Biomedical signal and image processing **(Course 055-355-5501)**

Lect. 3 Signal sampling

1-D sampling theorem.

Shifted bases representation of signal $a(x)$: $a(x) \cong a_{rec}(x) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_{rec}(x - k\Delta x)$;

$$\alpha_k = \int_{-\infty}^{\infty} a(x)\varphi_{discr}(x - k\Delta x)dx = \int_{-\infty}^{\infty} \alpha(f)\Phi_{discr}(f)\exp(-i2\pi fk\Delta x)dx =$$

$$\int_{-\infty}^{\infty} \alpha_{df}(f)\exp(-i2\pi fk\Delta x)dx = a_{df}(k\Delta x)$$

Fourier spectrum of such a signal $\alpha_{rec}(f) = \sum_{k=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \alpha_{df}(p)\exp(-i2\pi pk\Delta x)dp\right)\Phi_{rec}(f)\exp(i2\pi fk\Delta x) =$

$$\int_{-\infty}^{\infty} \alpha_{df}(p)\left(\sum_{k=-\infty}^{\infty} exp[i2\pi(f-p)k\Delta f]\right)dp\,\Phi_{rec}(f) = \Delta x\int_{-\infty}^{\infty} \alpha_{df}(p)\left(\sum_{m=-\infty}^{\infty} \delta\left[\left(f - p + \frac{m}{\Delta x}\right)\right]\right)dp\,\Phi_{rec}(f) =$$

$$\Delta x\left(\sum_{m=-\infty}^{\infty} \alpha_{df}\left[\left(f + \frac{m}{\Delta x}\right)\right]\right)\Phi_{rec}(f).$$

Signal sampling and reconstruction can be treated as periodical replication of spectrum of the signal subjected to filtering by the discretization device and separation of the restored signal spectrum by the filter $\Phi_{rec}(f)$.

In the ideal case, when $\varphi_r(x) = \mathrm{sinc}(\pi x/\Delta x)$, $\Phi_r(f) = rect((f + 1/2\Delta x)\Delta x)$,

$$a(x) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{sinc}[\pi(x - k\Delta x)/\Delta x],$$

$$\varphi_{disc}(x) = \frac{1}{\Delta x}\mathrm{sinc}(\pi x)$$

$$\alpha_k = \frac{1}{\Delta x}\int_{-\infty}^{\infty} s(x)\mathrm{sinc}[\pi(x - k\Delta x)/\Delta x]dx = \int_{-1/2\Delta x}^{1/2\Delta x} \alpha(f)\exp[-i2\pi f(x - k\Delta x)]df = a(k\Delta x)$$

$$\tilde{\tilde{\alpha}}(f) = \alpha(f), f \in [-1/2\Delta x, 1/2\Delta x]$$

$$a(x) = \left\{a_d(x) = \sum_{-\infty}^{\infty} a_k\,\delta(x - k\Delta x)\right\} \circ \sin c(\pi x/\Delta x) \Leftrightarrow \left\{\sum_{l=-\infty}^{\infty} \alpha\left(f - \frac{l}{\Delta x}\right)\right\} rect\left[\Delta x\left(f + \frac{1}{2\Delta x}\right)\right]$$

The sampling theorem: given signal samples, obtained with shift basis functions, the best approximation of the signal from its samples is that obtained through their sinc-interpolation.
Discretization interval and its relation to the signal spectrum limitation.

2-D sampling theorems.

Optimal selection of sampling rasters and dense packing of the signal spectrum in frequency domain.
Variety of 2-D sampling rasters.

Practical aspects of signal sampling, image rastering and reconstruction.

