently increasing the ratio $Q$ of the area of search to the signal extent.

It should also be noted that eq. (2.40) has a general, fundamental meaning. Note that $\log Q$ is the entropy of the results of our measurements. Therefore, eq. (2.40) gives an absolute lower bound for object signal energy per unit of measurement information in the presence of white Gaussian noise,

$$E_s/\log Q \geq N_0 \ln 2.$$ \quad (2.41)

2.6. LOCALIZATION RELIABILITY IN THE PRESENCE OF ADDITIVE WHITE GAUSSIAN NOISE: MORE ACCURATE ESTIMATION AND APPROXIMATION OF THE LOCALIZATION ERROR DISTRIBUTION DENSITY

As it was stated in §§ 2.1, 2.2 and 2.5, two characteristic sections may be separated in the probability density of the localization error. The section of small errors forms the main mode of the density and has an approximately Gaussian shape with almost uniform tails which stretch to the borders of the field of search. The dispersion of the main Gaussian mode was estimated in § 2.2, and the total volume of the tail section (i.e., the probability of anomalous localization errors) was estimated in § 2.5. In this section we shall further substantiate these estimations using the methods of the theory of Gaussian random processes. These methods were developed in the early forties by Rice [1944, 1945]. We shall also summarize the results presented earlier by formulating a unified expression for the localization error distribution density.

For simplicity we consider the one-dimensional case (Yaroslavsky [1970]):

$$B(x - x_0) = d(x - x_0) + n(x).$$ \quad (2.42)

Let $\hat{x}_0$ be the estimation of the object coordinate $x_0$, obtained as the coordinate of the highest maximum of the signal in the output of the matched filter. The probability density $p(\hat{x}_0)$ of $\hat{x}_0$ may be found from the probability density $p(R, \hat{x}_0)$ of the event, such that in the point $\hat{x}_0$, the random process

$$R(x) = R_s(x - x_0) + R_n(x)$$ \quad (2.43)

in the output of the matched filter has a local maximum with a value $R$, and from the conditional probability $P(X | R, \hat{x}_0)$ that the process values in all other local maxima over the interval of search, say $X$, do not exceed $R$,

$$p(\hat{x}_0) = \int_{-\infty}^{\infty} p(R, \hat{x}_0) P(X | R, \hat{x}_0) \, dR.$$ \quad (2.44)
The probability density \( p(R, \hat{s}_0) \) may be computed as

\[
p(R, \hat{s}_0) \, dx = -\int_{-\infty}^{\infty} p(R(\hat{s}_0)) = \mathcal{R}(R(\hat{s}_0), R'(\hat{s}_0)) \, dR' \, dR.',
\]

where

\[
R'(\hat{s}_0) = \frac{\partial R(x)}{\partial x} \bigg|_{x=\hat{s}_0}, \quad R''(\hat{s}_0) = \frac{\partial^2 R(x)}{\partial x^2} \bigg|_{x=\hat{s}_0}.
\]

Since \( n(x) \) is supposed to be a Gaussian process it follows that (Rice [1944])

\[
p(R, R_c, R_{20}) = \left(2\pi\sigma_0^2\right)^{1/2} \frac{\exp\left(-\frac{1}{2\sigma_0^2} \int_{-\infty}^{\infty} [f_x'(s)]^2 \, ds\right)}{
\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma_0^2} [f_x'(s)]^2 \right) \, ds}
\]

where \( f_x' \) is the fourth moment of the object signal power spectrum.

\[
E_x = \int_{-\infty}^{\infty} f_x'(s)^2 \, ds
\]

After substitution of eq. (2.46) into eq. (2.45a) and integration, we obtain

\[
p(R, \hat{s}_0) = \left(2\pi\sigma_0^2\right)^{1/2} \exp\left(-\frac{R(\hat{s}_0)}{2\sigma_0^2} \left(\frac{1}{E_x} \int_{-\infty}^{\infty} f_x'(s)^2 \, ds\right) - \frac{R(\hat{s}_0)}{2\sigma_0^2} \right)
\]

For relatively small localization errors, when

\[
(\delta_0 - \hat{s}_0)^2 \ll \frac{R(\hat{s}_0)}{R_c(\hat{s}_0)} \approx \frac{1}{4\pi^2 f_c^2}
\]

the following approximations can be used for \( R(\hat{s}_0) \) and \( R_c(\hat{s}_0) \)

\[
\begin{align*}
R(\hat{s}_0) & \approx R_c(\hat{s}_0) \\
R_c(\hat{s}_0) & = (\delta_0 - \hat{s}_0)R_c(\hat{s}_0) = 4\pi^2 f_c^2 (\delta_0 - \hat{s}_0)
\end{align*}
\]

After some transformations, we then obtain the formula

\[
p(R, \hat{s}_0) = \left(\frac{\pi}{2\pi}\right)^{1/2} \exp\left(-\frac{1}{2\sigma_0^2} \left(\frac{R(\hat{s}_0)}{2\sigma_0^2} \left(\frac{1}{E_x} \int_{-\infty}^{\infty} f_x'(s)^2 \, ds\right) - \frac{R(\hat{s}_0)}{2\sigma_0^2} \right)
\]

\[
\times \left[\exp\left(-\frac{1}{2\sigma_0^2} \left(\frac{1}{E_x} \int_{-\infty}^{\infty} f_x'(s)^2 \, ds\right) - \frac{R(\hat{s}_0)}{2\sigma_0^2} \right) \right]
\]

with

\[
\sigma_0^2 = \frac{1}{4\pi^2 f_c^2} E_x
\]

If the signal-to-noise ratio is large enough, i.e., \( (E_x/N_0) \gg 1 \), then \( p(R, \hat{s}_0) \) can be approximated by a very simple expression,

\[
p(R, \hat{s}_0) = \left(\frac{\pi}{2\pi}\right)^{1/2} \exp\left(-\frac{R(\hat{s}_0)}{2\sigma_0^2}\right)
\]

Exact calculation of the conditional probability \( P(A/R, \hat{s}_0) \) in the general case is a difficult problem, but a reasonably good approximation can be obtained for high signal-to-noise ratios, \( (E_x/N_0) \gg 1 \). In this case, the value
R of the local maximum in the point \( x_\delta \) close to \( x_0 \) is much higher than the noise variance, and the distribution of the quantity of local maxima on the level higher than \( R \) tends to the Poisson distribution [Cramer and Leuβett 1967]. In this limiting case, if we neglect the probability of more than one local maximum in the area occupied by the object, we find that

\[
P(\mathcal{A}, R; x_\delta) \cong \exp[-Q_\delta(X - \Delta x)],
\]

where \( X \) is the area of search, \( \Delta x \) is the area occupied by the object, and \( Q_\delta \) is the mean quantity per unit length of the process \( n(x) \) maxima, exceeding the level \( R \). According to Rice [1944]

\[
Q_\delta = (\bar{f})^{1/2} \kappa \left[ \frac{1}{\kappa} \exp \left[ -\frac{\bar{r} + \sqrt{E_0/N_0}}{2} \right] \phi \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}} \right] \right].
\]

For high signal-to-noise ratios, \( (E_0/N_0) \gg 1 \),

\[
\phi \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}} \right] \approx \frac{(\kappa - 1)^{1/2} \exp \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{2} \right]}{(2\pi)^{1/2} \kappa} \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}}.
\]

and

\[
Q_\delta \approx (\bar{f})^{1/2} \exp \left[ -\frac{\bar{r} + \sqrt{E_0/N_0}}{2} \right] \phi \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}} \right].
\]

Therefore, we have in this case

\[
P(\mathcal{A}, R; x_\delta) \approx \left[ 1 - \exp \left[ -\frac{\bar{r} + \sqrt{E_0/N_0}}{2} \right] \phi \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}} \right] \right] \frac{1}{\kappa^2}.
\]

where we denote

\[
\bar{Q} - 1 = (X - \Delta x)^{1/2} \left( \frac{\bar{f}}{\kappa} \right).
\]

