LECTURE 4.1

Image parameter estimation

4.1.1 Problem formulation

Parameter estimation is one of the most fundamental problems with many applications in signal processing in general and in image processing in particular (Fig. 4.1.1). The problem is formulated as follows (Fig. 4.1.2): given an observed input signal (image) samples $\{s_k^{inp}, k = 0,1,..., N-1\}$ that implicitly depend on a parameter r, determine this parameter with the highest possible accuracy assuming that the way in which the parameter determines the signal is known to the accuracy of some random factors involved in the signalling (imaging) process such as, for instance, signal (image) sensor random noise. The parameter to be estimated can take arbitrary values. One can assume that probabilities of these values $Pr(r)$ (or probability density $p(r)$ if the parameter takes values in a continuous interval) can be known in advance. They are called a priori probabilities (probability density). Because of the random factors involved in the problem, quality of the parameter estimation can be evaluated only in statistical terms of probability of error (for parameters that takes finite discrete set of values) or, for parameters with values in a continuous interval, probability density of the estimation error $p(r - \hat{r})$, where $\hat{r}$ is the parameter estimate. It is known in statistical theory of decision making and parameter estimation that the estimate that provides minimum to the probability of error is one that has maximal a posteriori probability $Pr(r / \{s_k^{inp}\})$ (probability that the parameter has a particular value r given the signal $\{s_k^{inp}\}$):

$$\hat{r}_{opt} = \underset{s(x,r) \circledR \hat{r}}{arg\,max}\; Pr(r / \{s_k^{inp}\}). \qquad (4.1.1)$$

This estimate is called the MAP estimate. Therefore, optimal parameter estimation device should be capable of measuring, given the observed signal $\{s_k^{inp}\}$, a posteriori probabilities of values of the parameter and of determining the value for which this probability is maximal. To enable this, one should specify how exactly the random factors involved in the observed signal (image) determine the input signal.

4.1.2 Additive white Gaussian noise (AWGN) model

One of the most widely accepted models of signal observation is that of additive signal independent white Gaussian noise

$$\{s_k^{inp}\} = \{s_k^{tr}(r) + n_k\}, \tag{4.1.2}$$

where $\{n_k\}$ are uncorrelated random values with distribution density:

$$p_n(n) = \frac{1}{\sqrt{2\pi\sigma}} exp\left(-\frac{n^2}{2\sigma_n^2}\right). \tag{4.1.3}$$

and $\{s_k^{tr}(r)\}$ are samples of a hypothetical "true" signal that is deterministically (uniquely) linked with the parameter r. From Eq. (4.1.2) one can find probability density of input signal samples given parameter r:

$$Pr\left(\{s_k^{inp}\}/\ r\right) = Pr\left(\{n_k = s_k^{inp} - s_k^{tr}(r)\}\right). \tag{4.1.4}$$

A posteriori probability $Pr\left(r/\{s_k^{inp}\}\right)$ can be then found from Bayes relationship of the probability theory:

$$Pr\left(r/\{s_k^{inp}\}\right) = \frac{Pr\left(r/\{s_k^{inp}\}\right)p(r)}{p\left(\{s_k^{inp}\}\right)} \propto Pr\left(\{n_k = s_k^{inp} - s_k^{tr}(r)\}\right) p(r). \tag{4.1.5}$$

Substitute Eq.(4.1.3) into Eq. (4.1.5) and obtain for uncorrelated additive noise:

$$Pr\left(r/\{s_k^{inp}\}\right) \propto exp\left(-\frac{1}{2\sigma_n^2}\sum_{k=0}\left|s_k^{inp} - s_k^{tr}(r)\right|^2 + ln\left(p(r)\right)\right) \propto$$

$$exp\left(-\frac{1}{2\sigma_n^2}\sum_{k=0}^{N-1}\left|s_k^{tr}(r)\right|^2 + \frac{1}{\sigma_n^2}\sum_{k=0}^{N-1} s_k^{inp} s_k^{tr}(r) + ln\left(p(r)\right)\right). \tag{4.1.6}$$

Therefore, MAP-estimate can be found as:

$$r_{MAP}^{opt} = \underset{\{s_k^{inp}\}\to r}{arg\,max}\left(exp\left(-\frac{1}{2\sigma_n^2}\sum_{k=0}^{N-1}\left|s_k^{tr}(r)\right|^2 + \frac{1}{\sigma_n^2}\sum_{k=0}^{N-1} s_k^{inp} s_k^{tr}(r) + ln\left(p(r)\right)\right)\right). \tag{4.1.7}$$

If a priori distribution of the parameter $p(r)$ is uniform, MAP estimate becomes what is called maximum likelihood (ML) estimate

$$r_{MAP}^{opt} = \arg\max_{\{s_k^{inp}\}\to r} \left(exp\left(-\frac{1}{2\sigma_n^2}\sum_{k=0}^{N-1}\left|s_k^{tr}(r)\right|^2 + \frac{1}{\sigma_n^2}\sum_{k=0}^{N-1} s_k^{inp} s_k^{tr}(r) \right)\right). \qquad (4.1.8)$$

ML estimate is also used when the parameter a priori distribution is not known.

Eqs. (4.2.7) and (4.1.8) define the structure of a parameter estimation device. They imply that this device should consist of a unit for computing correlation between input signal $\{s_k^{inp}\}$ and templates $\{s_k^{tr}(r)\}$ for different values of the parameter followed by a unit for generation an estimate by finding which correlator output is the highest (Fig. 4.1.2)

These results can be extended to multi component images such as, for instance, colour or multispectral images. Let images are represented by M components, for all components AWGN model is applied:

$$s_k^{inp,m} = s_k^{tr,m}(r) + n_k^m\,;\; m = 1,\ldots,M \qquad (4.1.9)$$

and all noise samples are uncorrelated. Then one can obtain that MAP-estimate of the parameter is defined by the equation:

$$r_{MAP}^{opt} = \arg\max_{\{s_{k,l}^{inp,m}\}\to r} \left(exp\left(-\sum_{m=1}^{M}\frac{1}{2\sigma_{n^m}^2}\sum_{k=0}^{N-1}\left|s_k^{tr,m}(r)\right|^2 + \sum_{m=1}^{M}\frac{1}{\sigma_{n^m}^2}\sum_{k=0}^{N-1} s_k^{inp,m} s_k^{tr,m}(r) + ln(p(r)) \right)\right)$$

(4.1.10)

4.1.3 Estimation of signal position: AWGN-model

An important special case of image parameter estimation is that of object localization. The goal of the localisation is determination of co-ordinates of a given target object in image. Let (x_0, y_0) are unknown co-ordinates of a target defined by its samples $\{s_k^{tr}(x_0, y_0)\}$ that are observed in a mixture with additive white Gaussian signal independent noise. (AWGN model). Then, with a account that target signal energy represented by the first term in the exponent in Eq. (4.1.17) does not depend on target co-ordinates, one can find from Eq. (4.1.17) that MAP co-ordinate estimation $(\hat{x}_0, \hat{y}_0)$ is defined by the following equation:

$$(\hat{x}_0, \hat{y}_0) = \arg\max_{(x_0, y_0)} \left\{ \frac{1}{\sigma_n^2}\sum_{k=0}^{N-1} s_k^{inp} s_k^{tr}(x_0, y_0) + ln(p(x_0, y_0)) \right\}, \qquad (4.1.11)$$

where $p(x_0, y_0)$ is co-ordinate a priori distribution density.

