Imaging Optics
and
Computational Imaging

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Imaging has always been the primary goal of informational optics. The whole history of optics is a history of creating and perfecting imaging devices. The main characteristic feature of the latest stage of the evolution of informational optics is integrating of physical optics with digital computers. With this, informational optics is reaching its maturity. It is becoming digital and imaging is becoming computational.
Image formation and processing capability of optics

- Image formation from light wave-front
- Image geometrical transformations
- Image integral transforms (Fourier & Fresnel, correlation and convolution)
- Image brightness point-wise manipulation (photographic alchemy & electro-optics)
New qualities brought in to imaging systems by digital computers

Flexibility and adaptability: no hardware modifications are necessary to reprogram digital computers to solving different tasks.

Digital computers integrated into optical imaging systems enable them to perform any operation needed.

Acquiring and processing quantitative data contained in optical signals and connecting optical systems with other informational systems and networks is most natural when data are handled in a digital form.

Low price: computers are much cheaper than optics.
Digital vs analog imaging: a tradeoff between good and bad features.

The fundamental limitation of digital signal processing is the speed of computations. What optics does in parallel and with the speed of light, computers perform as a sequence of very simple logical operations with binary digits, which is fundamentally slower whatever the speed of these operations is.

Optimal design of image systems requires appropriate combination of analog and digital processing using advantages and taking into consideration limitations of both.
Marriage analog electro-optical and digital processing requires appropriate linking analog and digital signals and transformations.
TO BE DISCUSSED:

- **Linking optical and digital image processing**
  - Principles of converting physical reality into digital signals
  - Discrete representations of imaging transforms
  - Resolving power of discrete Fourier analysis
  - Aliasing artifacts in numerical reconstruction of holograms
  - Building continuous image models
  - Signal numerical differentiation and integration

- **Computational imaging in examples**
  - Stabilization and super-resolution in turbulent video
  - Image recovery from sparse sampled data
  - Imaging without optics: optics less smart sensors

- **Conclusion: computational imaging and evolution of vision in nature**
References


Linking optical and digital image processing
Digital processors incorporated into optical information systems should be regarded and treated together with signal discretization and reconstruction devices as integrated analog units and should be specified and characterized in terms of equivalent analog transformations.

Discrete transformation corresponds to its analog prototype if both act to transform identical input signals into identical output signals.

Discrete representation of signal transformations should parallel that of signals.
Signal digitization: conversion of physical reality into a digital signal
SIGNAL DIGITIZATION

General quantization:

- Splitting signal space into “equivalence cells”
- Indexing “equivalence cells” by natural numbers
- Associating each index number with a “representative” signal of the corresponding equivalence cell
An example of general digitization: verbal description of the world

An estimation of the volume of image signal space:

\[ \gg 256 \times 1.000.000 \]
SOLUTION:
TWO STEPS SIGNAL DIGITIZATION:

Two step digitization:

Signal discretization by means of expansion over a set of basis functions

\[ a(x) = \sum_{k=0}^{N-1} \alpha_k \varphi_k^{(r)}(x) \]

\[ \alpha_k = \int_{X} a(x) \varphi_k^{(d)}(x) \]

Element-wise quantization of the representation coefficients \( \{ \alpha_k \} \) (general digitization in 1D signal space)
Signal discretization

Types of discretization basis functions:

- Shift (sampling) basis functions
  \[ \{\phi^{(s)}(x - k\Delta x)\}; \{\phi^{(r)}(x - k\Delta x)\} \]

- Scaled basis functions
  \[ \{\phi^{(d)}(kx)\}; \{\phi^{(r)}(kx)\} \]

- Shift&Scale basis functions:
  Wavelets
The sampling theorem

A “bandlimited” signal can be perfectly reconstructed from an infinite sequence of its samples if the inter-sample distance does not exceed \(1/2B\), where \(B\) is the highest frequency in the original signal.

The sampling theorem formulated in terms of the accuracy of reconstruction of continuous signals from their samples:

- The least square error approximation \(\tilde{a}(x)\) of signal \(a(x)\) from its samples \(\{a_k\}\) taken on a uniform sampling grid with sampling interval \(\Delta x\) is

\[
\tilde{a}(x) = \sum_{k=-\infty}^{\infty} a_k \text{sinc}[2\pi(x - k\Delta x)/\Delta x],
\]

provided that signal samples \(\{a_k\}\) are obtained as:

\[
a_k = \frac{1}{\Delta x} \int_{-\infty}^{\infty} a(x) \text{sinc}[2\pi(x - k\Delta x)/\Delta x] dx
\]

- The approximation mean square error is minimal in this case and is equal to:

\[
\int_{-\infty}^{\infty} |a(x) - \tilde{a}(x)|^2 dx = \int_{-1/\Delta x}^{-1/\Delta x} |\alpha(f)|^2 df + \int_{1/\Delta x}^{1/\Delta x} |\alpha(f)|^2 df = 2 \int_{1/\Delta x}^{1/\Delta x} |\alpha(f)|^2 df
\]

where \(\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx) dx\) is signal Fourier spectrum, \(f\) is frequency, and \(\text{sinc}(x) = \sin x / x\) is “Sinc-function”, the point-spread function of the ideal low-pass filter

\[
\text{sinc}(2\pi x / \Delta x) = \Delta x \int_{-1/\Delta x}^{1/\Delta x} \exp(-i2\pi fx) df
\]
Discrete representation of imaging transforms
**Discrete representation of the convolution integral**