By substituting eqs. (2.52) and (2.56a) into eq. (2.44), we obtain after changing variables according to eq. (2.51a):

\[
p(\bar{r}_0) = \frac{1}{(2\pi)^{1/2} \bar{r}^{1/2}} \left[ \frac{1}{(2\pi)^{1/2} \bar{r}^{1/2}} \right] \exp \left[ -\frac{1}{2} \frac{r_0^2}{\bar{r}} \right]
\]

\[
\times \left[ 1 - \exp \left[ -\frac{\bar{r} + \sqrt{E_0/N_0}}{2} \right] \phi \left[ \frac{\bar{r} + \sqrt{E_0/N_0}}{\kappa + \sqrt{E_0/N_0}} \right] \right] \frac{1}{\kappa^2}.
\]

Equation (2.57) has a simple physical meaning. First of all, it confirms the conclusion reached in § 2.2, which was that small localization errors are normally distributed with the variance as defined by eqs. (2.18a, b). It also specifies, by eq. (2.49), which errors can be regarded as small. Furthermore, the integral in the formula has the meaning of a complement to the integral of eq. (2.39a) for the estimation of the anomalous localization errors. The factor \( \bar{Q} \) in eq. (2.56a), defined by eq. (2.56b), represents a more substantiated estimation of a parameter \( \bar{Q} \) which was introduced qualitatively in § 2.5 as the ratio of the area of search to the area occupied by the object.

These results can be summarized by the following approximation formula for the distribution density of the localization errors \( n_\Delta \) and \( n_n \):

\[
p(n_\Delta, n_n) = \frac{1}{2\pi \sigma_\Delta \sigma_n} \left[ \frac{1}{2\pi \sigma_\Delta \sigma_n} \right] \exp \left[ -\frac{1}{2} \frac{(n_\Delta - \sigma_\Delta)^2}{\sigma_\Delta^2} - \frac{(n_n - \sigma_n)^2}{\sigma_n^2} \right]
\]

\[
\times \exp \left[ -\frac{1}{2} \frac{(n_\Delta - \sigma_\Delta)^2}{\sigma_\Delta^2} - \frac{(n_n - \sigma_n)^2}{\sigma_n^2} \right]
\]

\[
\left( \frac{n_\Delta - 2 \sigma_\Delta n_n + \sigma_n^2}{\sigma_\Delta^2} \right)^{1/2} + \frac{P}{S} \Delta S.
\]

which unifies the characterization of normal (first term in the sum) as well as anomalous (second term) noise components. Recall from § 2.5 that \( S \) is the area of the field of search and \( \Delta S \) is the area occupied by the object.

2.7. LOCALIZATION RELIABILITY IN THE PRESENCE OF ADDITIVE WHITE GAUSSIAN NOISE AND MULTIPLE OUTSIDE OBJECTS

Let us now suppose that the observed picture signal \( b(x, y) \) having the object signal \( a(x - x_0, y - y_0) \) to be located also contains additive white Gaussian noise \( n(x, y) \) and some quantity \( Q \) of outside objects signals.
\begin{equation}
(\mu_i(x, y) \text{ which do not overlap each other or the given object (Yaroslavsky [1972]).})
\end{equation}

\begin{align*}
b(x, y) & = a(x - x_0, y - y_0) + \sum_{i=1}^{n} \mu_i(x, y) + n(x, y).
\end{align*}

A prominent illustration of such a situation could be the task of automatic detection of a specific character on a page of printed text.

Since the signals of the outside objects are supposed not to overlap the signal of the object under search, it is clear that the filter which ensures the highest localization accuracy, i.e., the lowest normal error variance, will, in this case, be the matched one. The variance of normal errors will be defined by the same formulae (eq. 2.18a, b) as in the absence of outside objects. The presence of outside objects affects only the probability of anomalous errors.

To estimate the probability of anomalous errors, let us introduce some quantitative characteristics of outside objects. Let \( P(\theta) \) be the probability of appearance of exactly \( Q \) outside objects in the area of search. Let the outside objects form some \( i \) classes by the maximal value \( R_{\max} \) of the cross-correlation function between the given object signal and the signal of the outside object of class \( q \). Let \( P(Q_0, Q_1, ..., Q_i) \) be the joint probability of appearance of \( Q_i \) objects of the first class, \( Q_i \) objects of the second class, etc., given the total quantity of outside objects is \( Q \).

Evidently, in the presence of outside objects, anomalous errors will occur mainly due to false identification of the given object with one of the outside objects. Then the probability of anomalous errors may be found approximately as the complement to the probability that all signal values in the output of the matched filter in the points of location of the outside objects do not exceed the signal value in the location of the given object. Since the outside objects do not overlap each other or the located object, these values can be regarded as statistically independent. Therefore, by analogy with eq. (2.39b),

\begin{equation}
P_i = 1 - \Delta N_i \sum_{Q_i \geq 0} P(Q) \sum_{Q_0} P(Q_0, Q_1, ..., Q_i) \times \prod_{i=1}^{N_i} \left[ 1 - \left\{ \frac{\phi \left( \frac{E_0 - R_{\max}}{\sqrt{N_i E_0}} + n \right) \Psi_i \right\} \right].
\end{equation}

where \( \Delta N_i \) denotes averaging over the Gaussian variable \( n \) with zero mean and unity variance. Since the probability distribution of the appearance of

\begin{equation}
P_i = \frac{1}{Q} \sum_{Q_0} P(Q) \sum_{Q_1} P(Q_1, Q_2, ..., Q_i) \times \prod_{i=1}^{N_i} \left[ 1 - \left\{ \frac{\phi \left( \frac{E_0 - R_{\max}}{\sqrt{N_i E_0}} + n \right) \Psi_i \right\} \right].
\end{equation}

Thus for estimating the probability of anomalous errors it is necessary to know: (i) the variance \( \sigma^2 = N_i E_0 \) of the additive noise at the matched filter output, (ii) the differences between the values of the auto- and cross-correlation peaks, (iii) the probabilities of the outside object for each class, and (iv) the probability distribution of the total quantity \( Q \) of outside objects in the area of search.

Some more transparent formulae and practical recommendations can be obtained by considering the partial cases, when the probability distribution \( P(Q) \) is concentrated around some point \( Q_0 \) and \( i = 1 \). In this case,

\begin{equation}
P_i = \frac{1}{Q} \sum_{Q_0} P(Q) \sum_{Q_1} P(Q_1, Q_2, ..., Q_i) \times \prod_{i=1}^{N_i} \left[ 1 - \left\{ \frac{\phi \left( \frac{E_0 - R_{\max}}{\sqrt{N_i E_0}} + n \right) \Psi_i \right\} \right].
\end{equation}

This expression coincides with eq. (2.39b) for the probability of anomalous errors with the presence of outside objects being the only difference. Now the signal-to-noise ratio is hardly reduced by the cross-correlation peaks of the outside objects. Naturally, the probability of anomalous errors in the case of multiple outside objects also features threshold behavior.

\begin{equation}
\lim_{Q \to \infty} P_i = \begin{cases} 1, & \text{when } \frac{E_0 - R_{\max}}{\sqrt{N_i E_0}} \leq \sqrt{\frac{E_i}{Q_0}} \ln Q_0; \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

For instance, for \( Q_0 = 16 \) to 2048 (note that the sum of characters on a standard printed page is about 2000) and \( (E_0 - R_{\max}) \sqrt{N_i E_0} \geq 4.5 \), the quantity \( P_i \) is less than \( 10^{-7} \), but it increases rapidly with decreasing signal-to-noise ratio \( (E_0 - R_{\max}) \sqrt{N_i E_0} \). The probability of anomalous errors in the presence of multiple outside objects could therefore be very high and the reliability of object detection could consequently be very low.
In order to increase the reliability, it is necessary to increase the signal-to-noise ratio by suppressing the cross-correlation peaks for outside objects in relation to the auto-correlation peak of the object under localization. This requires an appropriate modification of the matched filter.