Applying to Eq. (4.1.11) sampling theorem, one can convert it into equation for corresponding continuous signals $s^{tr}(x - x_0, y - y_0)$ and $s^{inp}(x, y)$.

$$(\hat{x}_0, \hat{y}_0) = \underset{(x_0, y_0)}{arg\,max} \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^{inp}(x, y) s^{tr}(x - x_0, y - y_0) + N_0 \ln(p(x_0, y_0)) \right\}, \qquad (4.1.12)$$

where N_0 is noise spectral density.

Since target co-ordinates are shift parameters, computation target signal – input signal correlation function needed for finding optimal co-ordinate estimate $(\hat{x}_0, \hat{y}_0)$ can be implemented by input signal filtering by a linear filter with impulse response $s^{tr}(x, y)$. Such an implementation is called ***matched filtering*** ([1], Fig. 4.1.4). In digital image processing, matched filtering can be efficiently implemented in Fourier domain by means of Fast Fourier Transform algorithms. Frequency response $H_{opt}(f_x, f_y)$ of the matched filter can be found as complex conjugate to Fourier transform spectrum $\alpha(f_x, f_y)$ of the target signal.

$$H_{opt}(f_x, f_y) = \alpha^{*}(f_x, f_y) = \left(\int_{-\infty}^{\infty} s^{tr}(x, y) exp[i2\pi(f_x x + f_y y)] dxdy \right)^{*}. \qquad (4.1.13)$$

The matched filtering concept can be extended to target localization in multi-component images. As it follows from Eq. 4.1.10, for optimal localization of a target in multi-component images with uncorrelated additive Gaussian noise, one should sum up outputs of the corresponding component wise matched filters to obtain a resulting correlation signal in which position of maximum should be found and taken as the co-ordinate estimate.

4.1.4 Target location in correlated (non white) noise.

If additive observation noise is correlated (non white) the problem of optimal target location can be reduced to that in white noise, if one puts input signal that contain now white noise through a filter that converts non white noise with spectral density $N(f_x, f_y)$ into white one. This filter is called “whitening” filter (Fig. 4.1.6). Its frequency response is inversely proportional to square root of noise spectral density:

$$H_w(f_x, f_y) = 1/[N(f_x, f_y)]^{1/2}. \qquad (4.1.14)$$

Obviously, the same whitening filtering should be applied to the target signal as well. In the implementations, one can combine matched filtering and input and target signal whitenings into one filter with frequency response

$$H_{opt}(f_x, f_y) = a^*(f_x, f_y) / N_n(f_x, f_y). \qquad (4.1.15)$$

This filter is called ***"optimal" filter***. One can show ([1]) that the optimal filter provides maximum to the ratio of signal peak at its output in the point of the target location to standard deviation of the additive noise for any distribution of noise in the input signal. Optimal filter is most frequently a sort of a band pass filter as it is illustrated in Fig. 4.1.7 since it represents a compromise between the need of suppressing correlated noise with most of its energy concentrated in low frequencies while preserving substantial part of the target signal energy that may be spread more uniformly in the bandwidth.

4.1. 5 Performance characteristics of optimal localization devices. Accuracy and reliability of localization.

Complete statistical characterization of the localization device performance is provided by the distribution density of localization errors:

$$e_x = x_0 - \hat{x}_0 ; \; e_y = y_0 - \hat{y}_0 . \qquad (4.1.16)$$

However, analytical solution for the error distribution density is not known. One can nevertheless obtain reasonable good approximation to it by taking into account that two substantially different types of errors define the localization quality. Consider signal at the output of the matched filter (Fig. 4.1.8). It consists of two additive components: the matched filtered target signal one and matched filtered noise component Since matched filter impulse response is identical to the target signal, the matched filtered target signal is identical to the target signal autocorrelation function and the same is the correlation function of the matched filtered white noise. As one can see from the figure, localization error can be caused by two reasons which define two types of the error. One type of errors is caused by relatively small misplacements of the signal maximal peak from the position of the signal autocorrelation peak due to the noise present. These errors are called ***"normal errors"*** as one can show that they have normal distribution density ([2-5]). The second type of errors occur when a noise peak happens far away from the position of of the signal autocorrelation peak and this noise peak exceeds matched filter output signal in the target location point. These errors are called ***anomalous errors***.

As normal errors have normal distribution density, they can be characterized by their variance. In Appendix 1, a brief outline of the derivation of the normal error standard deviation is provided for 1-D signals. One can show that for 2-D signals in their continuous representation, variance of normal localization errors id defined by Eqs. 4.1.17- 4.1.23

$$\sigma_x^2 = \frac{1}{4\pi^2} \frac{\bar{f}_y^2}{\bar{f}_x^2 \bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2} \frac{N_0}{E_a}; \tag{4.1.17}$$

$$\sigma_y^2 = \frac{1}{4\pi^2} \frac{\bar{f}_x^2}{\bar{f}_x^2 \bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2} \frac{N_0}{E_a}; \tag{4.1.18}$$

$$\sigma_{xy}^2 = \frac{1}{4\pi^2} \frac{\bar{f}_{xy}^2}{\bar{f}_x^2 \bar{f}_y^2 - \left(\bar{f}_{xy}^2\right)^2} \frac{N_0}{E_a}. \tag{4.1.19}$$

where

$$E_a = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y . \tag{4.1.20}$$

$$\bar{f}_x^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x^2 \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}. \tag{4.1.21}$$

$$\bar{f}_y^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_y^2 \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}. \tag{4.1.22}$$

$$\bar{f}_{xy}^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x f_y \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\alpha\left(f_x, f_y\right)\right|^2 df_x df_y}. \tag{4.1.23}$$

Discrete analogues of these formulas one can find in Ref. [4].

