

Lect. 7 Discrete Fourier Transforms

Discrete Fourier Transform: discrete representation of the Fourier Integral

$$\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx) dx; \quad a(x) = \sum_k a_k \sin c[\pi(x - k\Delta x) / \Delta x]; \quad \alpha(f) = \sum_r f_r \sin c[\pi(f - r\Delta f) / \Delta f]$$

$$\Rightarrow \text{DFT: } \alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N}\right) \quad \text{IDFT: } a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{N}\right)$$

Basic properties: cyclicity, symmetries, shift and convolution theorems:

$$\alpha_r = \alpha_{(r) \bmod N}; \quad a_k = a_{(k) \bmod N}; \quad \{\pm a_{N-k}\} \Leftrightarrow \{\pm \alpha_{N-s}\}; \quad \{a_k = \pm a_{N-r}^*\} \Leftrightarrow \{\alpha_r = \pm \alpha_{N-r}\};$$

$$a_{(k+p) \bmod N} \Leftrightarrow \alpha_r \exp(-i2\pi rp / N); \quad \left\{ \sum a_n b_{(k-n) \bmod N} \right\} \Leftrightarrow \sqrt{N} \alpha_r \beta_r$$

$$\text{2-D DFT: } \alpha_{r,s} = \frac{1}{\sqrt{NM}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} a_{k,l} \exp\left(i2\pi \left(\frac{kr}{N} + \frac{ls}{M}\right)\right).$$

$$a_{k,l} = \frac{1}{\sqrt{NM}} \sum_{r=0}^{N-1} \sum_{s=0}^{M-1} \alpha_{r,s} \exp\left(-i2\pi \left(\frac{kr}{N} + \frac{ls}{M}\right)\right)$$

2-D cyclicity and symmetries.

Shifted Discrete Fourier Transforms:

If sampling positions in signal and Fourier domain are shifted by u and v with respect to origin of signal and its spectrum co-ordinate systems, respectively, discrete representation of the Fourier integral is:

$$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k^{u,v} \exp(i2\pi(k+u)(r+v) / N)$$

Removing irrelevant constant $\exp(i2\pi uv / N)$ gives rise to SDFT(u,v) and ISDFT(u,v):

$$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k^{u,v} \exp(i2\pi kv / N) \exp(i2\pi(k+u)r / N)$$

$$a_k^{u,v} = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r^{u,v} \exp(-i2\pi ru / N) \exp(-i2\pi rk(r+v) / N)$$

Important special cases of SDFT are:

Discrete cosine transform:

$$\text{DCT: } \text{SDFT}_{1/2,0}(a_k = a_{2N-1-k}): \quad \alpha_r = \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{(k+1/2)r}{N}\right); \quad \{\alpha_r = -\alpha_{2N-r}\}$$

$$\text{IDCT: } \text{ISDFT}_{1/2,0}(\alpha_r = -\alpha_{2N-r}): \quad a_k = \frac{1}{\sqrt{2N}} \left\{ \alpha_0 + 2 \sum_{r=1}^{N-1} \alpha_r \cos\left(\pi \frac{(k+1/2)r}{N}\right) \right\}$$

Discrete Sine Transform:

$$\text{DST: } \text{SDFT}_{1,1}(\{a_k = -a_{2N-1}\}): \quad \alpha_r = \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left(\pi \frac{(k+1)(r+1)}{N+1}\right); \quad \{\alpha_r = -\alpha_{2N-r}\}$$

$$\text{IDST: } a_k = \frac{1}{\sqrt{2N}} \sum_{r=0}^{N-1} \alpha_r \sin\left(\pi \frac{(k+1)(r+1)}{N+1}\right)$$

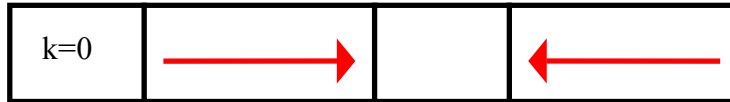
Energy compaction properties of DFT, DCT and DST: they redistribute signal energy into a small fraction of transform coefficients. DCT and DST remove signal discontinuities at boundaries and hence compact energy more efficiently than DFT. DFT, DCT and DST and Karhunen-Loew Transform

