L. Yaroslavsky. Biomedical signal and image processing (**Course 055-355-5501**)

Lect.2 Digital representation of signals.

Basic principles.

General quantization in signal space.

ε-net and ε-entropy. Estimation of informational volume of signals.

Two-step digitization: discretization + element-wise quantization.

Discretization as signal expansion over a set of basis functions:

$$a(x) = \tilde{a}(x) = \sum_{k=0}^{N-1} \alpha_r \varphi_{rest}(x,r);\quad \alpha_r = \int_X a(x)\varphi_{discr}(x,r)dx .$$

Classes of basis functions:

Shift basis functions: $\varphi(x,r) = \varphi_0(x - r\Delta x)$;

Rectangle b.f.: $\varphi_0(x) = rect(x/\Delta x)$; $a(x) = \sum_k \alpha_k rect((x - k\Delta x)/\Delta x)$; $\alpha_k = \frac{1}{\Delta x}\int_{k\Delta x}^{(k+1)\Delta x} a(x)dx$;

Sinc-b.f.: $\varphi_0(x) = \mathrm{sinc}(\pi x/\Delta x)$; $\mathrm{sinc}(\pi x/\Delta x) = \frac{\sin(\pi x/\Delta x)}{\pi x/\Delta x} = \Delta x\int_{-1/2\Delta x}^{1/2\Delta x}\exp(i2\pi f x)df$

$$a(x) = \sum_k \alpha_k \mathrm{sinc}(\pi(x - k\Delta x)/\Delta x);\quad \alpha_k = \frac{1}{\Delta x}\int_{-\infty}^{\infty} a(x)\mathrm{sinc}(\pi(x - k\Delta x)/\Delta x)dx =$$

$$= \int_{-\infty}^{\infty}\alpha(f)\mathrm{rect}((f + 1/2\Delta x)/\Delta x)\exp(-i2\pi f x)dx = \tilde{\alpha}(k\Delta x)$$

Triangle, cubic and other spline bases.

Multiplicative basis functions:

Exponential b.f.: $four(x,r) = \exp(i2\pi rx/X)$; Discretization by signal expansion to a Fourier series:

$$a(x) = \sum_r \alpha_r \exp(-i2\pi rx/X);\quad \alpha_r = \frac{1}{X}\int_{r-X/2}^{X/2} a(x)\exp(i2\pi rx/X)dx = \frac{1}{X}\alpha(r/X)$$

Walsh b.f.: $wal_r(x) = (-1)^{\sum_{m=0}^{\infty} r_m^{Gr}(x/X)_{m+1}} = \exp\left[i\pi\sum_{m=0}^{\infty} r_m^{Gr}(x/X)_{m+1}\right]$

Mixed scale/shift basis functions:

Wavelets: combination of shift and multiplicative functions. Global and local signal representation.

Haar wavelets : $har_r(x) = \frac{2^{m/2}}{\sqrt{X}}\, sign\left[\sin\left(2^{m+1}\pi\frac{x}{X}\right)\right] rect\left[2^m\frac{x}{X} - (r)_{\mathrm{mod}\,2^m}\right]$

m - most significant non zero bit in binary representation of r .

Signal space dimensionality and optimal bases.

Karhunen-Loeve basis and signal statistical ensemble covariance function $R_a(x,\xi)$

$$a(x) \approx \tilde{a}(x) = \sum_{r=0}^{N-1}\alpha_r\varphi_{KL}(x,r);\quad \int_X R_a(x,\xi)\varphi_{KL}(\xi,r)d\xi = \lambda_r\varphi_{KL}(x,r) ;$$