We come to the conclusions that: (i) in the presence of multiple outside objects it is impossible to achieve simultaneously a minimum of normal error variance and a minimum probability of anomalous errors with the same localization device, and (ii) the optimal localization can be obtained in two steps. The first step is optimal localization with a minimal probability of anomalous errors. Here a reliable but not accurate estimation of coordinates is obtained. The second step is localization with minimal variance of normal errors within the small area found in the first step. The optimal estimator for the second step and its quantitative characteristics were discussed in this section. In the next section, we shall discuss the design and features of the minimal anomalous error localization device.

§3. Localization of Objects on a Complex Background with a Minimum of Anomalous Errors

3.1. FORMULATION OF THE PROBLEM

In the present and subsequent subsections, we shall discuss the problem of object localization in pictures with a complex background, like photographs of natural scenes, aerial and space photographs of the Earth, etc. The main problem of localization in this general case will be the problem of anomalous errors of localization caused by the background outside objects.

We confine the analysis to one involving a localization device similar to that depicted in fig. 3, consisting of a linear filter and decision making unit determining the coordinates of the absolute signal maximum in the filter output (Yaroslavsky [1975, 1985]). Our aim is now to find the optimal linear filter ensuring a minimum probability of anomalous localization errors.

At the onset of this discussion, let us define exactly the notion of optimality. In order to allow for possible nonhomogeneity of the optimality criterion over the picture frame, let us assume that the picture is decomposed into \( Q \) fragments of area \( S_q, q = 0, 1, \ldots, Q - 1 \) (fig. 6). Let \( h(x, y) \) be a histogram of the picture signal \( h(x, y) \) in the filter output as measured for the \( q \)-th fragment over the area not occupied by the object (i.e., the background)

area, for the fixed background, and fixed sensor (imaging system) noise, provided that the object is located at the point with coordinates \((x_0, y_0)\). Further, let \( h(x, y) \) be the filter output in the object location (it may be assumed that \( h > 0 \) without restricting the generality of the argument). Since the localization device under consideration depends upon the coordinates of the desired object through those of the absolute maximum at the signal output, the integral

\[
P(x_0, y_0) = AV_{\text{ave}}AV_{\text{ave}} \int_{h>0} h(x, y) \, dh,
\]

then represents the average portion of the \( q \)-th fragment points that can be erroneously taken by the decision unit for the object coordinates. The symbols \( AV_{\text{ave}} \) and \( AV_{\text{ave}} \) represent averaging over the sensor's noise and over possible realizations of the background component of the picture.

Generally speaking, \( h(x, y) \) should be regarded as a random variable because it depends on such random factors as sensor noise, photographic environment, illumination, object orientation, neighboring objects, etc. In order to take these factors into consideration, we introduce as a priori probability density \( p(h) \) of \( h(x, y) \). Object coordinates should also be regarded as random. Moreover, the weight of the measurement errors in the localization problems
may differ over different picture fragments. To allow for these factors, we introduce weighting functions \(w_q(x_0, y_0)\) and \(W_q\) characterizing the a priori significance of localization within the \(q\)th fragment and for each \(q\)th fragment, respectively,

\[
\int_{x_0} w_q(x_0, y_0) \, dx_0 \, dy_0 = 1; \tag{3.2a}
\]

\[
\sum_{q=1}^{n} W_q = 1. \tag{3.2b}
\]

The performance of the localization device under consideration may then be described by a weighted mean with respect to \(p(b_0), w_q(x_0, y_0), \) and \(W_q\) of the integral of eq. (3.1):

\[
P = \int_{-\infty}^{\infty} p(b_0) \sum_{q=0}^{n-1} W_q \int_{x_0} w_q(x_0, y_0) \, dx_0 \, dy_0 \\
\times \int_{b_0}^{\infty} \mathcal{A}V_{\text{im}} \mathcal{A}V_{\text{eq}} f_q(b, x_0, y_0) \, db. \tag{3.3}
\]

A device which provides the minimum value of \(P\) will be regarded as optimal in average over sensor noise and the background component of the picture. If we seek a device which would be optimal for the fixed background part of the picture, we should eliminate \(\mathcal{A}V_{\text{eq}}\) (i.e., averaging over background) from eq. (3.3).

### 3.2. Localization of an Exactly Known Object for the Spatially Homogeneous Optimality Criterion

Assume that the object under search is exactly defined, which in the present context means that the response of any filter to this object may be exactly determined, or that \(p(b_0)\) is a delta function:

\[
p(b_0) = \delta(b - b_0). \tag{3.4}\]

Let \(f_q(b)\) be the histogram, averaged within each fragment over \((x_0, y_0)\)

\[
f_q(b) = \int_{x_0} w_q(x_0, y_0) f_q(b, x_0, y_0) \, dx_0 \, dy_0. \tag{3.5}\]

Equation (3.3), which defines the localization quality, then becomes

\[
P = \sum_{q=1}^{n-1} W_q \int_{b_0}^{\infty} \mathcal{A}V_{\text{im}} \mathcal{A}V_{\text{eq}} f_q(b). \tag{3.6}\]

Suppose that the optimality criterion is spatially homogeneous, i.e., that the weights \(W_q\) are independent of \(q\) and are equal to \(1/\mathcal{Q}\). Then

\[
f(b) = \sum_{q=1}^{Q-1} \frac{1}{\mathcal{Q}} f_q(b) \tag{3.7}
\]

represents the histogram of the filter output signal as measured over the whole picture and averaged with respect to the unknown coordinates of the object under localization.

By substituting eq. (3.7) into eq. (3.6), we obtain

\[
P = \int_{b_0}^{\infty} \mathcal{A}V_{\text{im}} \mathcal{A}V_{\text{eq}} f(b) \, db. \tag{3.8}
\]

Let us now determine the frequency response \(H(f_s, f_l)\) of a filter which provides a minimum value of \(P\). In the derivation, we shall follow the ideas of Yaroslavsky [1979, 1986] with minor additional substantiations.

The choice of \(H(f_s, f_l)\) affects both \(b_0\) and the histogram \(f(b)\). Since \(b_0\) is the filter response at the point of object location, it may be determined through the object Fourier spectrum \(a(f_s, f_l)\) as

\[
b_0 = \int_{-\infty}^{\infty} a(f_s, f_l) H(f_s, f_l) \, df_s \, df_l. \tag{3.9}
\]

The relationship between \(b_0(f_s, f_l)\) and \(H(f_s, f_l)\) is, generally speaking, of an involved nature. The explicit dependence of \(b_0\) on \(H(f_s, f_l)\) may be written only for the second moment of the histogram \(f(b)\) by making use of Parseval's relation for the Fourier transform,

\[
m^2 = \int_{-\infty}^{\infty} b^2 f(b) \, db = \frac{1}{S} \int_{-\infty}^{\infty} \left| a(f_s, f_l) \right|^2 H(f_s, f_l) \, df_s \, df_l, \tag{3.10}
\]

where \(S\) is the search area of the picture at the filter output without the area occupied by the object signal,

\[
a(f_s, f_l) f(b) = \sum_{k=1}^{n} a_k \delta(k f_s - f_l) \int_{x_0} w_k(x_0, y_0) \, dx_0 \, dy_0 = \sum_{k=1}^{n} a_k \delta(k f_s - f_l) \tag{3.11}
\]

and \(\sum_{k=1}^{n} a_k \delta(k f_s - f_l)\) is the Fourier spectrum of the picture with the signal in the area occupied by the desired object set to zero (i.e., the spectrum of the background component of the picture).