Problems

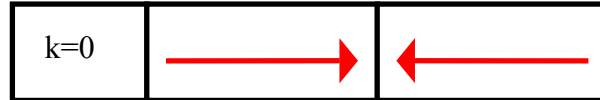
1. Describe distinctions between integral Fourier Transform and DFT and basic properties of DFT.
2. Explain Shifted Discrete Fourier Transform, DCT and DST and their relation to DFT

Home work: Investigate and explain DFT spectra of sinusoidal signals of different frequencies

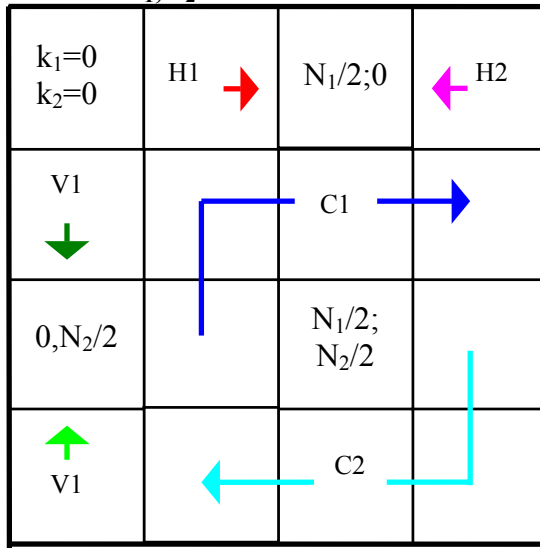
N - even number



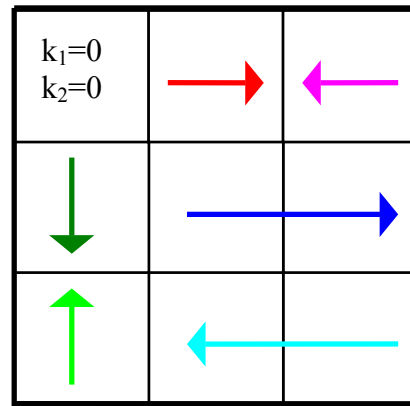
N - odd number



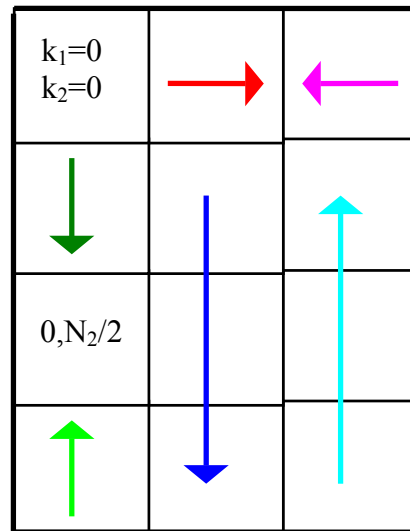
N_1, N_2 - even numbers



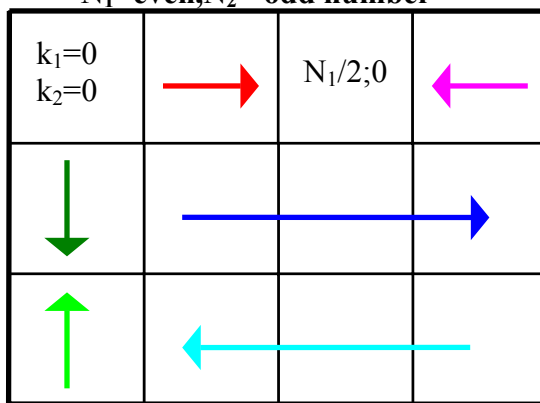
N_1, N_2 - odd numbers



N_1 - odd, N_2 - even

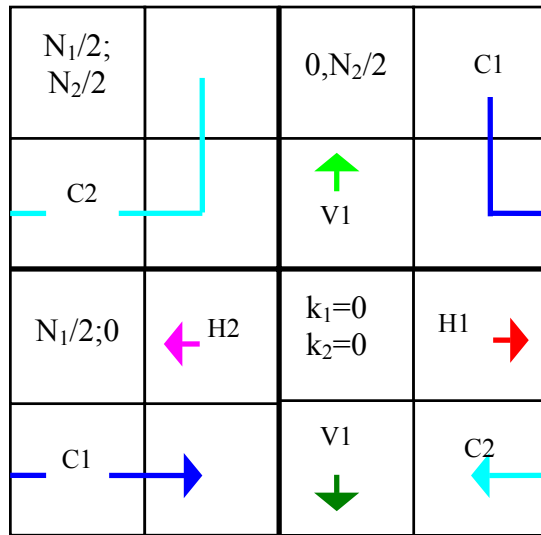
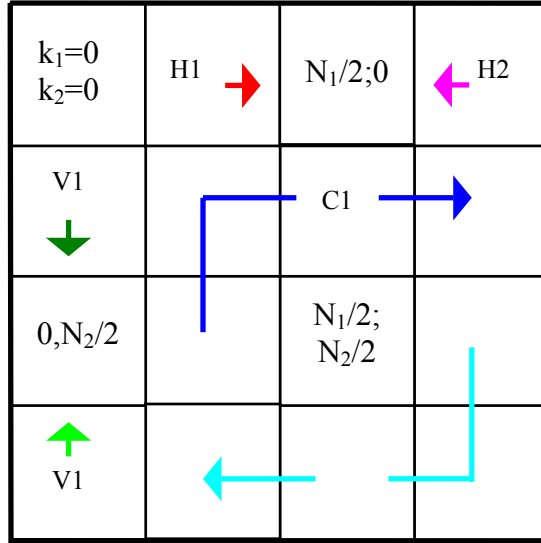