The convolution integral of a signal \(a(x)\) with shift invariant kernel \(h(x)\):

\[
b(x) = \int_{-\infty}^{\infty} a(\xi)h(x - \xi)\,d\xi
\]

Digital filter for samples \(\{a_k\}\) and \(\{b_k\}\) of input and output signals

\[
b_k = \sum_{n=0}^{N_h-1} h_n a_{k-n}
\]

\(N_h\) is the number of non-zero samples of \(h_n\)

Discrete impulse response \(\{h_n\}\) of the digital filter for input and output signal sampling bases \(\phi^{(i)}(x)\) and \(\phi^{(r)}(x)\) and sampling interval \(\Delta x\)

\[
h_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h[x - \xi - n\Delta x] \phi^{(r)}(\xi) \phi^{(i)}(x)\,dx\,d\xi,
\]

Overall point spread function of an analog filter equivalent to a given digital filter

\[
h_{eq}(x, \xi) = \sum_{k=0}^{N_b-1} \sum_{n=0}^{N_h-1} h_n \phi^{(r)}(x - n\Delta x) \phi^{(i)}[\xi - (k - n)\Delta x]
\]

\(N_h\) is the number of samples of the filter output signal \(\{b_k\}\) involved in reconstruction of analog output signal \(b(x)\), \(N_h\) is the number of samples of the digital filter PSF and \(\Delta x\) is the signal sampling interval

Overall frequency response \(H_{eq}(f, p)\)

\[
H_{eq}(f, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{eq}(x, \xi) \exp[i2\pi(f x + p \xi)]\,dx\,d\xi = \text{SV}(f, p)\,CFrR(p)\,\Phi^{(r)}(f)\,\Phi^{(i)}(p).
\]

\[
\text{SV}(f, p) = \sum_{k=0}^{N_b-1} \exp[i2\pi(f - p)k\Delta x]; \quad \Phi^{(r)}(f) = \int_{-\infty}^{\infty} \phi^{(r)}(x)\exp(i2\pi f x)\,dx
\]

\[
\Phi^{(i)}(p) = \int_{-\infty}^{\infty} \phi^{(i)}(x)\exp(i2\pi p x)\,dx;
\]

\[
CFrR(p) = \sum_{n=0}^{N_h-1} h_n \exp(i2\pi n\Delta x) = \sum_{r=0}^{N_h-1} \eta_r \text{sinc}[N\pi(f/\Delta f - r)]; \quad \eta_r = \frac{1}{\sqrt{N}} \sum_{n=0}^{N_h-1} h_n \exp\left[i2\pi \frac{n - (N - 1)/2}{N} r \right]
\]
## Discrete representation of 1D Fourier integral transform

### 1-D direct and inverse integral Fourier Transforms of a signal \( a(x) \)

\[
\alpha(f) = \int \limits_\infty^\infty a(x) \exp(i2\pi fx) \, dx \quad \quad \quad \quad \quad a(x) = \int \limits_\infty^\infty \alpha(f) \exp(-i2\pi fx) \, df
\]

### Direct and Inverse Canonical Discrete Fourier Transforms (DFT)

\[
\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left\{ i2\pi \frac{k}{N} \right\} \quad \quad \quad \quad \quad a_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \alpha_r \exp\left\{ -i2\pi \frac{j}{N} \right\}
\]

### Direct and Inverse Shifted DFTs (SDFT(u,v)):

**Sampling conditions:**

- Signal and signal sampling device coordinate systems as well as, correspondingly, those of signal spectrum and of the assumed signal spectrum discretization device, are shifted with respect to each other in such a way that signal sample \( a_0 \) and, correspondingly, sample \( \alpha_0 \) of its Fourier spectrum are taken in signal and spectrum coordinates at points \( x = -u\Delta x \) and \( f = -v\Delta f \).

**Signal “cardinal” sampling:** \( \Delta x = 1/N\Delta f \)

\[
\alpha_{u,v}^r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left\{ i2\pi \frac{k}{N} (r + v) \right\} \quad \quad \quad \quad \quad a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_{u,v}^r \exp\left\{ -i2\pi \frac{k}{N} (r + v) \right\}
\]

### Direct and Inverse Discrete Cosine Transform (DCT):

**Sampling conditions:**

- Special case of SDFT for sampling grid shift parameters: \( u = 1/2; v = 0 \)

Analog signal of final length is, before sampling, artificially padded with its mirror copy to form a symmetrical sampled signal of double length: \( a_{-2N} = a_{-N-1} \).

\[
\alpha_r = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left\{ \frac{\pi}{N} \frac{k + 1/2}{r} \right\} \quad \quad \quad \quad \quad a_k = \frac{1}{\sqrt{2N}} \left[ \alpha_r + 2 \sum_{r=0}^{N-1} \alpha_r \cos\left\{ \frac{\pi}{N} \frac{k + 1/2}{r} \right\} \right]
\]