Discretization errors; aliasing, strobe effect; moire effect.
Discretization in tomography: body projecting and sampling projections

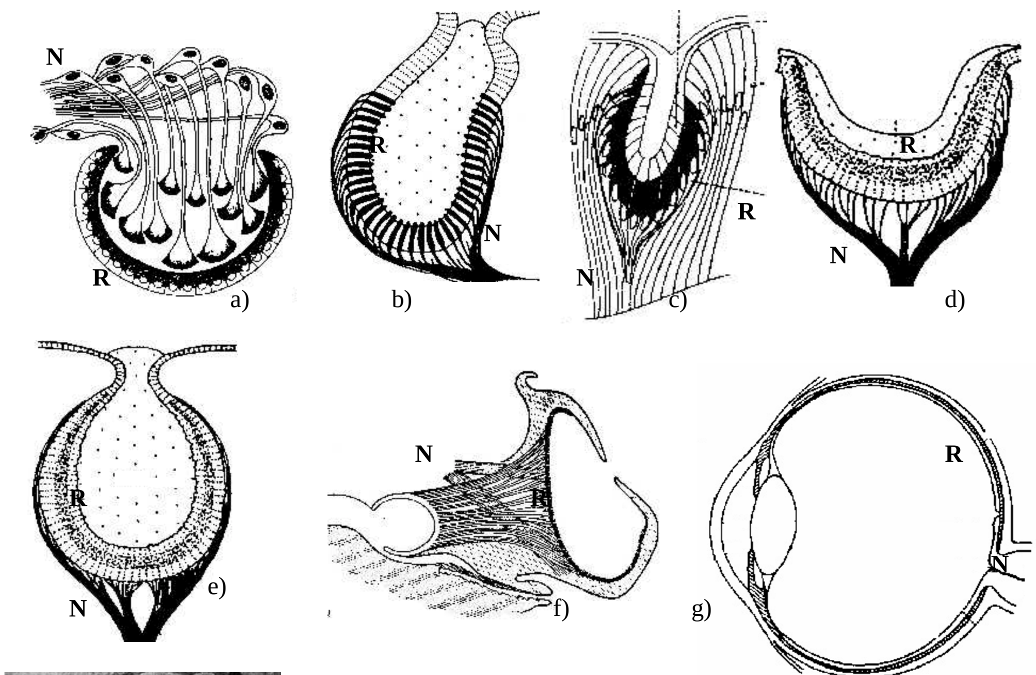
Problems

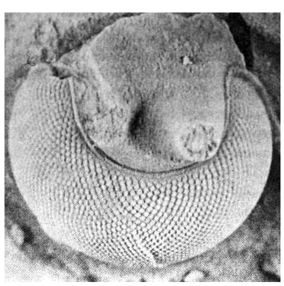
1. What happens with signal Fourier spectrum when the signal is sampled?
2. Prove and explain sampling theorem in signal and in spectral domain. Explain duality between periodical and sampled signals.
3. What is sinc-interpolation and in what sense it is optimal for signal restoration from sampled data?
4. For 2-D signals, explain the relationship between signal spectrum dense packing in spectral domain and discretization rasters. What types of 2-D rasters you know?
5. Classify 2-D signal discretization methods you know.
6. Explain principles of tomographic image reconstruction in terms of signal discretization

Home exercise: Suggest and simulate on Matlab a method for illustrating aliasing effects

Signal sampling in nature and technical device

The most widely used method for signal/image discretization is signal/image ***sampling*** by sensing them with a set of sensors that are equidistantly placed in signal/image coordinates. The sampling is carried out either by a mechanical or electronic signal/image scanning with a single sensor or with an array of sensors that work in parallel. The latter is exemplified by modern CCD and CMOS TV cameras. Remarkable enough that image sampling was first invented by the Nature in a form of a compound and retinal eyes





Retinal eyes (**R** - retina; **N** - nerves): flatwarm (a); bevolve mollusc (b); polychaet worm; limpet (d); abalone (d); nautilus pinhole (f); human eye (g). Compound eye: 400 million old fossilized trilobite eye (h). (Images (a-f,h) are adopted from R. Dawkins, Climbing Mount Unprobable, W.W.Norton Co, N.Y.,1998)

Different types of vision organs in the animal kingdom. In all types of eyes one can see devices for image discretization by sampling