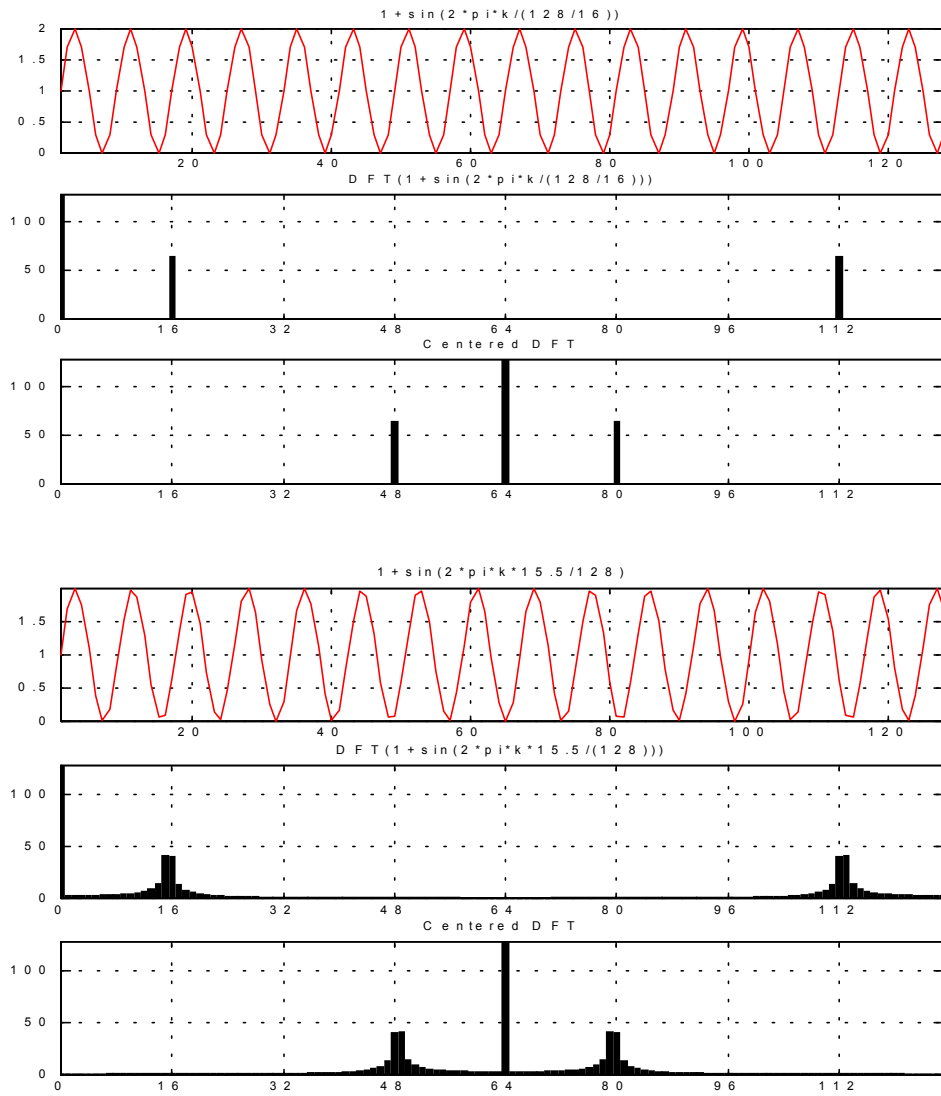
N_1 - even, N_2 - odd number



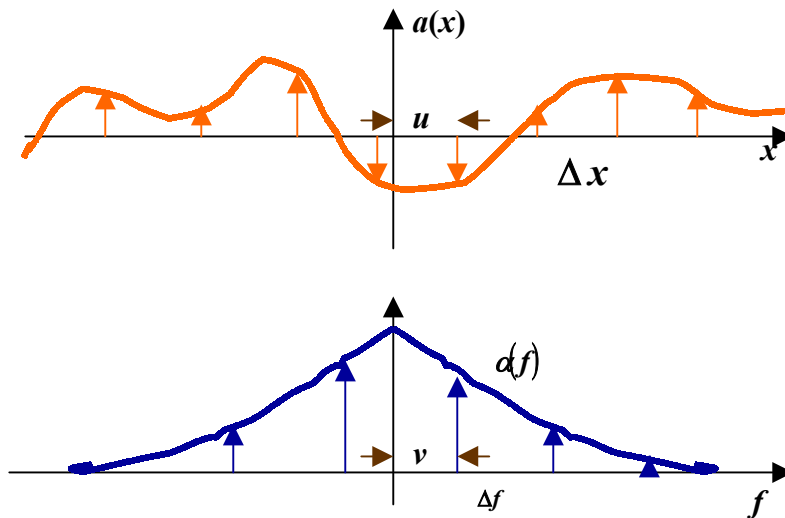
Types of the DFT spectra symmetry for one and two-dimensional real-valued signals