### Direct and Inverse Scaled Shifted DFTs (ScSDFT(u,v;\( \sigma \))

**Sampling conditions:**

- Sampling rate is \( \sigma \) times the cardinal rate: \( \Delta x = 1/\sigma N\Delta f \)

- Sampling shift parameters: \( u, v \neq 0 \)

\[
\alpha_{u,v}^r = \sum_{k=0}^{N-1} a_k \exp\left\{ i2\pi \frac{(k + u)(r + v)}{\sigma N} \right\} \quad \quad \quad \quad \quad a_k = \sum_{r=0}^{\sigma N-1} \alpha_{u,v}^r \exp\left\{ -i2\pi \frac{(k + u)(r + v)}{\sigma N} \right\}
\]
Discrete representation of 2D Fourier integral transform

2-D direct and inverse integral Fourier Transforms of a signal \( a(x, y) \)

\[
\alpha(f, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x, y) \exp[i2\pi(fx + py)] \, dx \, dy
\]

\[
a(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(f, p) \exp[-i2\pi(fx + py)] \, df \, dp
\]

2-D separable direct and inverse canonical DFTs:

Sampling conditions:
- Sampling in a rectangular sampling grid with cardinal sampling rates \( \Delta x = 1/N_1 \Delta f_x \), \( \Delta y = 1/N_2 \Delta f_y \)
- Zero sampling grid shift parameters

\[
a_{k,j} = \frac{1}{\sqrt{N_1N_2}} \sum_{l=0}^{N_1-1} \sum_{m=0}^{N_2-1} a_{l,m} \exp \left[ -i2\pi \left( \frac{kr}{N_1} + \frac{ls}{N_2} \right) \right]
\]

Scaled Shifted DFTs

Sampling conditions:
Sampling in a rectangular sampling grid. Sampling rates \( \Delta x = 1/\sigma_1 N_1 \Delta f_x \); \( \Delta y = 1/\sigma_2 N_2 \Delta f_y \)
Non-zero sampling grid shift parameters \((u,v)\) and \((p,q)\)

\[
a_{k,j} = \frac{1}{\sqrt{N_1N_2}} \sum_{l=0}^{N_1-1} \sum_{m=0}^{N_2-1} a_{l,m} \exp \left[ -i2\pi \left( \frac{kr}{N_1} + \frac{ls}{N_2} \right) \right]
\]

Rotated and Scaled DFTs

Sampling conditions:
- Sampling in a rectangular sampling grid in a rotated coordinate system \( \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \) with \( \theta \) as a rotation angle
- Sampling rates \( \Delta x = 1/\sigma N \Delta f_x \); \( \Delta y = 1/\sigma N \Delta f_y \); Non-zero sampling grid shift parameters \((u,v)\) and \((p,q)\)

\[
\alpha_{r,s}^{\theta} \propto \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp \left[ i2\pi \left( \frac{\tilde{k}r + \tilde{s}l}{\sigma N} \cos \theta - \frac{\tilde{k}r - \tilde{s}l}{\sigma N} \sin \theta \right) \right]; \quad \tilde{k} = k + u; \quad \tilde{r} = r + v;
\]

\[
\tilde{l} = l + p; \tilde{s} = s + q
\]
Point spread function of discrete Fourier analysis

\[ \alpha_r = \int_{-\infty}^{\infty} \alpha(f) h_{DFA}(f, r) df \]

Spectrum samples computed using ScDFT

Signal continuous spectrum

PSF of discrete spectral analysis

\[ h_{DFA}(f, r) = N \text{sincd} \left[ N \left( \frac{r}{\sigma} - f N \Delta x \right) \right] \Phi_s(f) \]

\[ \text{sincd}(N; x) = \frac{\sin x}{N \sin(x/N)} \]

- \( N \) is the number of signal samples
- \( \sigma \) – sampling scale parameter
- \( \Delta x \) – sampling interval
Point Spread Function and resolving power of discrete Fourier analysis
Discrete representation of Fresnel integral transform

Fresnel integral Transform

\[ \alpha(f) = \int_{-\infty}^{\infty} a(x) \exp\left(i2\pi \frac{(x-f)^2}{\lambda Z}\right) \]

Scaled Shifted Discrete Fresnel Transform

**Sampling conditions:**
Cardinal sampling rate \( \Delta x = 1/\sigma^N \Delta f \) with scale \( \sigma \)
Non-zero sampling grid shift parameters \((u,v) : w = u/\mu - v\mu\)

\[ \alpha_r^{(u,w)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[-i \pi \frac{(k\mu - r / \mu + w)}{\sigma N}\right] \]

Convolutional Discrete Fresnel Transform (ConvDFrT)