Therefore, we can only rely upon the, in probability theory, well known Tchebychev's inequality (e.g., Hald [1962]), which connects the probability
for some random variable \( x \) to exceed some threshold \( h_0 \) with its mean value \( \bar{x} \) and standard deviation \( \sigma \) such that:

\[
\text{Probability}(x > \bar{x} + h_0 \sigma) \leq \epsilon_0 \tag{3.12}
\]

Applying this relationship to eq. (3.8) we can write

\[
P = \int_{h_0}^{\infty} \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n \hat{h}(b) \, db < \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n \frac{m_0^2 - k^2}{(b_0 - b)^2}, \tag{3.13}
\]

where \( k \) is the mean value of the histogram \( \hat{h}(b) \), which by virtue of the properties of the Fourier transform can be calculated as

\[
k = \frac{1}{S_1} \int_{S_1} \omega(x_0, y_0) \, dx_0 \, dy_0. \tag{3.14}
\]

It follows from this equation that the mean value \( \bar{F} \) of the histogram over the background part of the picture is defined by the filter frequency response \( H(0, 0) \) at the point \( (f_x = 0, f_y = 0) \). The same value affects the mean value of the signals over all the pictures under filtering. This constant bias of the signal at the filter output is irrelevant for the device which localizes the signal maximum. Therefore, we can choose any value for \( H(0, 0) \), and hence disregard \( \bar{F} \) in eq. (3.13) without restricting the generality of the analysis. We come to the conclusion that to make the rate \( P \) of anomalous errors minimal we should design a filter such that

\[
P(f_x, f_y) = \arg \min_{m_x, m_y} \left\{ \frac{\mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n (m_0^2)}{B_0^2} \right\} \tag{3.15}
\]

The solution of this equation follows from Schwartz' inequality:

\[
H_{opt}(f_x, f_y) = \frac{a^*(f_x, f_y)}{\mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |a(f_x, f_y)|^2}, \tag{3.16}
\]

where the asterisk (*) denotes the complex conjugate.

One can express \( |a(f_x, f_y)|^2 \) through the spectrum of the observed picture \( \beta(f_x, f_y) \) and that of the desired object \( a(f_x, f_y) \),

\[
a_n(f_x, f_y) = \beta(f_x, f_y) - a(f_x, f_y) \exp[-j2\pi(f_x x_0 + f_y y_0)]. \tag{3.17}
\]

Then, substitution of eq. (3.16) into eq. (3.11) results in

\[
\mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |a(f_x, f_y)|^2 = \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |r(f_x, f_y)|^2 - W(f_x, f_y) \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |a(f_x, f_y)|^2 = -W(f_x, f_y) \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |\beta(f_x, f_y)|^2,
\]

where \( f \) is \( (f_x, f_y) \), and

\[
W(f_x, f_y) = \int_{S_1} \omega(x_0, y_0) \exp[-j2\pi(f_x x_0 + f_y y_0)] \, dx_0 \, dy_0 \tag{3.18}
\]

is the spectrum of the weight function \( \omega(x_0, y_0) \).

At this point, it is useful to make some simplifying assumptions. First, note that if the weight function \( \omega(x_0, y_0) \) is approximately uniform over the picture area, its Fourier transform \( W(f_x, f_y) \) will be a function which is essentially very close to zero over the entire frequency plane except in the very close vicinity of the point \( (f_x = 0, f_y = 0) \). Therefore, its influence does not extend very far from this point, and is thus irrelevant in filter design. In a first approximation, we can neglect the last two terms in eq. (3.17) for the denominator of the optimal filter frequency response (3.15).

Second, the ratio of the power spectrum of the total picture to that of the object under search is, in order of magnitude, equal to the squared ratio of their areas. Usually, the area occupied by the object is much less than that of the picture itself. Therefore, we can often also neglect the second term in eq. (3.17) and use the following approximation for the denominator of eq. (3.15),

\[
\mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |a_n(f_x, f_y)|^2 \approx \mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |\beta(f_x, f_y)|^2, \tag{3.19}
\]

which implies that

\[
H_{opt}(f_x, f_y) = \frac{a^*(f_x, f_y)}{\mathcal{A} \mathcal{V}_m \mathcal{A} \mathcal{V}_n |\beta(f_x, f_y)|^2}. \tag{3.20}
\]

At last we can see that very often we might be interested in the most reliable localization in the given picture, i.e., with fixed background rather than in average over all possible pictures. In this case we should omit averaging \( \mathcal{A} \mathcal{V}_m \). Then the optimal filter will be represented as follows

\[
H_{opt}(f_x, f_y) = \frac{a^*(f_x, f_y)}{\mathcal{A} \mathcal{V}_m |\beta(f_x, f_y)|^2}. \tag{3.21}
\]

The filter described by eq. (3.21) is obviously adaptive, since its frequency response is determined by the power spectrum \( |\beta(f_x, f_y)|^2 \) of the observed picture.
On the other hand, $A^2 \nu_\sigma|\hat{D}(f_s,f_r)|^2$ is just a statistical power spectrum of the picture ensemble. Therefore, the filter described by eq. (3.20), which involves averaging over realizations of the background, is nothing but an optimal filter (2.29) for non-white additive noise.

As described in §1, the object recognition problem is similar to that of localization, and anomalous localization errors are equivalent to false recognition errors. Therefore, a localization device which works with a minimum probability of anomalous errors will provide the highest discrimination capability in the object recognition task. The filter described by eq. (3.20) is therefore optimal for recognition as well. The only difference is that, in this case, the average power spectrum $A^2 \nu_\sigma \nu_\sigma |\hat{D}(f_s,f_r)|^2$ of the observed picture should be substituted by the averaged power spectra $A^2 \nu_\sigma |\hat{D}(f_s,f_r)|^2$ of all the objects to be discriminated from the given one,

$$H_m(f_s,f_r) = \frac{A^2 \nu_\sigma |\hat{D}(f_s,f_r)|^2}{A^2 \nu_\sigma |\hat{D}(f_s,f_r)|^2}$$ (3.22)

Averaging $A^2 \nu_\sigma$ in eq. (3.22) should be carried out over all the objects, numbered by $k$, with the averaging weights proportional to the probability of appearance of the corresponding false object or, more generally, to some measure of wrong identification of the given object with the corresponding false one.

From the preceding argument, we see that the ratio of the squared signal value in the optimal filter output at the point of actual location of the object to the variance of the output signal over the rest of the picture can be used as a measure of the localization device optimality. We shall refer to this ratio as the "signal-to-noise" ratio (SNR).

From the above, it follows that for the optimal adaptive filter described by eq. (3.21), the SNR is given by

$$\text{SNR} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{D}(f_s,f_r)|^2}{A^2 \nu_\sigma |\hat{D}(f_s,f_r)|^2} df_s df_r.$$ (3.23)

We shall illustrate the advantageous and characteristic features of the optimal filter of eq. (3.21) using results obtained by computer simulation. Simulation results for a one-dimensional signal are shown in fig. 7. The reference signal in this experiment was the fragment of the signal marked in the fig. 7 as "object". One can easily see from this figure that the conventional correlator is unable to discriminate the "object" from other pulses in the signal, because the correlation peak is lower than the cross-correlation ones. In contrast, the optimal filter does this successfully: signal outbursts from the heavy

![Fig. 7. Comparison of the conventional correlator and the optimal filter for localization of an "object" in a one-dimensional signal.](image)

pulses in the signal are suppressed considerably in comparison to the signal peak from the object.