Centering DFT spectrum

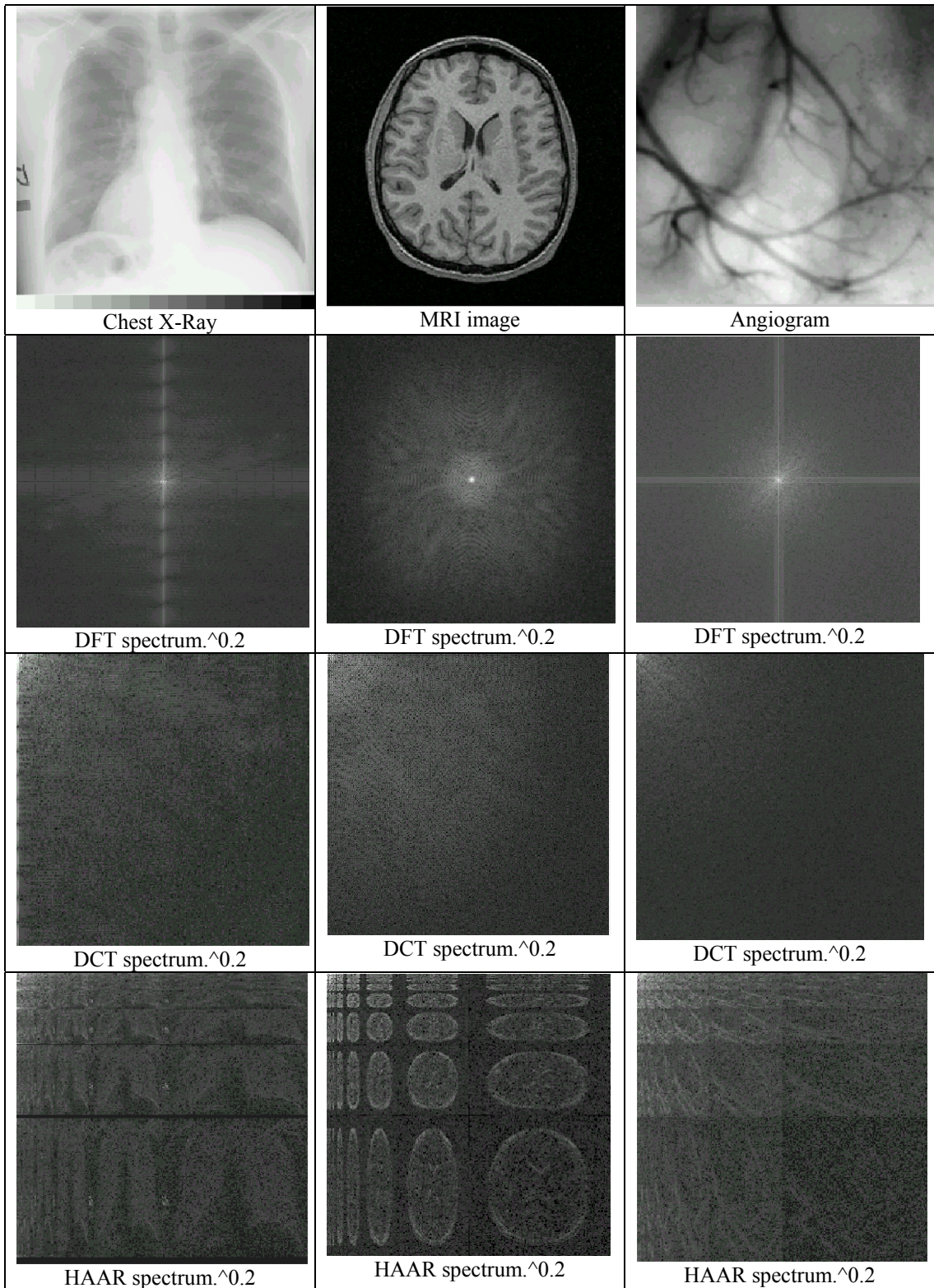




DFT spectra of sinusoidal signals with integer and noninteger number of periods

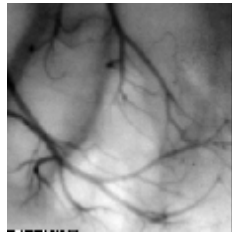


Shifted DFT: sampling diagram of continuous signal and its Fourier spectrum

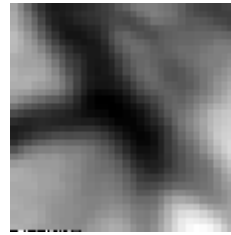