**Sampling conditions:** sampling rates: \( \Delta x = \Delta f \)

\[ \alpha_r = \frac{1}{N} \sum_{s=0}^{N-1} \left[ \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{k s}{N}\right) \right] \exp\left(-i \pi \frac{\mu^2 s^2}{N}\right) \exp\left(-i2\pi \frac{r s}{N}\right) = \]

\[ \frac{1}{N} \sum_{s=0}^{N-1} \left[ \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{k-r}{N} s\right) \right] \exp\left(-i \pi \frac{\mu^2 s^2}{N}\right) = \frac{1}{N} \sum_{k=0}^{N-1} a_k \left[ \sum_{s=0}^{N-1} \exp\left(-i \pi \frac{\mu^2 s^2}{N}\right) \exp\left(i2\pi \frac{k-r}{N} s\right) \right] = \]

\[ = \sum_{k=0}^{N-1} a_k \text{frincd}(N; \mu^2; r-k) \]
Frincd-function: DFT of “chirp”-function

\[
\text{frincd}(N; \mu^2; k) = \sum_{s=0}^{N-1} \exp\left(-i\pi \frac{\mu^2 s^2}{N}\right) \exp\left(i2\pi \frac{ks}{N}\right)
\]
Zones of applicability of Fourier and convolutional algorithms of reconstruction holograms recorded in near diffraction zone

Hologram $N$ samples; Camera pitch $\Delta f$

Object plane, $\mu^2 < 1$
- Fourier algorithm, reconstruction with aliasing
- Convolution algorithm, reconstruction without aliasing

Object plane, $\mu^2 = 1$
- Fourier and convolution algorithms: reconstructions are identical

Object plane, $\mu^2 > 1$
- Fourier algorithm, reconstruction without aliasing
- Convolution algorithm, reconstruction with aliasing

$\mu^2 = \frac{\lambda Z}{N \Delta f^2}$
Hologram reconstruction: Fourier algorithm vs Convolution algorithm

Fourier reconstruction of the central part of the hologram free of aliasing

Convolution reconstruction

Hologram courtesy
Dr. J. Campos, UAB,
Barcelona, Spain

Image is destroyed due to aliasing

Aliasing artifacts

All restorations are identical

Z=33m;
\( \mu^2 = 0.2439 \)

Z=83mm;
\( \mu^2 = 0.6618 \)

Z=136mm;
\( \mu^2 = 1 \)

Hologram courtesy Dr. J. Campos, UAB, Barcelona, Spain
Discretization aliasing artifacts in reconstruction of a hologram on different distances using Fourier reconstruction algorithm (left), the Fourier reconstruction algorithm with appropriate hologram masking to avoid aliasing (middle) and the Convolution reconstruction algorithm.

Hologram courtesy Dr. J. Campos, UAB, Barcelona, Spain.
BUILDING CONTINUOUS IMAGE MODELS
When working with sampled images in computers, one frequently needs to return back to their continuous originals.

Typical applications that require restoration of continuous image models are image geometrical transformations, image reconstruction from projections, multi-modality data fusion, target location and tracking with sub-pixel accuracy, image restoration from sparse samples and image differentiation and integration, to name a few.
Discrete sinc-interpolation: a gold standard for image resampling
Image resampling assumes reconstruction of the continuous approximation of the original non-sampled image by means of interpolation of available image samples to obtain samples in-between the given ones.

In some applications, for instance, in computer graphics and print art, simple interpolation methods, such as nearest neighbor or linear (bilinear) interpolations, can provide satisfactory results. However, all these methods add interpolation error to reconstructed continuous image models, thus introducing signal distortions additional to those caused by the primary image sampling.

A discrete signal interpolation method that is capable of secure continuous image restoration without adding any additional interpolation errors is the discrete sinc-interpolation.
Discrete sinc-interpolation is a discrete analog of the continuous sinc-interpolation, which secures error free reconstruction of “band-limited” signals from their samples and least mean square error reconstruction of arbitrary continuous signals, provided infinitely large number of signal samples is available.

Discrete sinc-interpolation does the same for discrete signals.
How can one design a perfect resampling filter?

- For the purposes of the design of the perfect resampling filter, one can regard signal co-ordinate shift as a general resampling operation.

- Signal resampling is a linear signal transformation. It can be fully characterized by its point spread function (PSF) or, correspondingly, by its overall frequency response.
The optimal shifting re-sampling filter is the filter that generates a shifted copy of the input signal with preservation of the analog signal spectrum in its base band defined by the signal sampling rate and by the number of available signal samples.

According to this definition, overall continuous frequency response $H^{(intp)}(p)$ of the optimal $\delta x$-shifting filter for the coordinate shift $\delta x$ is, by virtue of the Fourier transform shift theorem,

$$H^{(intp)}(p) = \exp(i2\pi p \delta x)$$
According to the discrete representation of the convolution integral, discrete frequency response coefficients \( \{ \eta^{(\text{intp})}_{r,\text{opt}}(\delta \tilde{x}) \} \) (DFT of its discrete PSF) of the optimal \( \delta \tilde{x} \)-shift re-sampling filter must be taken as samples, in sampling points \( \{ r/N\Delta x \}; r = 0,1,..., N - 1 \), of its overall continuous frequency response

\[
H^{(\text{intp})}(p) = \exp(i2\pi p \delta \tilde{x})
\]