An experiment similar to the computer simulation just described was carried out by Varenslovsky (1975, 1979) using a real picture. An aerial photograph (fig. 8) digitized over a square raster of 512 x 512 pixels was used in experiments on localization of 20 test (5 x 5) pixels, which were uniformly dark marks superimposed upon the picture. The disposition of the marks is shown in fig. 9 by numbered squares. As can be seen from this scheme, the test marks were located in structurally different areas of the aerial photograph in order to evaluate the correlator and optimal filter performance under different background conditions. The contrast of the marks was about 25% of the video signal magnitude range. For synthesis of the filter represented by eq. (3.21), the observed power spectrum of the picture $|\hat{D}(f_s,f_r)|^2$ was used as a zero-order approximation for the averaged-over-noise spectrum $A^2 \nu_\sigma |\hat{D}(f_s,f_r)|^2$.

A comparison of the output signals for the conventional matched filter and the optimal one is presented in fig. 10. Figure 10 shows (in the downward direction) the graphs of the lines of the initial picture and the outputs of the conventional correlator and optimal filter going through the centers of marks (12) and (15) in fig. 8. One can easily see in the graph of correlator output the auto-correlation peaks of the test spots and false cross-correlation peaks,
including those exceeding the auto-correlation one. These false peaks result in false decisions. Comparison of this graph with the lower one in fig. 10, which presents the same line of the signal at the output of the optimal filter, shows how much the optimal filter facilitates the task of spot localization for the decision making unit.

The optimal adaptive filter can be implemented both digitally and optically. In digital implementation, one can use fast algorithms based on fast Fourier transforms for signal convolution. For optical implementation, a coherent optical system with a non-linear medium placed in the Fourier plane was proposed (Yaroslavsky [1973, 1976]; fig. 11). In this system when being exposed to a power spectrum of the picture, the non-linear medium implements a denominator of the optimal filter (3.21), if the transparency of the medium is inversely proportional to the intensity of the incident light. Experiments by Dudin, Kryshatik and Yaroslavsky [1977] with negative recording of the power spectrum of the observed picture on a photographic material with $\gamma = 1$ have confirmed the radical improvement in the reliability of object localization in aerial photographs.

3.3. LOCALIZATION OF INEXACTLY KNOWN OBJECTS
(Yaroslavsky [1979, 1986])

In the case of an inexact object, there exists uncertainty about the object parameters; i.e., the probability density $q(h_0)$ cannot be regarded
as a delta function. Therefore, the optimal estimator must provide the minimum of the integral

$$P_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A V_m A V_n \tilde{R}(b) \, db \, dp(b)$$

where \( \tilde{R}(b) \) is defined by eq. (3.5).

Two different possibilities should now be considered. Each possibility depends on implementation restrictions.

(a) An estimator with selection. Decompose the interval of possible values of \( b_0 \) into sub-intervals within which \( p(b_0) \) may be regarded as constant. Then,

$$\bar{P}_i \geq \sum_k p_k \int_{a_k}^{b_k} A V_m A V_n \tilde{R}(b) \, db,$$

where \( b_k \) is a representative of the \( k \)th interval and \( p_k \) is the area under \( p(b_0) \) over \( k \)th interval. Since \( p_k \geq 0 \), then \( \bar{P}_i \) is minimal if

$$P_{i}^k = \int_{a_k}^{b_k} A V_m A V_n \tilde{R}(b) \, db$$

is minimal. The problem reduces to that of localization of an exactly known object, with the only difference being that now a set of optimal filters given by

$$H_{i}^{w}(b_0, b_0) = \frac{2V_m}{A V_m A V_n} (\tilde{R}(b_0, b_0))^2$$

should be generated for each “representative” of all possible object variations. The same argument also applies to the adaptive filters of the type given by eq. (3.21).

Of course, the generation and fabrication of multiple filters (as well as multiple filtering itself) requires additional time and hardware, which can
be unacceptable. In this case, the only alternative is adjustment of the filter to an averaged object.

(b) Estimator adjusted to an averaged object. If the variance of the uncertain object parameters is not too large, one can solve the problem as though the object is exactly known, albeit at the expense of a higher rate of anomalous errors. The optimal filter in this case should be corrected with due regard to the object parameter dispersion. In order to show this, change the variables \( b_1 = b - b_0 \) and the order of integration in eq. (3.24),

\[
P_b = \int_0^\infty \int_{-\infty}^\infty p(b_0, b_1) f(b_0 + b_1) \, db_0.
\]

The internal integral in eq. (3.28) is a convolution of distributions, or a distribution of the difference of two variables \( b \) and \( b_0 \). Denote this distribution by \( \eta_{b_0}(b_1) \). Its mean value is equal to the difference of mean values \( b_0 \) and \( b_0 \), i.e., \( b_0 - b_0 \), and the variance is equal to the sum of variances of these distributions; i.e., \( \sigma^2 = \sigma_{b_0}^2 + \sigma_{b_1}^2 \), where \( \sigma_{b_0}^2 \) is the variance of the distribution \( p(b_0) \). Therefore,

\[
\eta_{b_0}(b_1) = \int_0^\infty p(b_0, b_1) \, db_0 = \int_{-\infty}^{\infty} \eta_{b_0}(b_1 + b_0) \, db_0.
\]

The problem has thus been reduced to that of § 3.2. One can use eq. (3.15) for optimal filter frequency response with appropriate modifications of its numerator and denominator. Since \( \eta_{b_0}(b_1) \) is a mean value of the filter output in the point of the object located over the distribution \( p(b_0) \), the complex conjugate spectrum \( \hat{s}^*(f_x, f_y) \) of the object in the numerator of eq. (3.15) should be substituted by the average over the \( p(b_0) \) of the complex conjugate object spectrum \( \hat{s}^*(f_x, f_y) \). The denominator of eq. (3.15) should be modified by addition of the variance of the object spectrum,

\[
|\eta_{b_0}(f_x, f_y)|^2 = A^2 \text{Var}(\hat{s}(f_x, f_y) + \hat{\eta}(f_x, f_y))^2.
\]

We can now write the following expression for the optimal-in-average filter frequency response,

\[
R_{opt}(f_x, f_y) = \frac{\hat{s}^*(f_x, f_y)}{A^2 \text{Var}(\hat{s}(f_x, f_y))^2 + \text{Var}(\hat{\eta}(f_x, f_y))^2}.
\]

The variance of the object spectrum \( |\eta_{b_0}(f_x, f_y)|^2 \), being of the order of magnitude of the object power spectrum \( |\hat{s}(f_x, f_y)|^2 \), is evidently considerably less than the power spectrum of the total picture. Consequently, for the denominator of eq. (3.31) we may use the same approximation as that used in eq. (3.20)

\[
R_{opt}(f_x, f_y) = \frac{\hat{s}^*(f_x, f_y)}{\text{Var}(\hat{s}(f_x, f_y))}. \tag{337}
\]

By eliminating averaging over the background, we obtain the adaptive filter

\[
R_{opt}(f_x, f_y) = \frac{\hat{s}^*(f_x, f_y)}{\text{Var}(\hat{\eta}(f_x, f_y))}. \tag{333}
\]

As already mentioned, such an averaged filter cannot provide as large a "signal-to-noise" ratio as a filter adjusted for the exactly known object. Indeed, for each specific object with spectrum \( \hat{s}(f_x, f_y) \), the following inequality for the "signal-to-noise" ratio \( \text{SNR} \) for the filter (3.32) takes place,

\[
\text{SNR} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{s}^*(f_x, f_y) \hat{s}(f_x, f_y)}{\text{Var}(\hat{\eta}(f_x, f_y))^2} \, df_x \, df_y
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{s}^2(f_x, f_y)}{\text{Var}(\hat{\eta}(f_x, f_y))^2} \, df_x \, df_y. \tag{334}
\]

Note that the right-hand part of the inequality is the "signal-to-noise" ratio for the optimal filter matched to the object.