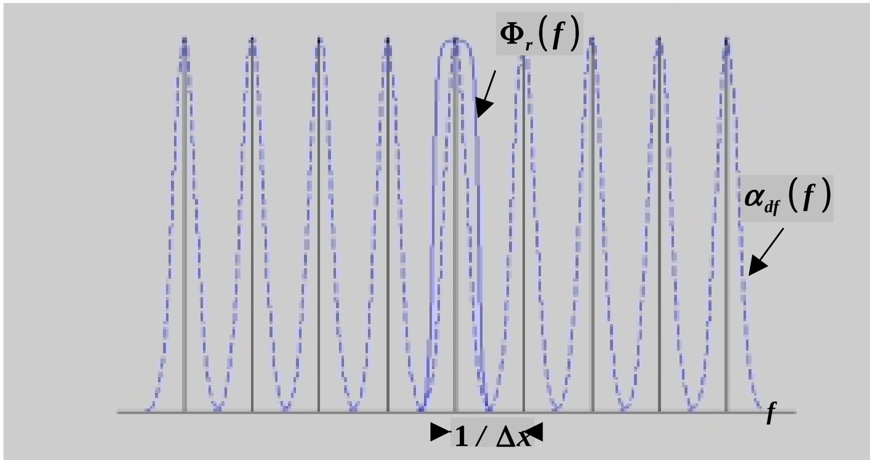


Illustration of periodical replication of signal spectrum in signal sampled representation.

Images before and after sampling (rectangle raster)

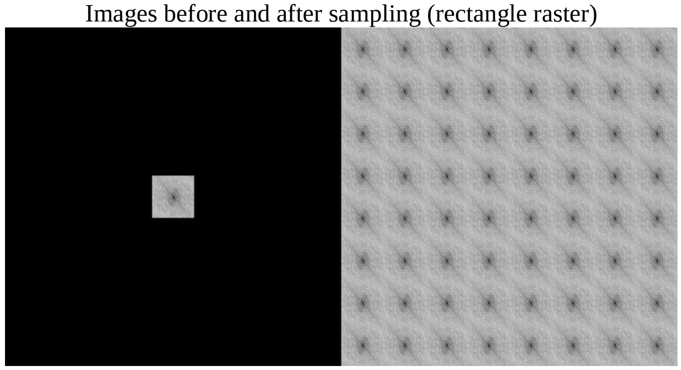
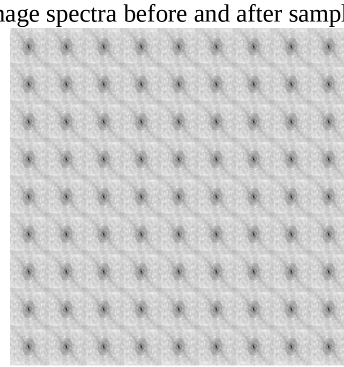
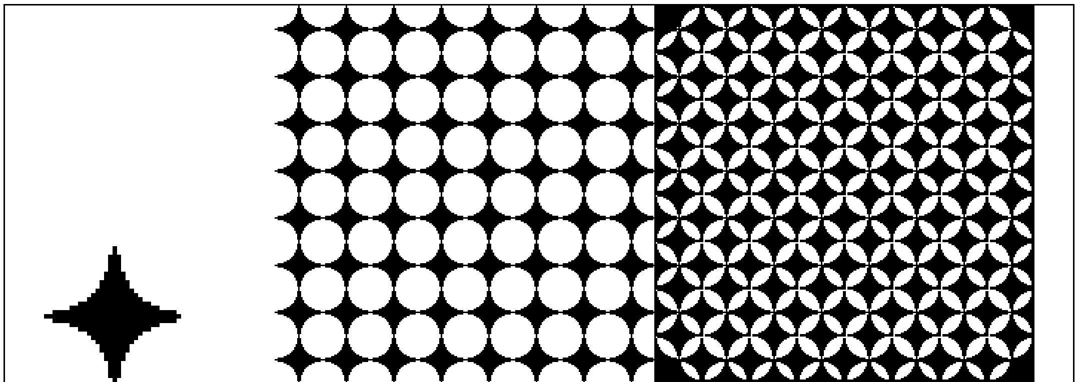