Variance of normal errors can be regarded as a measure of ***the localization accuracy***. Anomalous errors define the ***localization reliability***. They can be characterized by the following reasoning. If the additive noise is stationary (spatially homogeneous) anomalously high noise peak can occur with equal probability everywhere outside actual target object location. Therefore, probability density of this component of the localization error can be regarded uniform, and the errors can be completely characterized by the probability of the event that such an error happens. This probability can be found as complement to the probability that noise values outside the signal autocorrelation peak do not exceed matched filter output signal in the point of target location ([2-4]). The latter can be evaluated from the assumption that matched filter noise values measured at the distance of its correlation interval (which is of the order of magnitude of the target signal correlation interval that is approximately equal to the target size) are uncorrelated. In this way one can arrive at the following formula for the probability of anomalous (false detection) errors ([2-4]):

$$P_{ae} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{\nu^2}{2}\right) \left\{1 - \left[\Phi\left(\sqrt{\frac{E_a}{N_0}} + \nu\right)\right]^{Q-1}\right\} d\nu \qquad (4.1.24)$$

where Q is the number of possible target non overlapping position:

$$Q \cong \frac{Image\ Size}{Target\ Size}, \qquad (4.1.25)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{\nu^2}{2}\right) d\nu, \qquad (4.1.26)$$

and E_a target signal energy is defined by Eq. 4.1.20.

Fig. 4.1.9 shows an experimental distribution density of the localization error in which one can clearly see a Gaussian shaped mode around zero error that represents normal errors and uniform tales that can attributed to anomalous errors.

An important property of the probability anomalous error is that when the number Q of possible non overlapping positions of the target in the input image becomes very large, the probability of anomalous errors as a function of signal-to noise ratio $E_a / 2N_0$ exhibits threshold behaviour ([2-4]):

$$\lim_{Q \to \infty} P_{ae} = \begin{cases} 1, if\ E_a / 2N_0 \le lnQ \\ 0, if\ E_a / 2N_0 > lnQ \end{cases}. \qquad (4.1.27)$$

In this equation, lnQ can be regarded as a measure of uncertainty in the target position within the input image. Therefore one can interpret Eq. 4.1.28 as a lower bound on signal-to-noise ratio per "nit" (a substitute for bit for natural logarithm measure of the uncertainty) of information regarding possible target position. This property is illustrated by experimental plots in Fig. 4.1.10.

Eqs. 4.1.17 – 4.1.23 that describe the variance of normal errors and Eqs. 4.1.24 – 4.1.27 that describe probability of anomalous errors imply that with respect to the target size there is a trade-off between localization accuracy and reliability. It follows from these equations that the smaller is the target size, the lower is variance of normal localization errors (the higher is localization accuracy) and the higher is probability of anomalous errors (the lower is the localization reliability)

4.1.6 Multiple foreign non overlapping object model

In applications, a model of Eq. 4.1.2 with a single object observed in the mixture with additive noise is very frequently too simplistic to describe a real situation. In some of such cases, a more realistic model that assumes that the image contains, in adition to the target object and observation noise, some number of foreign objects that are non overlapping with the target object. A typical situation that can be modelled in this way is character recognition in printed text. Since foreign objects are supposed do not overlap with the target, from the previous analysis one can conclude for multiple non overlapping model that:

- Localization accuracy depends solely on target signal-to-noise ratio in the same way as for single object model of Eq. 4.1.2.
- Localization reliability is determined here mostly by the presence of foreign objects and their cross correlation with the target object. Since cross correlations between target and foreign object can be very substantial, probability of anomalous errors may be very high and even equal to one unless special measures are taken.

Fig. 4.1.11 compares localization of an object (character ”O”) on a uniform background and within printed text that contains other characters as well. One can see from the figure that even if standard deviation of additive noise in the text image is about 12 times higher then that in the image with single character, matched filter that is optimal for single object model, has several false detection (anomalous error) in the text image.

The case of multiple non overlapping objects is a special case of a general model of a target object observed in clutter of other foreign objects. As one can the main problem of target localization is here the problem of minimization of false detection (anomalous localization errors). This problem is addressed in Lect. 4.2.