Images and their DFT, DCT and HAAR spectra

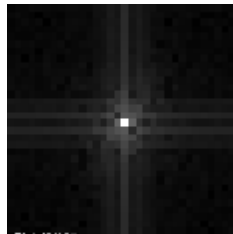
Displcsp.m:Input image



Selected fragment



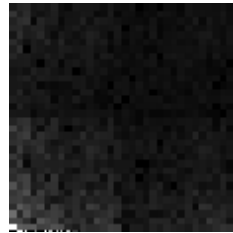
DFT spectrum of the fragment



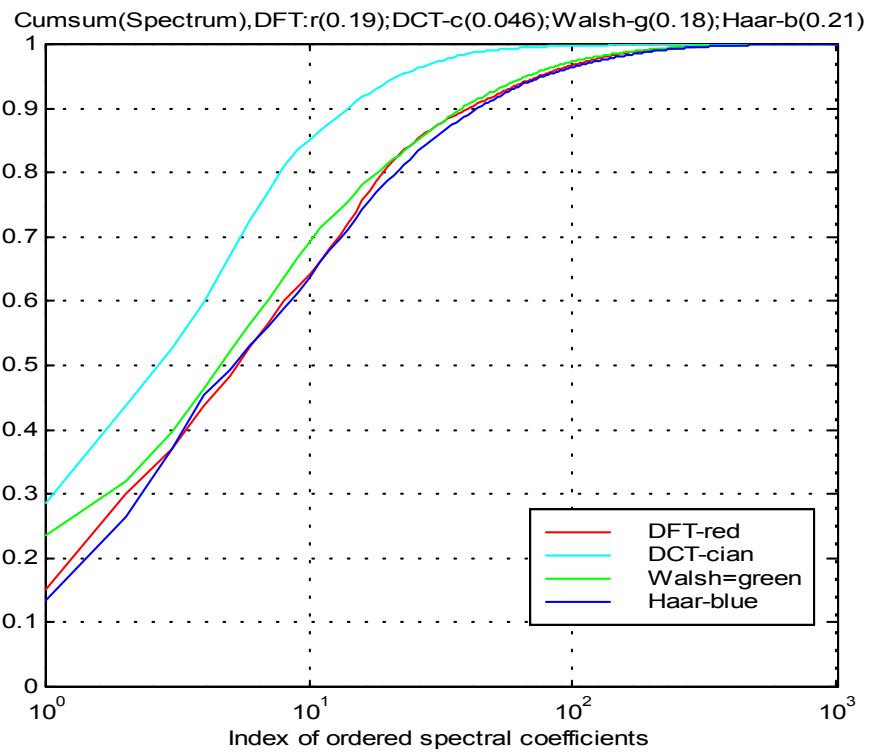
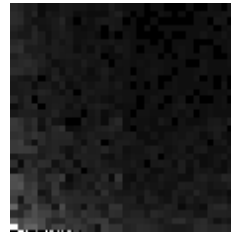
DCT spectrum of the fragment