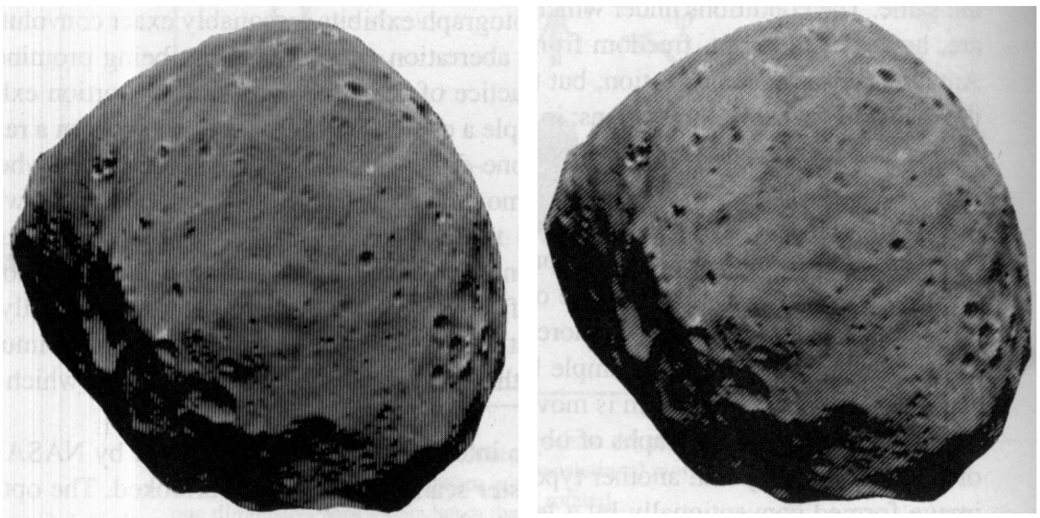
Image spectra before and after sampling



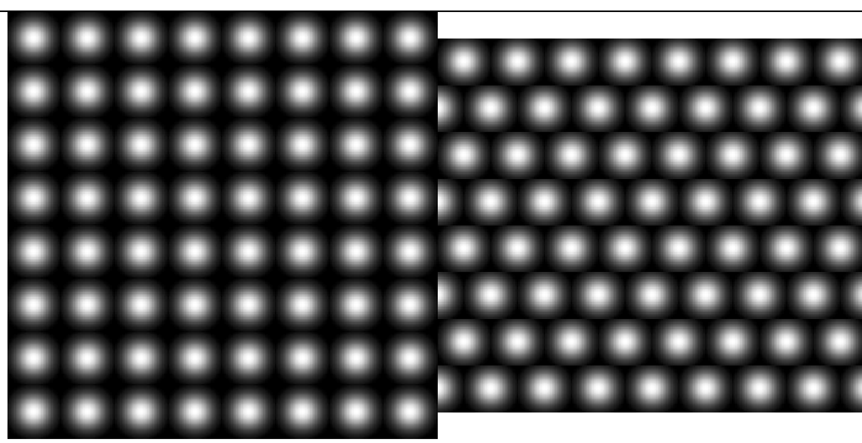
Aliasing due to undersampling



Spectrum and 2 methods of its periodical expansion

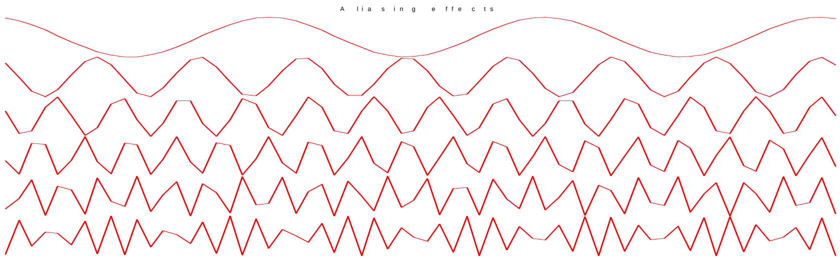


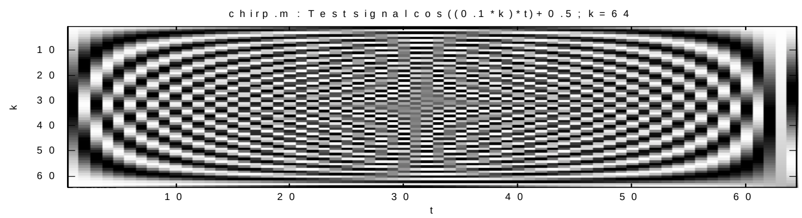
An image of the asteroid Caspra taken in space (NASA photo). Note that the screen is less obtrusive when sloping 45^{o}.

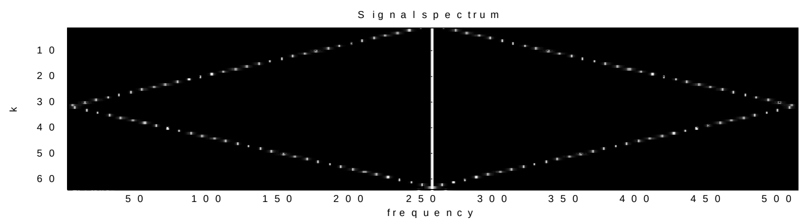


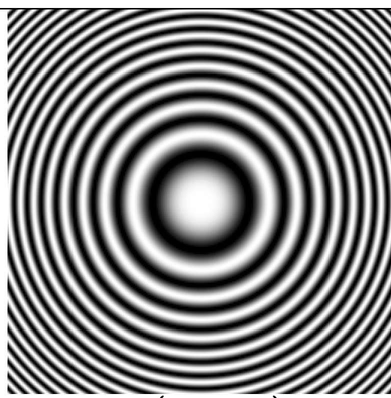