which, for the ideal signal sampling and reconstruction devices, coincides, within the signal base band, with the filter overall frequency response
Therefore for odd $N$

$$\eta^{(intp)}_{r, opt}(\delta x) = \frac{1}{\sqrt{N}} \exp\left(i 2\pi \frac{r \delta x}{N \Delta x}\right), \quad r = 0, 1, \ldots, (N - 1)/2$$

$$\eta^{(intp)}_{r, opt}(\delta x) = \eta^{*(intp)}_{N - r, opt}(\delta x), \quad r = (N + 1)/2, \ldots, N - 1$$

and for even $N$:

$$\eta^{(intp)}_{r, opt}(\delta x) = \begin{cases} 
\exp\left(i 2\pi \frac{r \delta x}{N \Delta x}\right), \quad &r = 0, 1, \ldots, N/2 - 1 \\
A \cos\left(\pi \frac{\delta x}{\Delta x}\right), &r = N/2 
\end{cases}$$

$$\eta^{(intp)}_{r, opt}(\delta x) = (\eta^{(intp)}_{N - r, opt}(\delta x))^*, \quad r = N/2 + 1, \ldots, N - 1$$

The following three options for $A$ are:

Case_0: $A=0$, Case_1: $A=1$, Case_2: $A=2$. 
For odd $N$, point spread function of the optimal $\delta\xi$-resampling (fractional shift) filter is

$$h_n^{\text{intp}}(\delta\xi) = \text{sincd}\{N, \pi[n - (N - 1)/2 - \delta\xi/\Delta x]\}.$$ 

For even $N$, Case_0 and Case_2, optimal resampling point spread functions are

$$h_n^{\text{intp0}}(\delta\xi) = \text{sincd}\{N; N - 1; \pi[n - (N - 1)/2 - \delta\xi/\Delta x]\}$$

and

$$h_n^{\text{intp2}}(\delta\xi) = \text{sincd}\{N; N + 1; \pi[n - (N - 1)/2 - \delta\xi/\Delta x]\} ,$$

correspondingly, where a modified sincd-function $\text{sincd}$ is defined as

$$\text{sincd}(N; M; x) = \frac{\sin(Mx/N)}{N \sin(x/N)}$$

Case_1 is just a combination of Case_0 and Case_2:

$$h_n^{\text{intp2}}(\delta\xi) = [h_n^{\text{intp0}}(\delta\xi) + h_n^{\text{intp2}}(\delta\xi)]/2 = \text{sincd}(\pm 1; N; x) = \left[\text{sincd}(N - 1; N; x) + \text{sincd}(N + 1; N; x)\right]/2.$$
Discrete sinc-functions
Frequency response of the discrete sinc-interpolators
The above results can be formulated as a theorem:

For analog signals defined by their $N$ samples, discrete sinc-interpolation is the only discrete convolution based signal re-sampling method that, for odd $N$, does not distort signal spectrum samples in its base band specified by the signal sampling rate; for even $N$, discrete sinc-interpolation distorts only the highest $N/2$-th frequency spectral component.
Implementation issues:

The described $\mathcal{F}$-resampling filter that implements discrete sinc-interpolation is designed in DFT domain. Therefore it can be straightforwardly implemented using Fast Fourier Transform with the computational complexity of $O(\log N)$ operations per output signal sample, which makes it competitive with other less accurate interpolation methods.

From the application point of view, the only drawback of such an implementation is that it tends to produce signal oscillations due to the boundary effects caused by the circular periodicity of convolution implemented in DFT domain. These oscillation artifacts can be virtually completely eliminated, if discrete sinc-interpolation convolution is implemented in DCT domain.
DFT-based vs DCT-based discrete sinc-interpolation

Sample indices

DFT-based interpolation

Initial signal samples

DCT-based interpolation

DFT-based discrete sinc-interpolation

DCT-based discrete sinc-interpolation
Interpolation accuracy comparison:
16x18° - image rotation

Test image

NearNeighb, T=7.27

Bilinear, T=11.1

Bicubic, T=17.7

Discrete sinc, T=14.2

Image recovery and, more generally, sign problems that are among the most fundamental involve every known scale—from the determining the structure of unresolved segment of the tiniest molecules. Stated in its recovery problem is described like this: Given at produced g. Unfortunately, when stated otherwise can be said. How is g related to f? Is g known, can g be be used to furnish an estimate of f? If g is corrupted by noise, does the noise persist, can we ameliorate the effects of the noise? If g is corrupted by noise, does the noise persist, can we ameliorate the effects of the noise? Even if g unique algorithm for computing f from g? What about? Can it be usefully incorporated in our This (and others) are the kinds of questions we ask ourselves. It is the purpose of this book.
Discrete sinc-interpolation vs spline (Mems531) interpolation: Image 1000x18° rotation

Image recovery and, more generally, sign problems that are among the most fundamental involve every known scale—from the determining the structure of unresolved element of the tiniest molecules. Stated in its recovery problem is described like this: Given at produced $g$. Unfortunately, when stated more can be said. How is $g$ related to $f$? Is $g$ itself, can $g$ be used to furnish an estimate $f$ of $f$? If $g$ is corrupted by noise, does the noise play a role, can we ameliorate the effects of the noise? How radical changes in $f$? Even if $g$ unique algorithm for computing $f$ from $g$? What about? Can it be usefully incorporated in our These (and others) are the kinds of questions we will ask in the moment of this book.