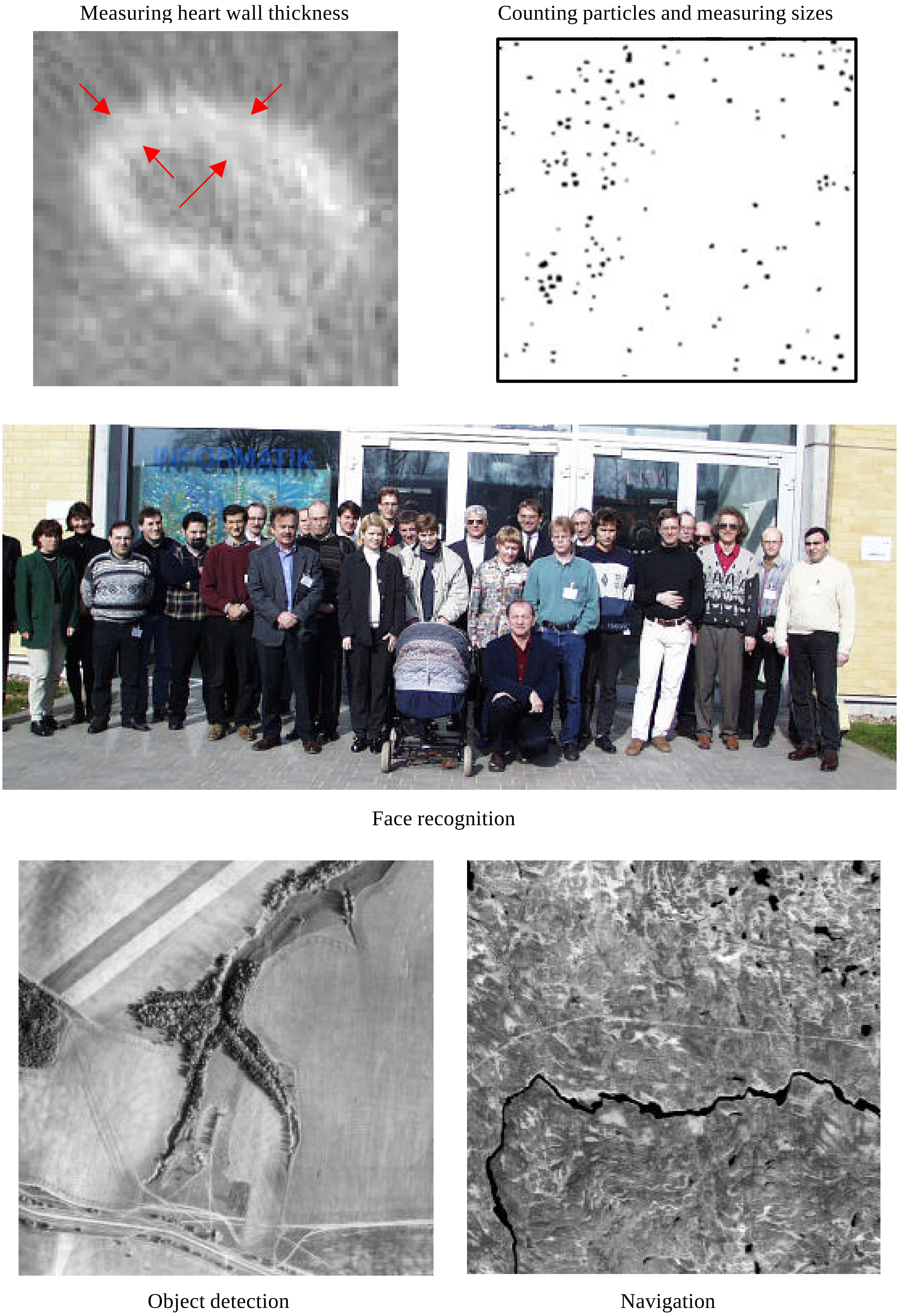


Fig. 4.1.1 Examples of applications that require image parameter estimation

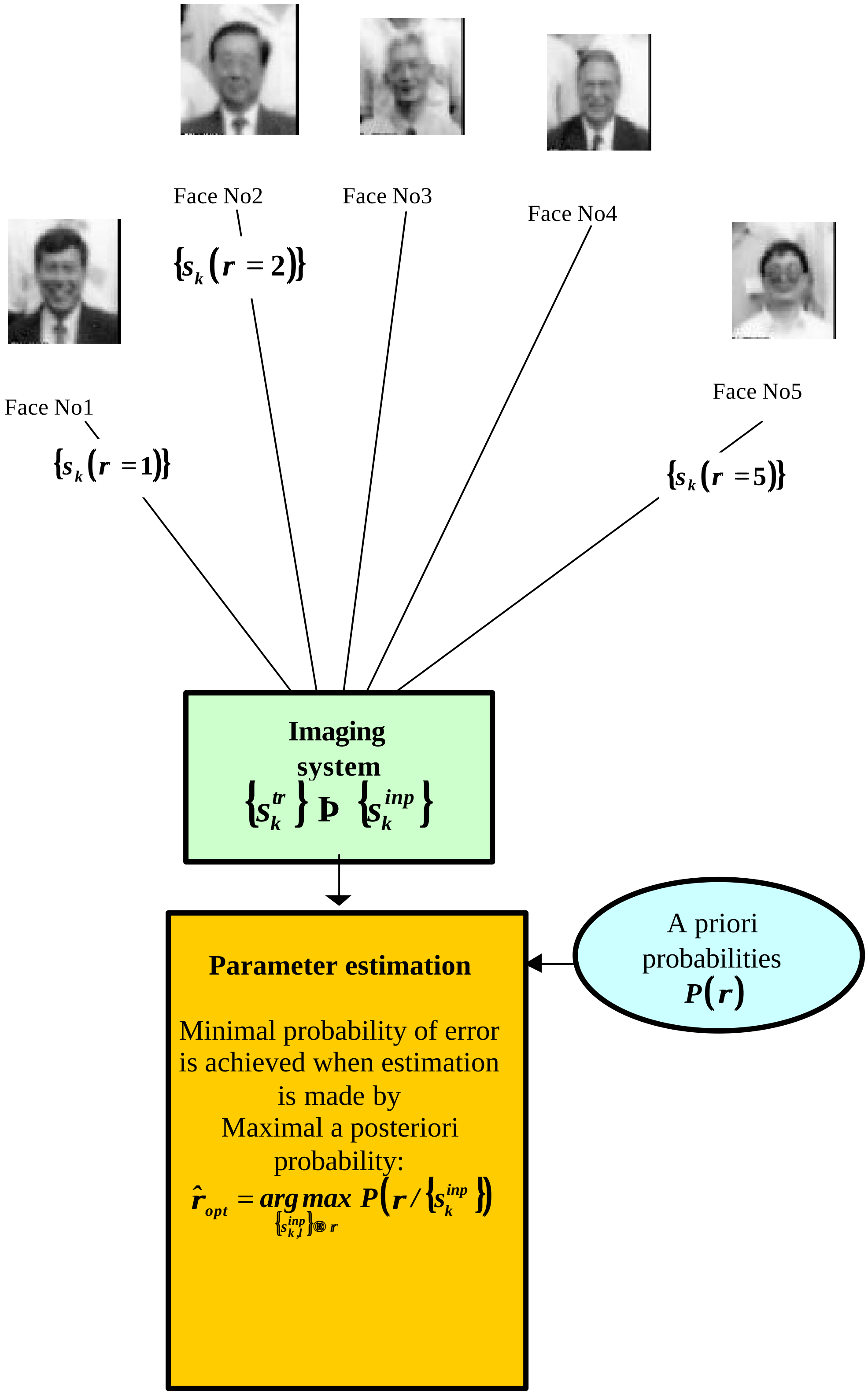


Fig. 4.1.2 Principle of optimal parameter estimation:

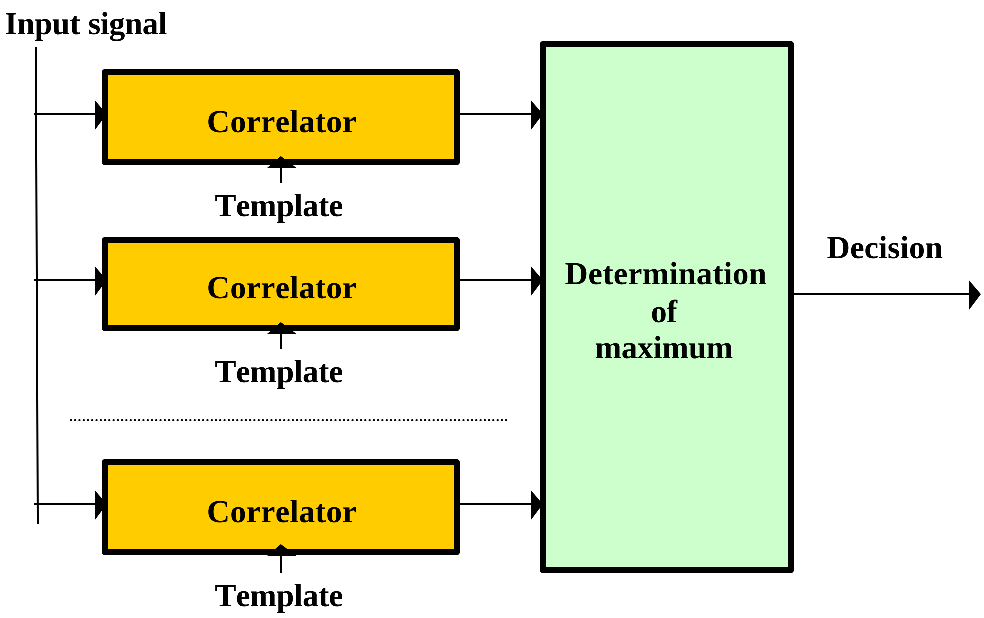


Fig. 4.1.3 Optimal device for estimation of a parameter with quantized values (object recognition)

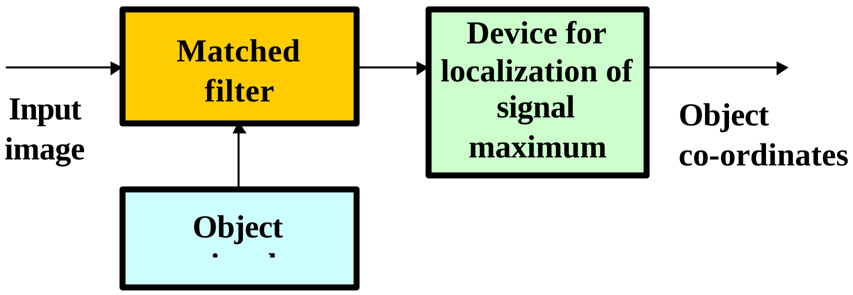


Fig. 4.1 4 Matched filtering for object localization

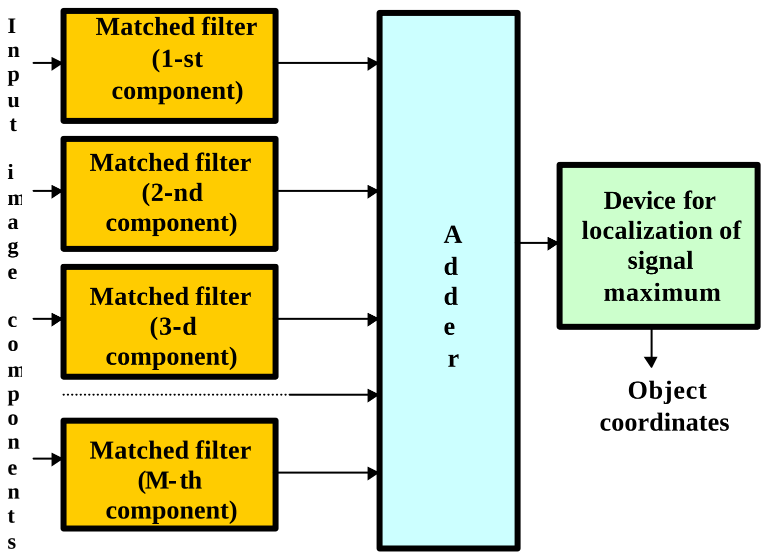


Fig. 4.1.5 Block-diagram of the optimal device for localizing objects in multi component images with additive white Gaussian noise in each channel with no inter channel correlations