3.4. RELIABLE LOCALIZATION FOR SPATIALLY INHOMOGENEOUS OBJECTS

(Turovskii, 1979, 1980)

Let us now abandon the assumption of § 3.2 concerning the spatial homogeneity in the optimality criterion. One of the following two ways to attain the minimum of \( P \) eq. (3.3) may then be chosen, depending on the implementation constraints.

(a) Localization device with a re-adjustable filter. Under a given non-negative \( W_b \), the minimum of \( P \) is attained at the minima of all

\[
P_b = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_x(x_v, y_v) \, dx_v \, dy_v \, db_b
\times \int_{-\infty}^{\infty} A^2 \text{Var}(\hat{s}(f_x, f_y))^2 \, df_x \, df_y \, db_b. \tag{335}
\]

This means that the filter should be re-adjustable for each fragment, filtering being carried out in the fragments within which the averaging in eq. (3.35)
is done. For each fragment, the optimal filter is determined through eqs. (3.13), (3.20), (3.21), (3.32) and (3.33) on the basis of measurements of the observed local spectra of fragments (with allowance for the above reservations about the influence of the object spectrum on the observed picture spectrum).

According to eq. (3.3), the fragments do not overlap; this corresponds to the fragment-by-fragment mode of processing. This can be extended naturally to the sliding processing mode based upon an estimate of the current local power spectrum of the picture. Note also that with both fragment-wise and sliding processing, the re-adjustable filter response does not depend on the weights \(W_e\).

(b) Localization device with a fixed filter. When there is no possibility of using an adjustable filter with fragment-wise or sliding processing, an alternative way is to design an estimator adjusted to the power spectrum of picture fragments averaged over \(W_e\). Indeed, it follows from eq. (3.3) that

\[
P = \int_{-\infty}^{\infty} \rho(b) \, db \times \int_{-\infty}^{\infty} \left( \sum_{e=0}^{Q-1} W_e \sum_{x_0 \in \mathcal{N}_{b_0}(b, x_0, y_0)} w_e(x_0, y_0) \, dx_0 \, dy_0 \right) \, db
\]

\[
= \int_{-\infty}^{\infty} \rho(b) \, db \int_{-\infty}^{\infty} \tilde{e}(b) \, db,
\]

where \(\tilde{e}(b)\) is a histogram averaged over \(W_e\) and \(w_e(x_0, y_0)\). By analogy with eq. (3.31) one may conclude from this that

\[
\tilde{e}(b) = \frac{\Delta V}{\Delta V_{\text{in}}} \, \frac{\Delta V_{\text{in}}}{\Delta V_{\text{out}}} \, |\tau_e(f_x, f_y)|^2 + |\tau_{\text{ref}}(f_x, f_y)|^2 \]

where

\[
\Delta V_{\text{out}} = \sum_{e=0}^{Q-1} W_e |\tau_e(f_x, f_y)|^2,
\]

and \(\tau_e(f_x, f_y)\) is the spectrum of the \(e\)th fragment background component. Thus, the frequency response of the optimal filter in this case depends on the weights \(W_e\) and is determined by the averaged power spectrum of all fragments.

3.5. RELIABLE LOCALIZATION IN BLURRED PICTURES

In many practical situations the picture in which an object should be located is defocused or blurred by the imaging system. The approach presented can be extended to this case as well (Yaroslavsky [1979, 1986]).

Let the picture be distorted by a linear, spatially invariant system with frequency response \(H(f_x, f_y)\). Obviously, the optimal localization device should be adjusted in this case to an object that was subjected to the same distortions as the observed picture. For instance, the adaptive filter described by eq. (3.21) should be modified as follows:

\[
H_{\text{opt}}(f_x, f_y) = \frac{H^*(f_x, f_y) |\tilde{e}(f_x, f_y)|^2}{\Delta V_{\text{in}} |H(f_x, f_y)|^2}.
\]

The physical meaning of this formula will be more evident if we represent \(H_{\text{opt}}(f_x, f_y)\) in the following equivalent form

\[
H_{\text{opt}}(f_x, f_y) = \frac{H^*(f_x, f_y) |\tilde{e}(f_x, f_y)|^2}{\Delta V_{\text{in}} |H(f_x, f_y)|^2 |\tilde{e}(f_x, f_y)|^2}
\]

with \(|\tilde{e}(f_x, f_y)|^2\) being the power spectrum of a hypothetical non-distorted picture. Thus, the optimal filter (3.38) may be regarded as consisting of two filters in cascade

\[
H_{\text{opt}}(f_x, f_y) = H_{\text{opt}1}(f_x, f_y) \cdot H_{\text{opt}2}(f_x, f_y).
\]

The first filter

\[
H_{\text{opt}1}(f_x, f_y) = \frac{H^*(f_x, f_y) |\tilde{e}(f_x, f_y)|^2}{\Delta V_{\text{in}} |H(f_x, f_y)|^2}
\]

\[
= \frac{H^*(f_x, f_y) |\tilde{e}(f_x, f_y)|^2}{[H(f_x, f_y)]^2[\tilde{e}(f_x, f_y)]^2}.
\]

\((N(f_x, f_y))\) is the spectral density of additive sensor noise) is nothing but an adaptive filter for correcting picture blur (Yaroslavsky [1987]). The second filter

\[
H_{\text{opt}2}(f_x, f_y) = \frac{2|\tilde{e}(f_x, f_y)|}{|\tilde{e}(f_x, f_y)|^2}.
\]

is simply the optimal localization filter for a non-distorted picture. This means that the optimal filter (3.38) may be treated as one performing deblurring and then optimal filtering of the resultant deblurred picture.

It will now be instructive to estimate how much picture blur affects the
localization reliability, which can be characterized by the attainable "signal-to-noise" ratio at the optimal filter output. By analogy with eq. (3.32) we can write

$$\text{SNR} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ N_0(f_x, f_y) \right] \left[ |H_0(f_x, f_y)|^2 \right] \, df_x \, df_y,$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ |H_0(f_x, f_y)|^2 \left[ |H_0(f_x, f_y)|^2 + N_0(f_x, f_y) \right] \right] \, df_x \, df_y,$$  \hspace{1cm} (3.43)

from which it is evident that if the frequency response of the imaging system does not fall to zero and the noise level is not too high, picture blur does not deteriorate the performance of the optimal filter to a large extent.

3.6. OPTIMAL LOCALIZATION IN MULTICOMPONENT PICTURES WITH CLUTTERED BACKGROUND

Though derivation of the structure of an optimal device for object localization in multicomponent pictures requires bulky computations. Therefore, we shall proceed from the analogy between optimal localization in pictures with additive Gaussian noise and that in single-component pictures with cluttered background. Following this analogy, we can conclude that the device for localization in multicomponent pictures with cluttered background should contain: (i) a unit for decorrelating the picture components, (ii) optimal filters in each channel, (iii) an adder for summation of the channel optimal filter outputs, and (iv) a device for localization of the signal maximum at the output of the adder. Thus, we modify the device of fig. 4 for object localization in multicomponent pictures with non-correlated noise in each channel in a way shown in fig. 12.

The decorrelating unit in this device is intended for suppressing the interchannel cross-correlation terms in the adder output. It can be implemented as a transcoder multiplying the picture component samples by a decorrelating matrix. The alternative way to perform component decorrelation is to carry out whitening of the component Fourier transform spectrum. The decorrelation operator DECORR, which must be applied in this case to the picture component samples \( \{b_{ix}(x, y)\} \), can be expressed as

$$\text{DECORR} = \text{DFT}^{-1} \left\{ \frac{\text{DFT} \{b_{ix}(x, y)\}}{\text{DFT} \{b_{ix}(x, y)\}} \right\} \hspace{1cm} (3.44)$$

where DFT(\cdot) and DFT\(^{-1}\)(\cdot) are direct and inverse discrete Fourier trans-

![Fig. 12. Block diagram of the optimal device for localization of objects in multicomponent pictures with cluttered background.]

forms over the variable \( m \), the number of the components. The optimal filters \( H_m(f_x, f_y) \) in the channels can be defined according to eq. (3.21) as

$$H_m(f_x, f_y) = \frac{\text{FT}_r \{ \tilde{A}_m(x, y) \}^*}{|\text{FT}_r \{ \tilde{A}_m(x, y) \}|^2},$$  \hspace{1cm} (3.45)

where \( \tilde{A}_m(x, y) \) and \( \tilde{A}_m(x, y) \) are the output of the decorrelating unit for the object and picture signals.