Test image

Test image low-pass filtered to 0.4 of its bandwidth
Discrete sinc interpolation vs spline (Mems531) interpolation: Rotation error DFT spectra comparison (image 10x36° rotation)

Pseudo-random test image

Test image DFT spectrum

Bicubic interpolation

Mems531 interpolation

Discrete sinc interpolation

DFT spectra of rotated image error (dark- small errors; bright – large errors)
Case study: Image numerical differentiation and integration

Signal numerical differentiation and integration are operations that are defined for continuous signals and require measuring infinitesimal increments or decrements of signals and their arguments. Therefore, numerical computing signal derivatives and integrals assumes one or another method of building continuous models of signals specified by their samples through explicit or implicit interpolation between available signal samples.
In numerical mathematics, alternative methods of numerical computing signal derivatives and integrals are commonly used that are implemented through signal discrete convolution in the signal domain:

\[
\hat{a}_k = \sum_{n=0}^{N_h-1} h_n^{\text{diff}} a_{k-n} ; \quad \bar{a}_k = \sum_{n=0}^{N_h-1} h_n^{\text{int}} a_{k-n}.
\]

Commonly, the simplest differentiating kernels of two and five samples are recommended

D1: \( h_n^{\text{diff}(1)} = [-0.5, 0, 0.5] \);  D2: \( h_n^{\text{diff}(2)} = [-1/12, 8/12, 0, -8/12, 1/12] \).

Most known numerical integration methods are the Newton-Cotes quadrature rules. The three first rules are the trapezoidal, the Simpson and the 3/8 Simpson ones defined, for \( k \) as a running sample index, as, respectively:

\[
\bar{a}_0^{(r)} = 0, \quad \bar{a}_k^{(r)} = \bar{a}_{k-1}^{(r)} + \frac{1}{2} (a_{k-1} + a_k) ; \quad \bar{a}_1^{(s)} = 0, \quad \bar{a}_k^{(s)} = \bar{a}_{k-2}^{(s)} + \frac{1}{3} (a_{k-2} + 4a_{k-1} + a_k) .
\]

\[
\bar{a}_0^{(3/8s)} = 0, \quad \bar{a}_k^{(3/8s)} = \bar{a}_{k-3}^{(3/8s)} + \frac{3}{8} (a_{k-3} + 3a_{k-2} + 3a_{k-1} + a_k) .
\]
**Synthesis of perfect differentiation and integration filters**

Differentiation and integration are shift invariant linear operations. Hence methods of computing signal derivatives and integrals from their samples can be conveniently designed, implemented and compared in the Fourier transform domain.

Signal differentiation and integration can be regarded as signal linear filtering with filter frequency responses, correspondingly

\[ H_{\text{diff}}(f) = -i2\pi f \quad \text{and} \quad H_{\text{int}}(f) = i/2\pi f \]

Then coefficients \( \{ \eta_r^{(\text{diff})} \} \) and \( \{ \eta_r^{(\text{int})} \} \) of discrete frequency responses of numerical differentiation and integration digital filters defined as samples of corresponding continuous frequency responses are:

\[
\eta_r^{(\text{diff})} = \begin{cases} 
-i2\pi r / N, & r = 0,1,..., N / 2 - 1 \\
-\pi / 2, & r = N / 2 \\
i2\pi(N - r) / N, & r = N / 2 + 1,..., N - 1
\end{cases}; \quad \eta_r^{(\text{int})} = \begin{cases} 
0, & r = 0 \\
iN / 2\pi, & r = 1,..., N / 2 - 1 \\
-\pi / 2, & r = N / 2 \\
iN / 2\pi(N - r), & r = N / 2 + 1,..., N - 1
\end{cases}
\]

for even \( N \) and

\[
\eta_r^{\text{diff}} = \begin{cases} 
-i2\pi r / N, & r = 0,1,..., (N - 1)/ 2 - 1 \\
i2\pi(N - r) / N, & r = (N + 1)/ 2,..., N - 1
\end{cases}; \quad \eta_r^{(\text{int})} = \begin{cases} 
N / 2\pi, & r = 0,1,..., (N - 1)/ 2 - 1 \\
iN / 2\pi(N - r), & r = (N + 1)/ 2,..., N - 1
\end{cases}
\]

for odd \( N \).
Frequency responses of numerical differentiation methods

Frequency responses of numerical integration methods
Differentiation error comparison

Differentiation error, 100 realizations; No ApodMask

Different error StdDev over signal central three quarters

Differentiation error

Sample index

Signal bandwidth (in fraction of the base band)
Resolving power of numerical integrators

Test signal

“Ideal” integration

Trapezoidal integration

Cubic spline integration

DFT/DCT integration
DCT based versus conventional differentiation-integration methods: signal restoration error

Comparison of standard deviations signal restoration error after iterative successive 75 differentiations and integrations applied to a test signal for DCT-based differentiation and integration methods and for D2 differentiator and trapezoidal rule integrator, respectively: a), c) – initial (blue curves) and restored (red dots) signals; b), d) – restoration error standard deviation vs the number of iterations
Accurate numerical differentiation and integration: implementation issues

One can show that numerical differentiation and integration according above equations imply the discrete sinc-interpolation of signals. Being designed in DFT domain, the differentiation and integration filters can be efficiently implemented in DFT domain using Fast Fourier Transforms with the computational complexity of the algorithms of $O(\log N)$ operations per signal sample.