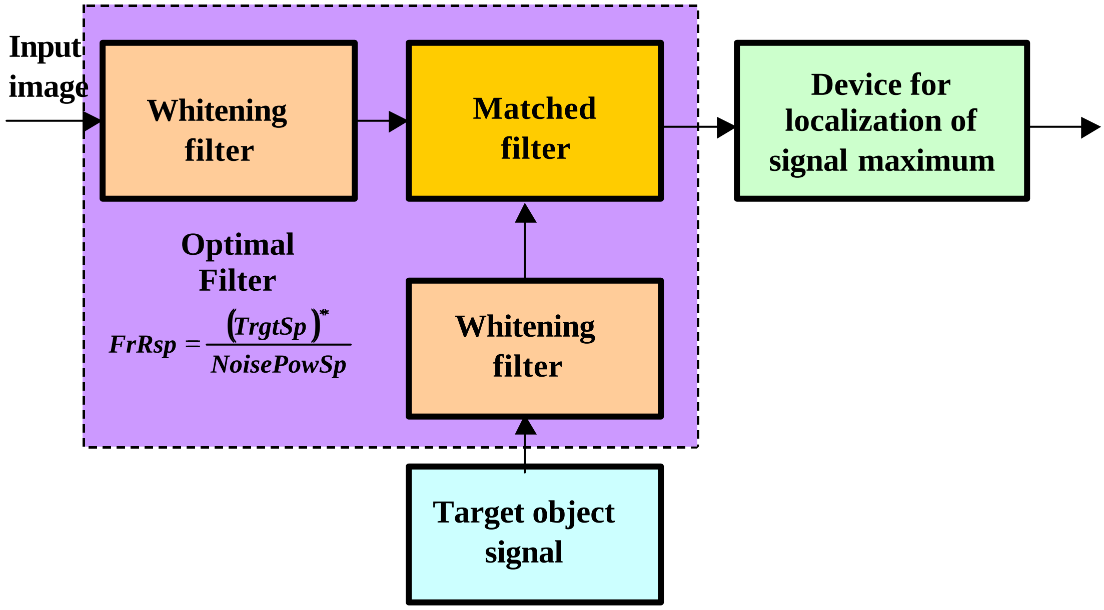


Fig. 4.1.6 "Optimal" filter concept for non "white" noise

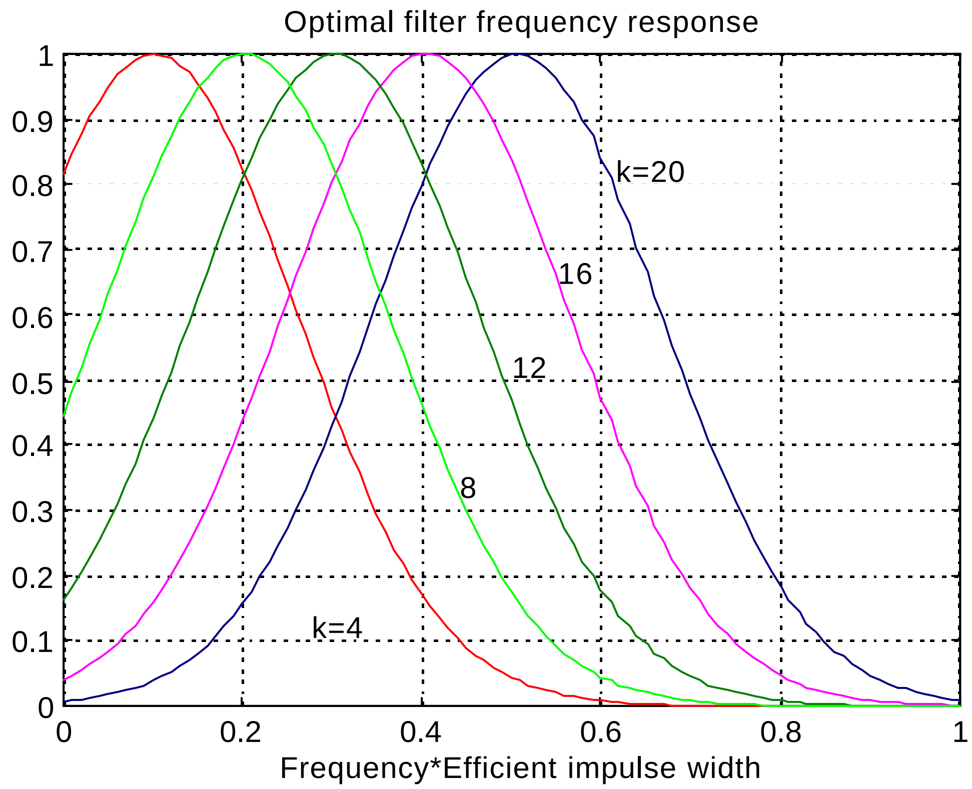
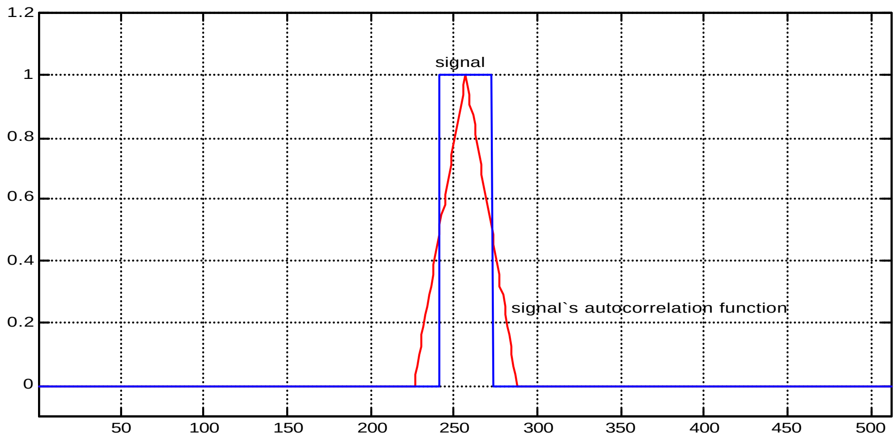
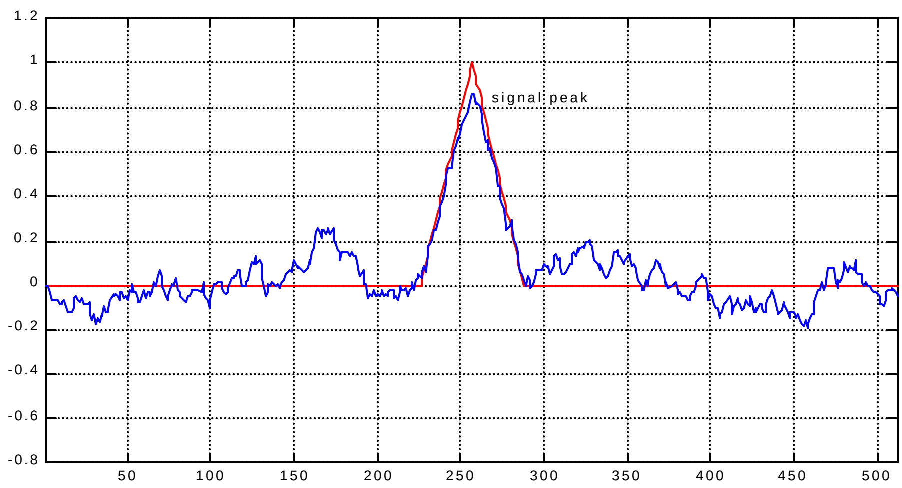


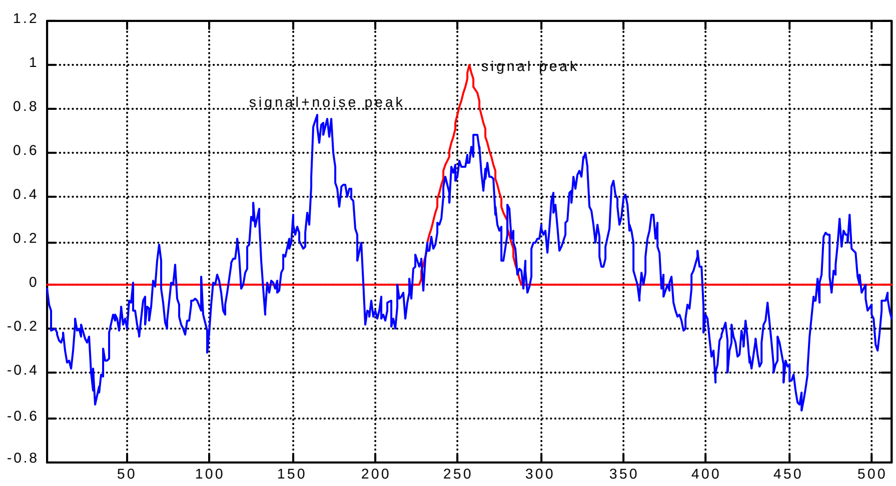
Fig. 4.1.7 Frequency response of the filter optimal for Gaussian-shaped impulse $s^{tr}(x) = exp\left(-\pi x^2 / 2S_a^2\right)$ and non-white noise with exponentially decaying power $N_n(f) = \exp\left(-f^2 / F_n^2\right)$ spectrum; $k = S_a F_n$



A rectangle target impulse and its autocorrelation function



Correlator’s output: the case of “normal” error (top: no noise, bottom)



Correlator’s output: the case of anomalous error

Fig. 4.1 8 Explanation of the origine of normal and anomalous localization errors.