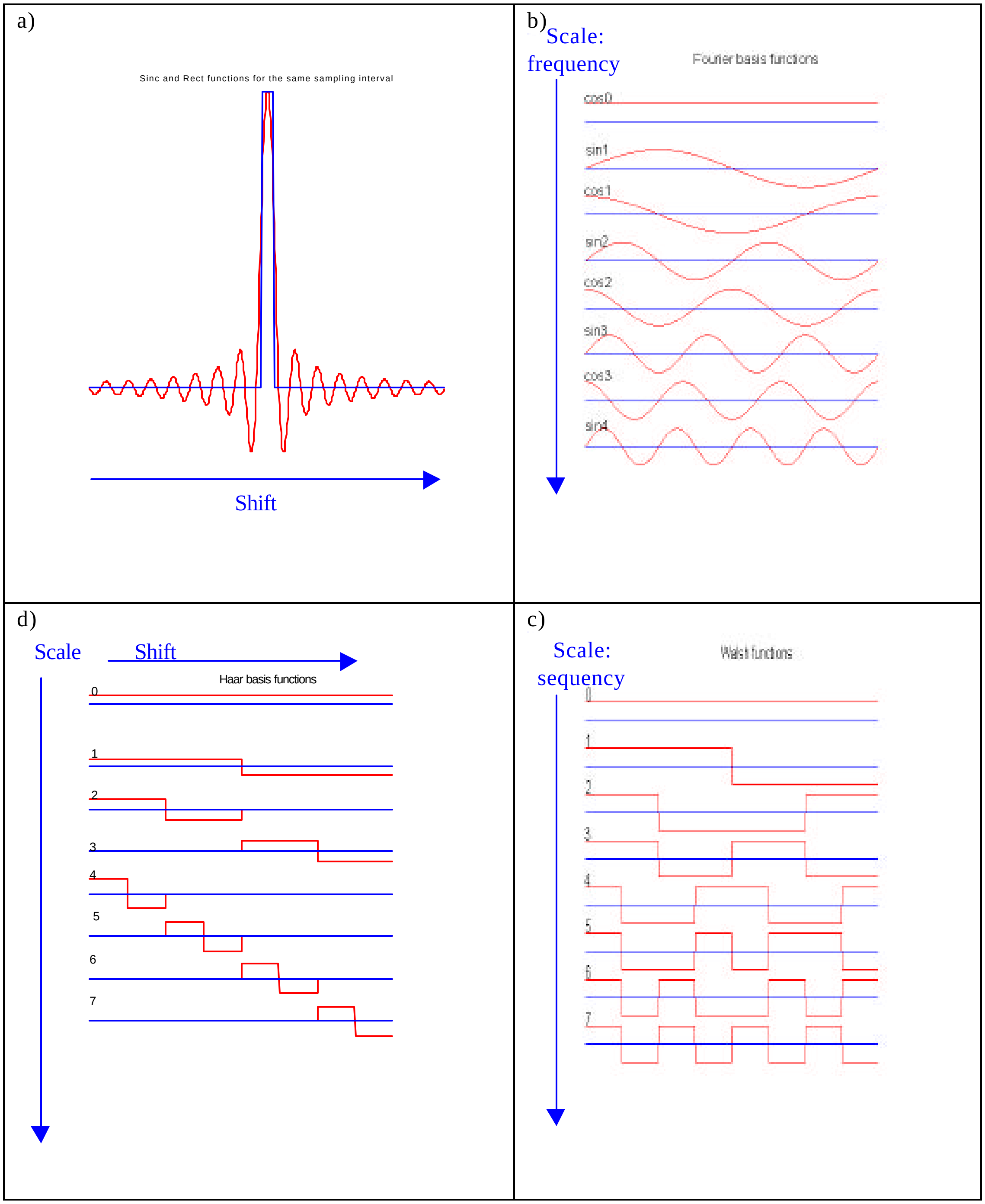
$$\|\varepsilon\|^2 = \|a(x) - \tilde{a}(x)\|^2 = \sum_{r=N}^{\infty}\lambda_r$$

Problems:

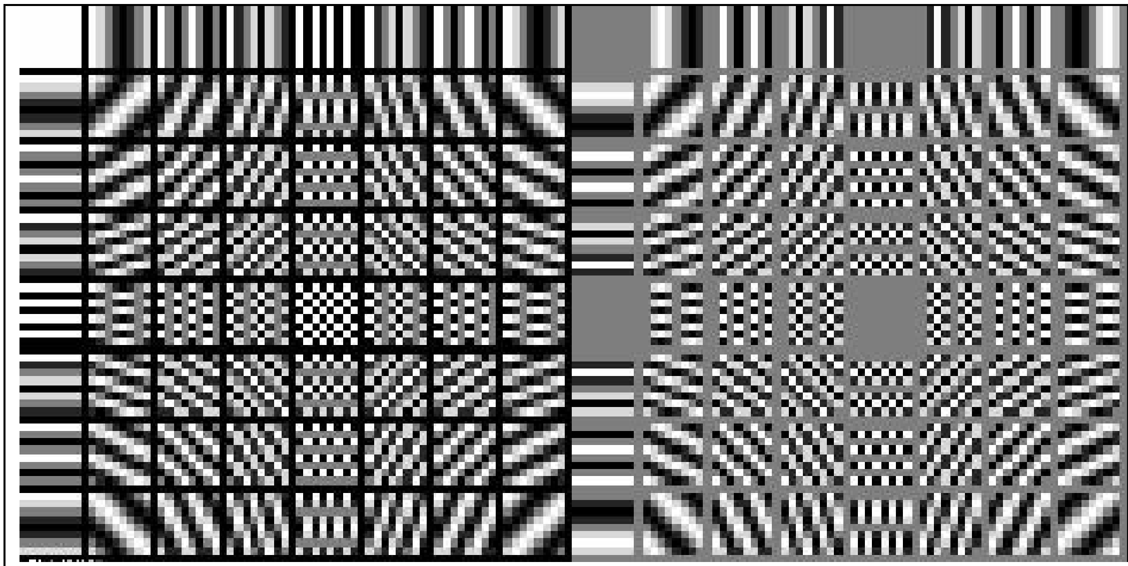
1. What are general signal space quantization, signal discretization and element wise quantization?
2. Formulate basic requirements to the discretization and reconstruction basis functions.
3. Describe basic classes of discretization basis functions. Give examples and explain basic properties of the basis functions.
4. What is dimensionality of the signal space and what is optimal discretization basis? Why Karhunen-Loeve basis is frequently regarded optimal?

Home work HW_1. Compare different shift basis functions (rect, sinc, triangle, spline) with the same discretization interval and their Fourier spectra

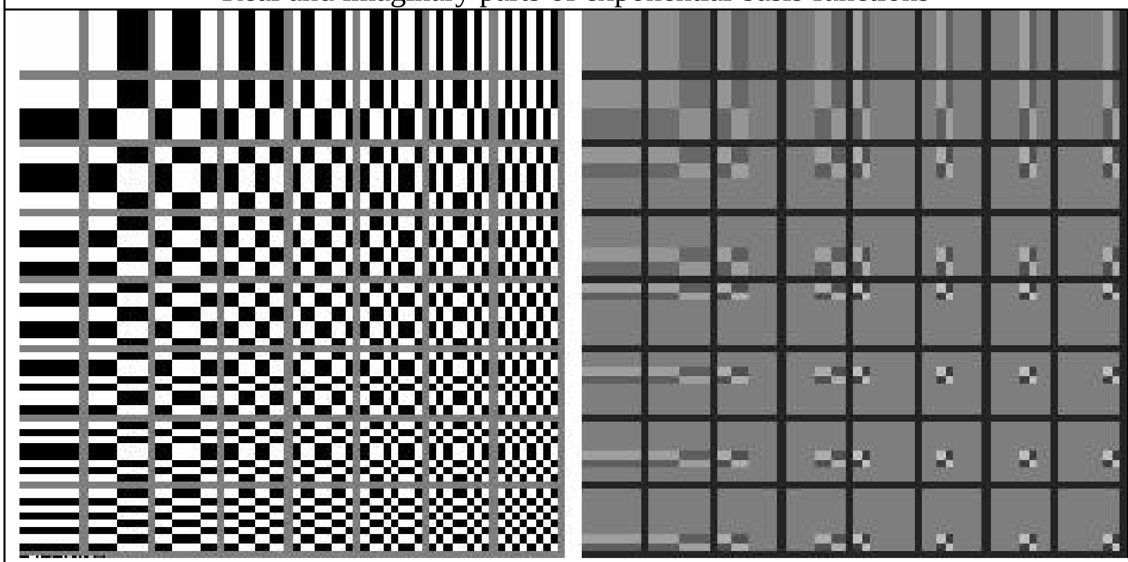
EXAMPLES OF BASIS FUNCTIONS:
a) shift b.f; b),c) – multiplicative (scale) b.f; d) – shift/scale b.f.