3.7. PHASE-ONLY, BINARY PHASE-ONLY, MINIMUM AVERAGE CORRELATION ENERGY, ENTROPY-OPTIMIZED AND OTHER FILTERS FOR OPTICAL PATTERN RECOGNITION, RELIABLE LOCALIZATION AND PICTURE CONTOURS

Very soon after Van der Lugt's [1964] introduction of holographic matched spatial filters and optical correlators in pattern recognition, it was recognized that the ability of matched filters to discriminate effectively between objects of different classes is far less ideal than their ability to
combat additive Gaussian noise. Recognition of this fact stimulated the
search for filters with better discrimination ability.

One of the oldest and most popular ideas is that first proposed by
Lowenthal and Belvaux [1967], and involves preprocessing of both the
picture and the reference object by an appropriate differential operator
before they are correlated. This can be accomplished by a corresponding
spatial filtering in coherent light, e.g., by using opaque stops or slits centered
on the optical axis, or by introducing into the coherent optical correlator
an additional spatial filter. If the transmittance of the filter is made propor-
tional to the distance from the optical axis, the filter gives a good approxima-
tion to the first derivative of the signal. Another simple method for increasing
discrimination by differentiation involves using the non-linearity of the
medium on which an holographic matched filter is recorded. It was noted
by Van der Lugt [1968] and by Bums, Dickinson and Warraniewicz [1968]
that overexposing the spectrum information from the target, of which the
filter is being made, can result in high pass filtering similar to spatial
differentiation. The so-called joint transform correlators, where formation
of the matched filter and the filtering operation are carried out simulta-
neously by using a non-linear medium in the Fourier plane of the coherent
optical correlator (Weaver and Goodman [1966]), were recently demon-
strated by Judith [1989] to have better discrimination capability than the
matched filter if the non-linearity of the media is properly chosen.

Fabrication of matched filter requires, in general, recording of both ampli-
tude and phase information. This is not a trivial process, especially if the
filter is to be a computer-generated synthetic one (as, e.g., an averaged filter,
or the filter for localization or recognition of an object given only by its
mathematical description, etc.). To simplify filter synthesis, Horner and
Gianino [1964] proposed the use of the so-called phase-only filter (POF)
instead of the matched filter. The POF is a filter with a constant amplitude
transmittance; the phase transmittance is equal to that of the matched filter.
The next attempt to simplify filter synthesis involved the so-called binary
phase-only filter (BPOF), where only the sign of the real and imaginary
parts of the matched filter complex transmittance is recorded (Horner and
Leger [1985]). Despite their having been introduced primarily to avoid the
synthesis of complex filters and to increase the filter efficiency in using light
energy in optical correlators, it was soon apparent that both POF and
BPOF have much better discrimination ability than a conventional matched
filter. This fact, along with their simplicity of implementation, contributed
greatly to the popularity of the POFs.

III. {3} LOCALIZATION OF OBJECTS ON A COMPLEX BACKGROUND

The better performance of the POF is usually explained by modeling this
filter as a combination of the matched filter with frequency response $a(f)$
and the inverse filter $1/|a(f)|$ (see, e.g., Chiajaka-Macukow and Nitka
[1987]):

$$POF(f) = a(f)/|a(f)|.$$  (3.46)

The inverse filter makes the amplitude response of the POF constant. Being
reciprocal to the magnitude of the reference object spectrum, the inverse
filter acts as a high pass or differentiating filter since the magnitude of the
picture spectrum usually decreases as the spatial frequency increases. But
why should enhancement of the picture's higher spatial frequencies resulting
from high-pass filtering or differentiating improve the discrimination capa-
abdility of the filter? And if an improvement can be achieved, by how much and
in what concrete ways should the higher frequencies be enhanced? Without
having answered these questions, many different improvements of the POF
were proposed:

(i) Amplitude compensated matched filters (ACMF) (Mu, Wang and Wang
[1988])

$$ACMF(f) = z^*(f)|a(f)|^2;$$  (3.47)

(ii) Amplitude-modulated POF (APPOF) (Awal, Karim and Jahan
[1990])

$$AMPOF(f) = z^*(f)|a(f)|^2 + \epsilon,$$  (3.48)

where $\epsilon$ is a small constant introduced to avoid the need to divide by zero;

(iii) POF with improved signal-to-noise ratio (POF/SNR) (Vijaya Kumar
and Zouhair Bahri [1989])

$$POF/SNR(f) = \begin{cases} z^*(f) & \text{for } f \neq F, \\ |a(f)|^2 & \text{for } f = F, \\ 0 & \text{otherwise}, \end{cases}$$  (3.49)

where the area $F$ is chosen by some optimization procedure to improve
signal-to-noise ratio;

(iv) A family of ternary matched filters (TMF) (e.g., Dickey, Vijaya Kumar,
Romero and Connely [1990]), which are the natural generalization of the
BPOF. The real and imaginary parts of the matched filter transmittance are
quantized on three levels $\{1, -1, 0\}$ instead of two, and the location of the
areas where transmittance of the filter must be zeroed is again chosen by an
optimization procedure to improve the signal-to-noise ratio;
Optimal BPOF (OPOF) (Farn and Goodman [1988]):

$$\text{OPOF}(f) = \exp\left[\left(\phi_1 c(f) + \phi_2 (1 - c(f))\right)^2\right].$$

(3.50)

where \(c(f)\) is a binary function and \(\phi_1\) and \(\phi_2\) are constants chosen from the standpoint of optimization of the filter figure of merits.

It is clear that the problem of filter optimality, in terms of discrimination capability, has been addressed by many researchers. To complete this short review, we shall mention two more approaches to filter discrimination capability optimization.

Mahalanobis, Vijaya Kumar and Casassus [1987] have recently introduced so-called "minimum average correlation energy" (MACE) filters. MACE filters produce sharp output correlation peaks and maximize the ratio of the squared peak value of the correlation function to the average correlation plane energy. In principle, MACE-filters are designed for identification (or detection) of multiple targets in the presence of virtually different types of uncertainty of their a priori description. However, for an exactly known object they coincide with the ACMF filters. Mahalanobis and Casassus [1991] demonstrated good performance of the MACE filters in actual experiments. Remarkably, they still needed some preprocessing of the input images in the form of edge enhancement, which modulates the correlation between the images under recognition. The need for preprocessing is, of course, not surprising because the set of possible images to be discriminated is in no way involved in the design of the MACE filter.

Fleischer, Mahlab and Shamir [1990] and Mahlab, Fleischer and Shamir [1990] have proposed another criterion for the synthesis of the optimal filter. They require a strong, narrow peak for a match between the input and filter function as contrasted with uniform signal distribution for a pattern to be rejected. For a pattern to be rejected, the signal distribution over the picture area is treated as a probability density distribution, so the requirement of uniform distribution is equivalent to the requirement of maximization of the entropy of this distribution. This is why filters obtained according to this criterion are referred to as entropy optimized filters (EOF).

Fleischer, Mahlab and Shamir [1990] showed by computer simulation that the discrimination power of the EOF is much better than that of the POF, and is comparable with that of MACE filters. It was also observed experimentally that a dominant feature of the EOF is a substantial enhancement of the high-frequency components of the images. Another important feature of the EOF is its adaptivity, since the entropy criterion also takes into account the patterns to be rejected. Despite these attractive features, optimi-

zation by the entropy criterion practically never yields a purely constant signal outside of the detection peak. The question therefore remains: Is it possible to further reduce the probability of a wrong classification which will take place when the remaining outbursts of the filter output signal (plus sensors noise, which is always present in real signals) exceed the detection peak?