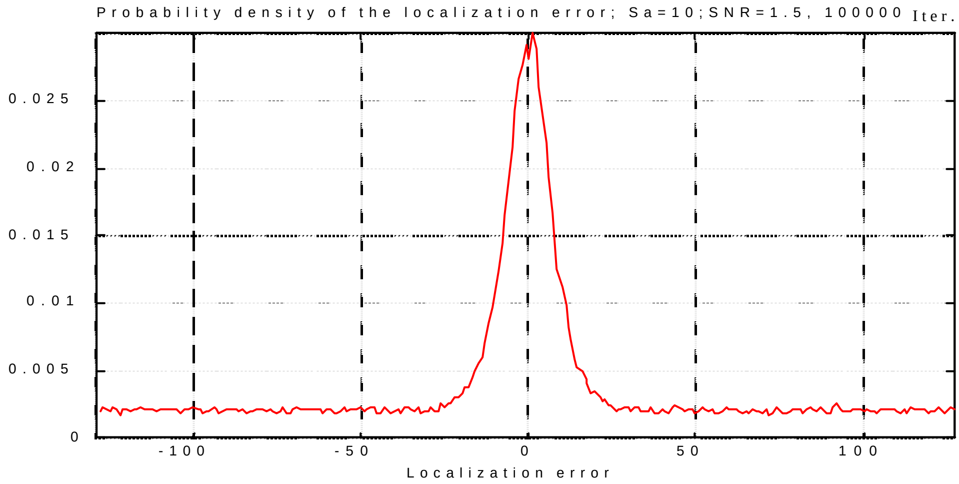


Fig. 4,.1.9 Experimental distribution density of the localization error for Gaussian-shaped impulse ($\sigma_a = 10$) and $\sqrt{E_a / N_0} = 1.5$ (100000 realizations)

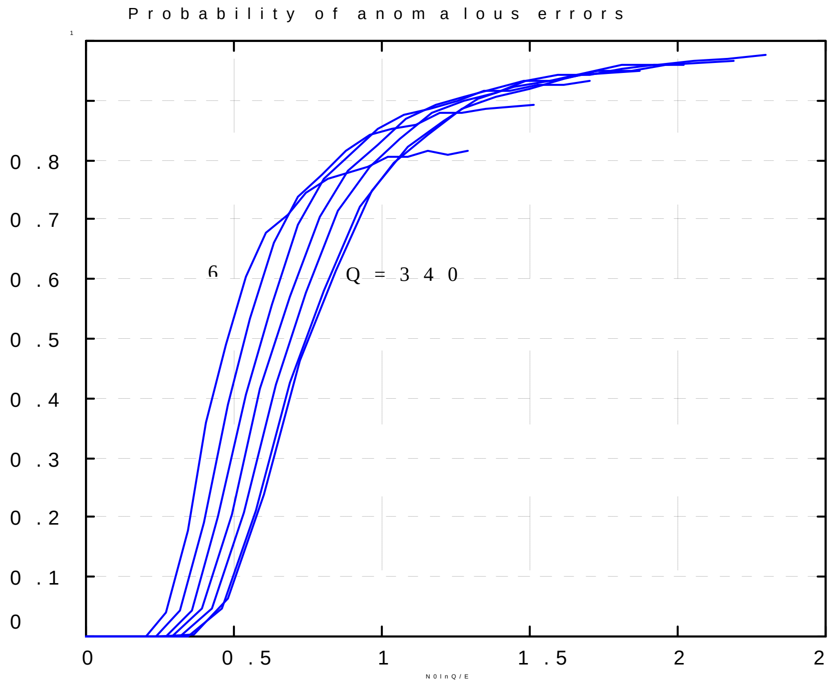
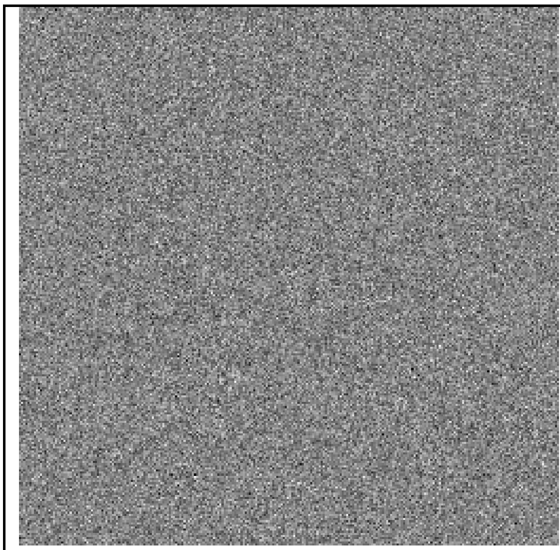
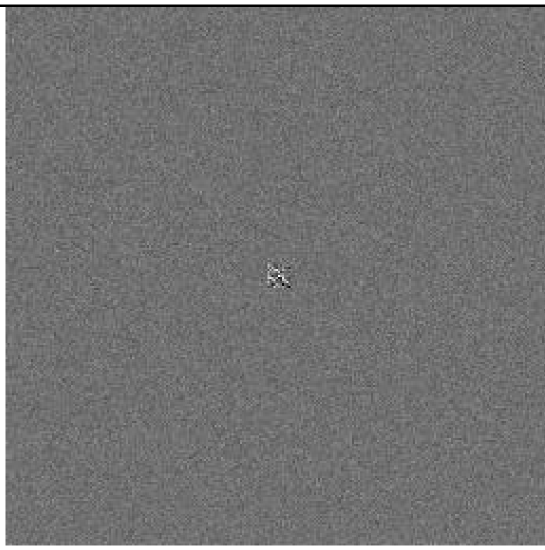