Spectra packing for rectangle and hexagonal discretization rasters

Aliasing effects in sinusoidal signals

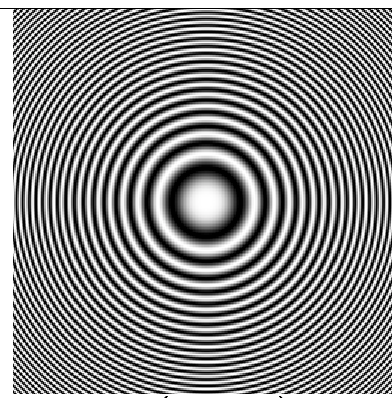




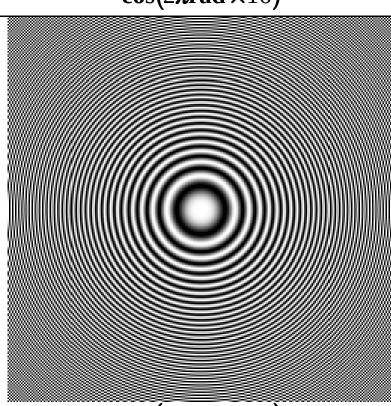




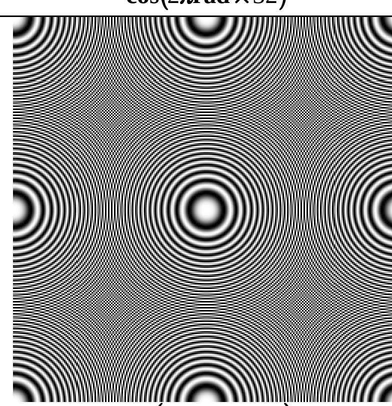
$\cos(2\pi \mathrm{rad} \times 16)$



$\cos(2\pi \mathrm{rad} \times 32)$



$\cos(2\pi \mathrm{rad} \times 64)$



$\cos(2\pi \mathrm{rad} \times 128)$

Optimal sampling/reconstruction and aliasing effects in images (I)