All of the filters reviewed in this section can be shown to approximate more-or-less the optimal filters of eqs. (3.21), (3.33) and (3.37a). The formulae for the optimal filters can also provide the answers to questions concerning the importance of high frequencies and contours for pattern recognition. To demonstrate this, we represent the basic formula (eq. (3.21)) for the optimal filter in the following way:

$$H_{opt}(f) = \frac{1}{(\bar{S}V_{min}[R(f)]^2)^{1/2} (\bar{S}V_{max}[R(f)]^2)^{1/2}}.$$

(3.51)

In such a representation, the optimal filter is regarded as consisting of two filters in cascade. The first one, which is represented by the first factor in eq. (3.51), may be referred to as a whitening filter. This is an appropriate description, because in the case of simultaneous observation of both the object and outside objects within the same picture, this filter makes the picture signal spectrum almost uniform in the filter output. The second filter (the second factor in eq. (3.51)) is obviously a matched filter for the object, pre-distorted by the same whitening operator. In the case of separate observation of patterns in pattern recognition, the whitening acts as an orthogonalization procedure, as described by Caufield and Maloney [1969].

If the observed signal contains only the object signal without any wrong patterns, the second filter in eq. (3.51) will be just a phase-only filter, and the entire filter (3.51) will be just an amplitude-compensated phase-only filter, as well as a MACE filter for a single object. As filter output we shall have, of course, a delta function. Further, in pattern recognition the main danger of false recognition is connected to the patterns which closely resemble the given one. In this case, the phase-only filter presents a quite reasonable approximation to the second filter in eq. (3.51), depending on the similarity between the patterns; the other filters mentioned above approximate the optimal filter itself. However, it is dissimilarities in spectra that are most important for the discrimination of patterns. The optimal filter takes advantage of these dissimilarities, while other filters do not.

Superiority of the optimal filter to, for instance, POF can be illustrated by the simulation results presented in fig. 13 for the signal of fig. 7.
(Yaroslavsky [1992c]). From top to bottom, the graphs show the output signals of the phase-only and optimal filters. In this one-dimensional simulation experiment the number of signal samples was chosen to be 128. Filtering was carried out by the FFT-technique. As an estimate of the denominator of eq. (3.51) the squared spectrum modulus of the total signal was chosen and smoothed by the convolution with a rectangle window of five samples for averaging over computer round-off noise.

An important feature of the optimal filter (3.51) is its adaptivity in application for target detection, since its frequency response is determined by the power spectrum of the observed picture. The whitening operation is therefore also adaptive. Due to the fact that usually (but not always) the picture spectrum decreases with increasing spatial frequencies, whitening results in enhancement of high frequencies (visually, this can be observed as edge enhancement). This is why the recommendations about enhancement of high frequencies work. But in contrast to these ad hoc recommendations, eq. (3.51) explicitly says to what degree, and in which specific way, this enhancement must be done for each specific picture.

Figures of the initial signal, of the whitening filter output signal and of the whitening filter pulse response are shown in fig. 14. Figure 14 illustrates the edge enhancement feature of the optimal filter for the same signal and object as represented in fig. 7. However, if the object to be detected was masked, for instance, by an high-frequency grid, the whitening would not enhance but rather attenuate the high frequencies of the grid. The same phenomena is illustrated by fig. 15 for the aerial photo of fig. 8 and by fig. 16 for the test pictures of geometrical figures and characters. It is clearly seen from these figures that whitening automatically suppresses all powerful or frequently occurring features of the objects. Examples of these features include low-frequency components of the pictures and even some edges if they occur very often in different patterns, such as vertical and horizontal lines in geometrical figures and characters. On the other hand, whitening enhances dissimilarities in the patterns, like circumferences and corners in geometrical figures and corners and inclined fragments of the characters.

Equations (3.21) and (3.51) also provide a rational explanation of the importance of different frequency bands in the signal. It was shown in § 3.2 (eq. (3.23)) that the signal-to-noise ratio at the output of this filter is equal to:

$$\text{SNR} = \int_{\mathcal{F}} \frac{|\tilde{d}(f)|^2}{\tilde{V}_{\text{rms}}(f)^2} \, df,$$

i.e., equal to the energy of the whitened spectrum of the object. Consequently, the higher the energy of the whitened signal, the better the signal-to-noise ratio. It is this relationship that measures the importance of the signal frequencies. In the next subsection we shall discuss how it can be exploited.
for selection of the reference objects from the point of view of their potential localization reliability.

3.8. SELECTION OF REFERENCE OBJECTS FROM THE STANDPOINT OF LOCALIZATION RELIABILITY

There exist numerous applications in which the reference object is not assigned and must therefore be selected. This is the case in aerial photo image registration and matching with a map; similar problems occur in stereogrammetry, in robot vision, etc. The question is how to make this choice to best advantage.

The literature on stereogrammetry recommends to take as reference objects those fragments of the aerial photo that have pronounced local peculiarities such as crossroads, river bends, separately situated buildings, etc. In the literature on pattern recognition and computer vision one can find recommendations to choose as reference objects those fragments of the picture where some informative functions such as local variance of the signal, variance of local gradients, or other similar measures of the fragment detailedness have extreme values. Almost all these recommendations are of a qualitative nature.

The analysis presented in this review provides a more quantitative and accurate approach to this problem, and gives a reasonable explanation of the various recommendations which have been offered. Indeed, eq (3.52) can be regarded as a precise performance measure for optimal localization. It follows from this equation for the maximum possible “signal-to-noise” ratio in the output of the optimal linear filter that the best reference objects will be those picture fragments which have maximal variance of the “whitened” power spectrum. Due to this feature they provide the greatest response of the optimal filter and, consequently, a minimal rate of the false identification errors.

Therefore, the reference objects with slowly decreasing spectra (i.e., picture fragments) which are visually estimated as containing the most intensive contours will be the best ones. Experimental verification of this conclusion was obtained by Yaroslavsky [1986]. The results of this investigation are
illustrated in figs. 17 and 18, where the "goodness" of the fragments of 16 × 16 pixels in terms of eq. (3.52) is represented by the darkness of the pixels. It may be readily seen that where the picture has some sharply pronounced local peculiarities (brightness overfalls, textures, etc.), the best fragments are found. Mostly they are located in the places that one would call contours. But it is seen from the figures that not all fragments which we would qualify as containing contours will be the best reference objects even if the fragments contain intensive brightness overfalls. For instance, the vertical edges in fig. 17 or the line of the coast in fig. 18 are not excellent reference objects in spite of their good contrast. This is because there are many similar fragments in the pictures, which means that it would be very difficult to distinguish them one from another. This performance measure is very sensitive to the dissimilarities between picture fragments and could be effectively used for automatic selection of the reference objects in pictures.

The corresponding algorithm for choice of relevant objects requires rather cumbersome computations. Therefore, computationally simpler algorithms approximating the exact one are of interest. Experiments by Belinsky and Yaroslavsky [1980] have shown that algorithms for computation of the local signal variance or local mean of the magnitudes of video signal gradients which are computationally very simple may be used as such approximating algorithms. But with these algorithms we, of course, lose the adaptivity inherent in the more comprehensive algorithms.

§ 4. Conclusion

The theory of optimal localization devices, which consist of an optimal adaptive linear filter and a point-wise decision making unit, seems now to
be more or less complete. Nonetheless, experimental experience and intuitive conjectures suggest that in the future, supplementing linear filtering with some adaptive, non-linear processing promises further and perhaps radical improvements in localization reliability.

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