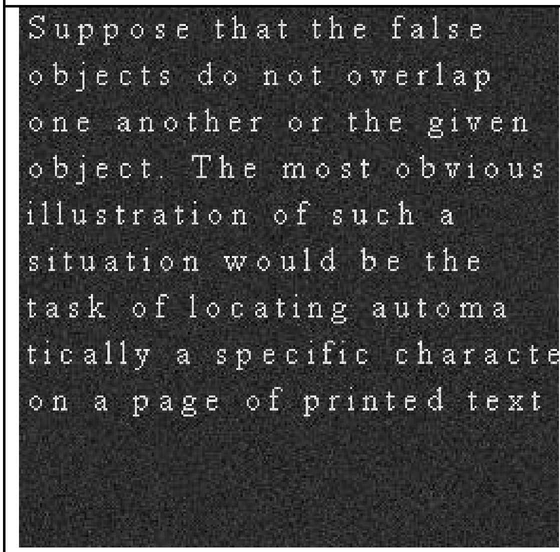
Fig. 4.1.10 Probability of anomalous errors as a function of normalized noise-to-signal ratio $\sqrt{N_0 \ln Q / E_a}$ for localization of rectangle impulses of 2;5;11;21;41;81;161 samples within an interval of 1024 samples (10000 realizations). The theoretical threshold value of the normalized noise-to-signal ratio is $\sqrt{2} / 2 \approx 0.707$



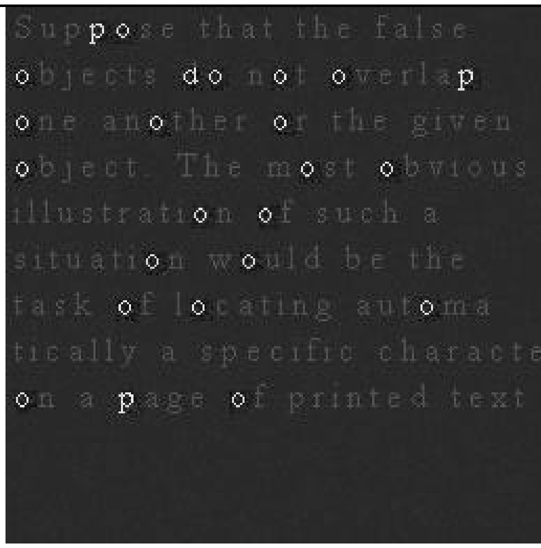
Input image with character "O" in its centre. Standard deviation of the observation noise 175/256



Localization result



Input image: a fragment of a printed text. Standard deviation the observation noise 15/256.



Localization result

Fig. 4.1.11 Comparison of localization of a target on a uniform background and in the presence of multiple non overlapping objects.

References:

1. A. Van der Lugt, Signal Detection by Complex Spatial Filtering, IEEE Trans., IT-10, 1964, No. 2, p. 139
2. L.P. Yaroslavsky, The Theory of Optimal Methods for Localization of Objects in Pictures, In: Progress in Optics, Ed. E. Wolf, v. XXXII, Elsevier Science Publishers, Amsterdam, 1993
3. L. Yaroslavsky, M. Eden, Fundamentals of Digital Optics, Birkhauser, Boston, 1996
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Appendix . Variance of the localization errors (1-D case).

Consider output signal

$$Corr_{sn}(x) = conv\left[\left(s^{tr}(x - x_0) + n(x)\right), h(x)\right] . \qquad \text{(A 4.1.1)}$$

of the filter with impulse response $h(x)$ and input signal $s^{tr}(x - x_0) + n(x)$. For normal localization errors, one should assume that this signal has its maximum in a point very close to the point with co-ordinate x_0. In this point, output signal derivative is equal to zero:

$$\frac{dCorr}{dx}\Big|_{x=\hat{x}_0} = \frac{dCorr_{sh}}{dx}\Big|_{x=\hat{x}_0} + \frac{dCorr_{nh}}{dx}\Big|_{x=\hat{x}_0} = 0 , \qquad \text{(A 4.1.2)}$$

where $Corr_{sh} = conv\left[s^{tr}(x - x_0), h(x)\right]$ and $Corr_{nh} = conv\left[n(x), h(x)\right]$.

Let $\hat{x}_0$ is a position of maximum of the filter output signal (co-ordinate estimate) and ε is the localization error:

$$\hat{x}_0 = x_0 + \varepsilon . \qquad \text{(A 4.1.3)}$$

Since x_0 is, by assumption, the target position

$$\frac{dCorr_{sh}(x)}{dx}\Big|_{x=x_0} = 0 . \qquad \text{(A 4.1.4)}$$

By expanding Eq.4.1.2 into Taylor series around point x_0, obtain

$$\varepsilon \frac{d^2 Corr_{sh}}{dx^2}\Big|_{x=x_0} + \frac{dCorr_{nh}}{dx}\Big|_{x=\hat{x}_0} = 0 , \qquad \text{(A4.1.5)}$$

from which it follows that

$$\left\langle \varepsilon^2 \right\rangle \left| \frac{d^2 Corr_{sh}}{dx^2}\Big|_{x=x_0} \right|^2 = \left\langle \left| \frac{dCorr_{nh}}{dx} \right|^2 \right\rangle . \qquad \text{(A4.1.6)}$$

and

$$\sigma_\varepsilon^2 = \left\langle \varepsilon^2 \right\rangle = \frac{N_0}{4\pi^2 \bar{f}_2^2 E_a} \frac{\int_{-\infty}^{\infty} f^2 \left|\alpha(f)\right|^2 df \int_{-\infty}^{\infty} f^2 \left|H(f)\right|^2 df}{\left(\int_{-\infty}^{\infty} f^2 \alpha(f) H(f) df\right)^2} . \qquad \text{(A4.1.7)}$$

where $\langle * \rangle$ denotes statistical averaging (mathematical expectation) over the noise ensemble.

It follows from Shwarz inequality that:

$$\sigma_{\varepsilon}^{2} \geq \frac{N_0}{4\pi^2 \bar{f}_2^{\,2} E_\alpha} \qquad \text{(A4.1.8)}$$

where

$$\alpha(f) = \int_{-\infty}^{\infty} s(x) \exp(i2\pi f x)\, dx \,; \qquad \text{(A4.1.9)}$$

$$H(f) = \int_{-\infty}^{\infty} h(x) \exp(i2\pi f x)\, dx \,; \qquad \text{(A4.1.10)}$$

$$E_\alpha = \int_{-\infty}^{\infty} |\alpha(f)|^2\, df \,; \qquad \text{(A4.1.11)}$$

$$\bar{f}_x^{\,2} = \frac{\int_{-\infty}^{\infty} f^2 |\alpha(f)|^2\, df}{\int_{-\infty}^{\infty} |\alpha(f)|^2\, df} \,. \qquad \text{(A4.1.12)}$$

Inequality of Eq. 4.1.9 becomes the equality when the filter is matched to the target signal:

$$H(f) = \alpha^*(f). \qquad \text{(A4.1.13)}$$