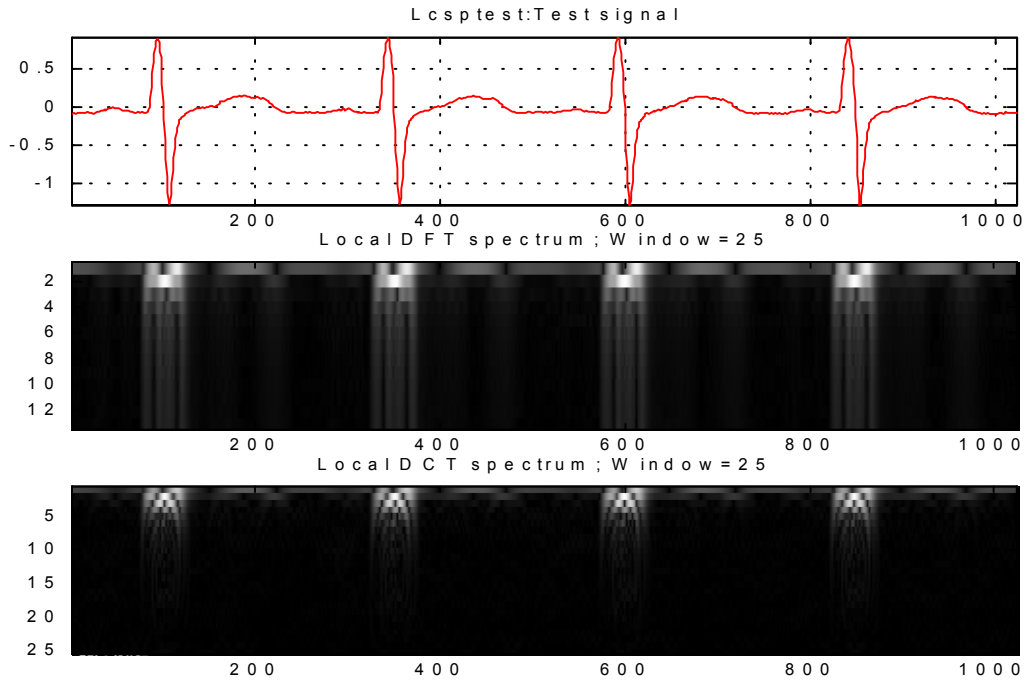
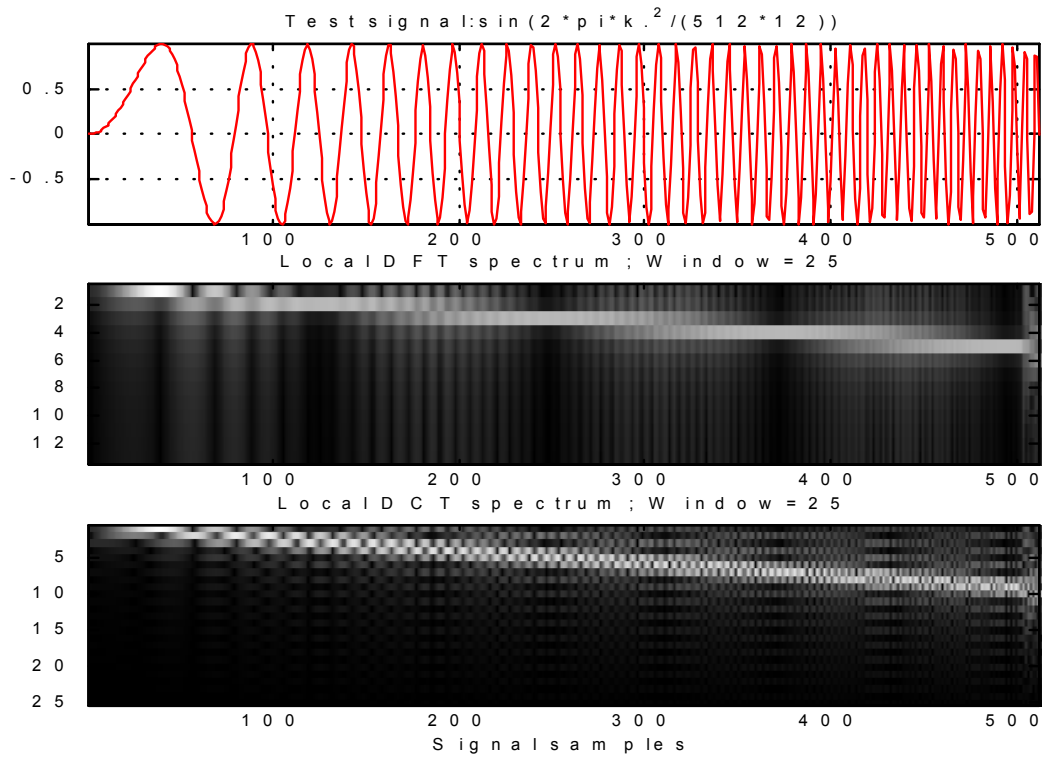
Walsh spectrum of the fragment



Haar spectrum of the fragment



Comparison of energy compaction properties of DFT, DCT, Walsh and Haar Transforms



Time-frequency DFT and DCT representation of a chirp signal (upper) and a electrocardiogram (lower)