Initial and 2-4-8 decimated sinc- (left) and NN-(right) interpolated images

S. Dali’s picture “Gala Nude Looking at the See” and discretization aliasing effects

Optimal sampling/reconstruction and aliasing effects in images (II)

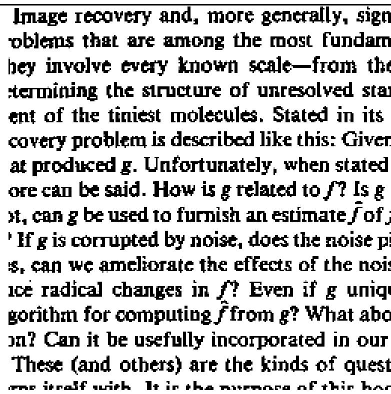


Image recovery and, more generally, sign
oblems that are among the most fundam
hey involve every known scale—from the
termining the structure of unresolved star
ent of the tiniest molecules. Stated in its
covery problem is described like this: Given
at produced g. Unfortunately, when stated
ore can be said. How is g related to f? Is g
ot, can g be used to furnish an estimate $\hat{f}$ of
' If g is corrupted by noise, does the noise pl
s, can we ameliorate the effects of the nois
ice radical changes in f? Even if g uniqu
gorithm for computing $\hat{f}$ from g? What abo
on? Can it be usefully incorporated in our
These (and others) are the kinds of quest