Likewise all DFT based discrete sinc interpolation algorithms, DFT-based differentiation and integration algorithms, being the most accurate in term of representation of the corresponding continuous filters within the signal base band, suffer from boundary effects. Obviously, DFT based differentiation is especially vulnerable in this respect.

This drawback can be efficiently overcome by means of even extension of signals to double length through mirror reflection at their boundaries before applying above described DFT based algorithms. For such extended signals, DFT based differentiation and integration are reduced to using fast DCT algorithm instead of FFT:

$$\{\tilde{a}_k\} = -\frac{2\pi}{N\sqrt{2N}} (-1)^k \sum_{r=1}^{N-1} (N - r) a^{(DCT)}_{N-r} \cos \left( \pi \frac{k + 1/2}{N} r \right);$$

$$\{\tilde{a}_k\} = \frac{\sqrt{N}}{2\pi\sqrt{2}} (-1)^k \sum_{r=1}^{N-1} \frac{a^{(DCT)}_{N-r}}{N - r} \cos \left( \pi \frac{k + 1/2}{N} r \right)$$

with the same computational complexity of $O(\log N)$ operations per signal sample.
COMPUTATIONAL IMAGING IN EXAMPLES
Case study: Real time stabilization and super-resolution of turbulent videos
The video stabilization algorithm

Input video sequence

Stable scene estimation by means of temporal averaging of video frames

Motion field estimation by means of “elastic” registration of frames

Frame wise scene segmentation to stable and moving objects by means analysis of the motion field

Formation of stabilized video by point-wise means of switching between input signal and stable scene signal according to the segmentation segmentation

Output video sequence
The super-resolution algorithm

Input video sequence

Input frame sub-sampling → Reference frame formation → Segmented motion field (frame-wise displacement maps)

Segmented displacement map controlled replacing reference frame samples of stable scene by samples of input (non-stabilized) frames from a selected time window, moving object samples being taken from the sub-sampled input frames

Image recovery from the map of sparse samples

Output stabilized and resolution enhanced video sequence
Super-resolution in turbulent videos

Low resolution frames upper right; image fused by elastic image registration from 50 frames (bottom right); a result of iterative interpolation of the middle image after 50 iterations (bottom left).
Turbulent video stabilization and super-resolution
Image recovery from sparse data
and
the discrete sampling theorem
Shannon-Kotelnikov’s sampling theorem tells how to optimally sample continuous signals and reconstruct them from the result of sampling with minimal MSE error. This optimal sampling assumes a uniform sampling grid.

In many applications sampled data are collected in an irregular fashion or are partly lost or unavailable. In these cases it is required to convert irregularly sampled signals to regularly sampled ones, or to restore missing data.
We address this problem in a framework of the discrete sampling theorem for “band-limited” discrete signals that have a limited number of non-zero transform coefficients in the domain of a certain orthogonal transform.
Basic assumptions:

- Continuous signals are represented in computers by their samples.
- Let the number of signal samples on a regular sampling grid that are believed to fully represent the original continuous signal is $N$.
- Let available be $K<N$ samples of this signal, taken at arbitrary positions of the signal regular sampling grid.
- The goal of the processing is generating, out of this “incomplete” set of samples, the complete set of $N$ signal samples with the best possible accuracy.
- For definiteness, we will use restoration mean square error for evaluating signal approximation accuracy.
Let \( \Lambda_N \) be a vector of \( N \) samples \( \{a_k\}_{k=0,\ldots,N-1} \), which completely define a discrete signal, \( \Phi_N \) be an \( N \times N \) orthogonal transform matrix
\[
\Phi_N = \{\varphi_r(k)\}_{r=0,\ldots,N-1}
\]
and \( \Gamma_N \) be a vector of signal transform coefficients \( \{\gamma_r\}_{r=0,\ldots,N-1} \) such that
\[
\Lambda_N = \Phi_N \Gamma_N = \left\{ \sum_{r=0}^{N-1} \gamma_r \varphi_r(k) \right\}_{k=0,\ldots,N-1}
\]