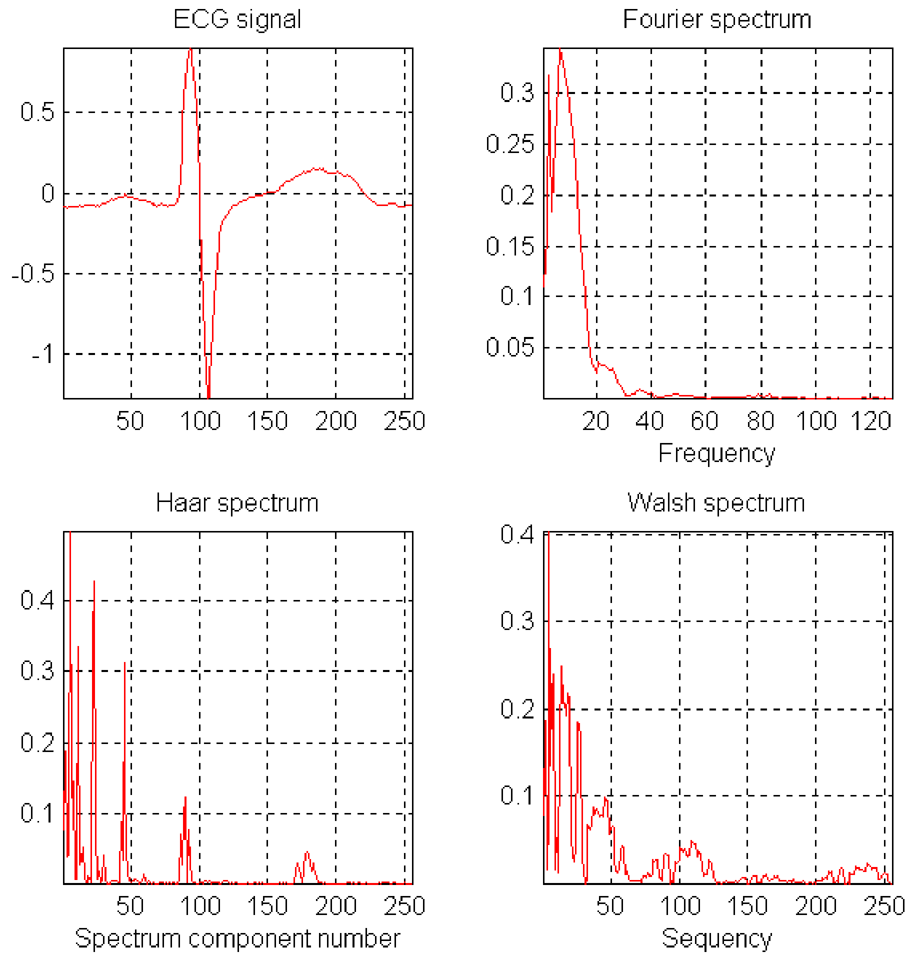
2-D Basis Functions



Real and imaginary parts of exponential basis functions



Walsh (left) and Haar (right) basis functions



Left-Right,Top-Bottom:Image and its Fourier,Walsh and Haar spectra