Initial image

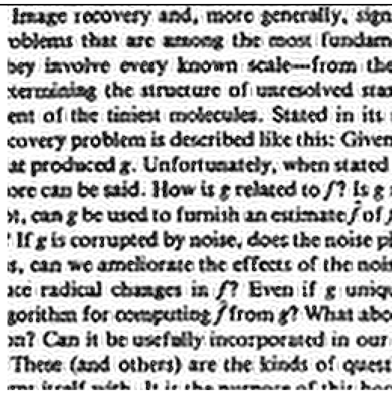
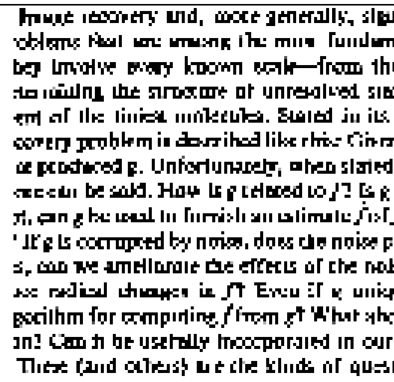
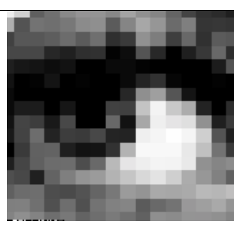
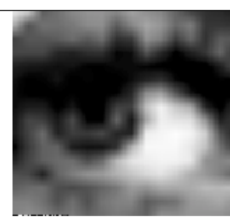
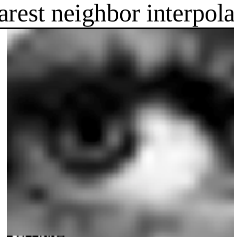
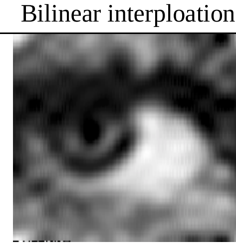
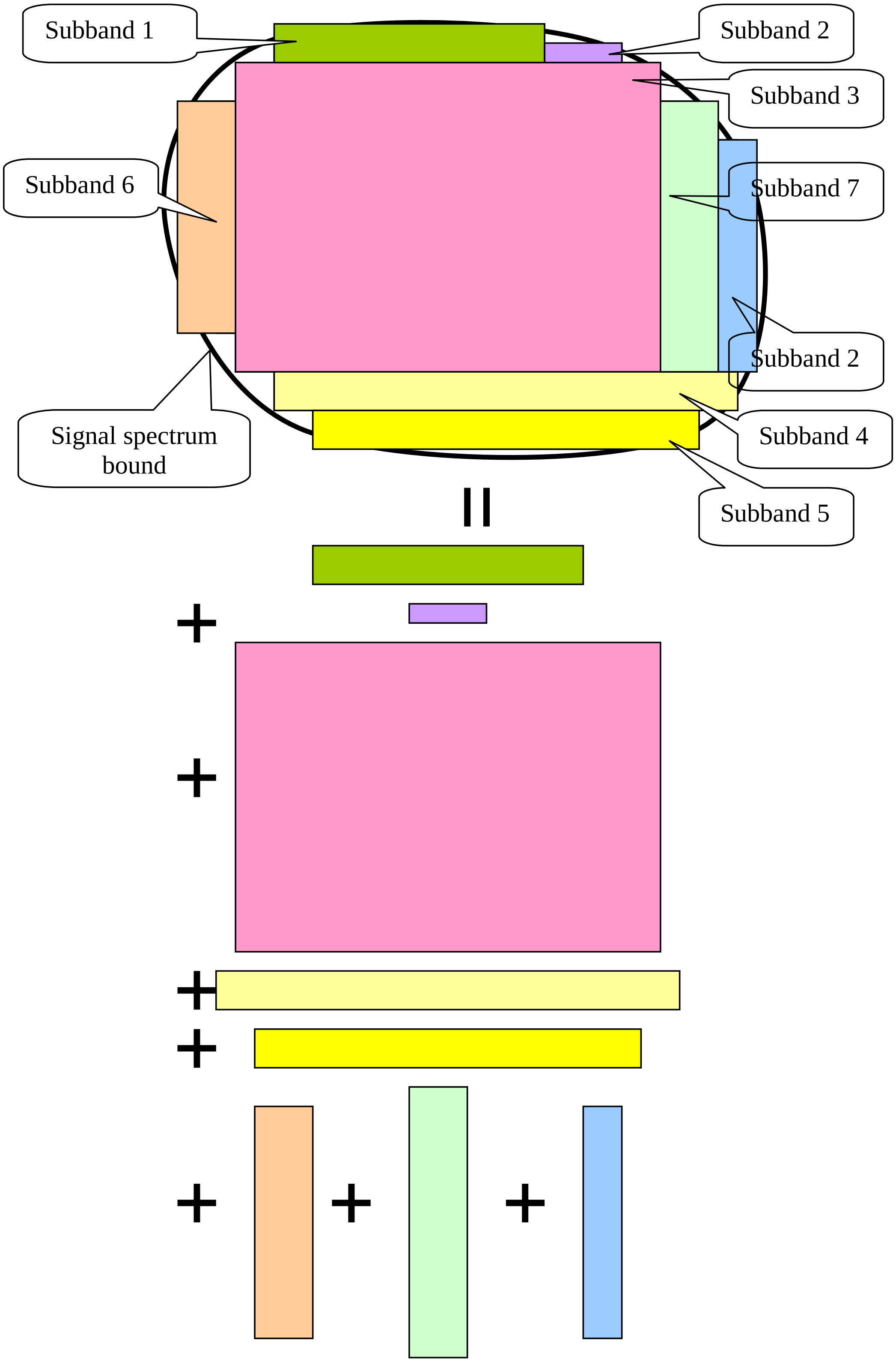
 Optimally sampled (1:2) image	 Decimated (1:2) image

Image interpolation (resampling) methods

 Nearest neighbor interpolation	 Bilinear interploation
 Bicubic interpolation	 Sinc-interpolation

Signal discretization and "sub-band decomposition"



Signal decomposition into sub-bands allows to minimize the number of signal samples while using regular sampling means