Assume now that available are only \( K < N \) samples \( \{a_k\}_{k \in \bar{K}} \), where \( \bar{K} \) is a \( K \)-size non-empty subset of indices \( \{0,1,\ldots,N-1\} \). These available \( K \) signal samples define a system of equations:
\[
\left\{ a_k = \sum_{r=0}^{N-1} \gamma_r \varphi_r(k) \right\}_{k \in \bar{K}}
\]
for signal transform coefficients \( \{\gamma_r\} \) of certain \( K \) indices \( r \).
Select now a subset \( \tilde{R} \) of \( K \) transform coefficients indices \( \{\tilde{r} \in \tilde{R}\} \) and define a “\( KofN \)”-band-limited approximation to the signal as the

\[
\hat{A}_{N}^{BL} = \left\{ \hat{a}_k = \sum_{\tilde{r} \in \tilde{R}} \gamma_{\tilde{r}} \varphi_{\tilde{r}}(k) \right\}
\]

Rewrite this equation in a more general form:

\[
\hat{A}_{N}^{BL} = \left\{ \hat{a}_k = \sum_{r \in \tilde{R}} \tilde{\gamma}_r \varphi_r(k) \right\}
\]

and assume that all transform coefficients with indices \( r \notin \tilde{R} \) are set to zero:

\[
\tilde{\gamma}_r = \begin{cases} 
\gamma_r, & r \in \tilde{R} \\
0, & r \notin \tilde{R} 
\end{cases}
\]

Then the vector \( \tilde{A}_K \) of available signal samples \( \{a_{\tilde{k}}\}_{\tilde{k} \in \tilde{K}} \) can be expressed in terms of the basis functions \( \{\varphi_r(k)\}_{r \in \tilde{R}} \) of transform \( \Phi_N \) as:

\[
\tilde{A}_K = KofN_\Phi \cdot \tilde{\Gamma}_K = \left\{ \tilde{a}_{\tilde{k}} = \sum_{\tilde{r} \in \tilde{R}} \gamma_{\tilde{r}} \varphi_{\tilde{r}}(\tilde{k}) \right\}
\]

and the vector \( \tilde{\Gamma}_K = \{\tilde{\gamma}_r\} \) of signal non-zero transform coefficients can be found as

\[
\tilde{\Gamma}_K = \{\tilde{\gamma}_r\} = KofN^{-1}_\Phi \cdot \tilde{A}_K
\]

In \( L2 \) norm, by virtue of the Parceval’s theorem, the band-limited signal \( \hat{A}_{N}^{BL} \) approximates the complete signal \( A_{N1} \) with mean squared error

\[
MSE = \left\| A_{N} - \hat{A}_{N} \right\| = \sum_{k=0}^{N-1} |a_k - \hat{a}_k|^2 = \sum_{r \notin \tilde{R}} |\gamma_r|^2
\]
**Statement 1.** For any discrete signal of \( N \) samples defined by its \( K \leq N \) sparse and not necessarily regularly arranged samples, its band-limited, in terms of a certain transform \( \Phi_N \), approximation can be obtained with mean square error

\[
MSE = \| A_N - \hat{A}_N \| = \sum_{k=0}^{N-1} |a_k - \hat{a}_k|^2 = \sum_{r \in R} |\gamma_r|^2
\]

provided that positions of the samples secure the existence of the matrix \( K_{ofN}^{-1}_\phi \) inverse to the sub-transform matrix \( K_{ofN}_\phi \) that corresponds to the band-limitation. The approximation error can be minimized by using a transform with the best energy compaction property.

**Statement 2.** Any signal of \( N \) samples that is known to have only \( K \leq N \) non-zero transform coefficients for certain transform \( \Phi_N (\Phi_N \text{ - “band-limited” signal}) \) can be fully recovered from exactly \( K \) of its samples provided the positions of the sample secure the existence of the matrix \( K_{ofN}^{-1}_\phi \) inverse to the transform sub-matrix \( K_{ofN}_\phi \) that corresponds to the band-limitation.
Analysis of transforms: DFT

Low-pass DFT band-limited signals:
\[
\tilde{r}_{LP} \in \tilde{R}_{LP} = \{[0,1,\ldots,(K-1)/2,N-(K-1)/2,\ldots,N-1]\}
\]

\[K_{ofN_{DF}^{LP}} \text{ DFT } \text{-trimmed matrix } K_{ofN_{DF}^{LP}} = \left\{ \exp\left(\frac{i2\pi \tilde{k}\tilde{r}_{LP}}{N}\right) \right\} \]

is a Vandermonde matrix, and, as such, it can be inverted

**Theorem 1.**

*Low-pass DFT band-limited signals of N samples with only K nonzero low frequency DFT coefficients can be precisely recovered from exactly K of their samples taken in arbitrary positions.*

High-pass DFT band-limited signals:
\[
\tilde{r}_{HP} \in \tilde{R}_{HP} = \{(N-K+1)/2,(N-K+3)/2,\ldots,N-K-1)/2\}
\]

\[K_{ofN_{DF}^{LP}} \text{ DFT } \text{-high-pass trimmed matrix } K_{ofN_{DF}^{HP}} = \left\{ \exp\left(\frac{i2\pi \tilde{k}\tilde{r}_{HP}}{N}\right) \right\} \]

is a Vandermonde matrix, and, as such, it can be inverted

**Theorem 2.**

*High-pass DFT band-limited signals of N samples with only K nonzero high frequency DFT coefficients can be precisely recovered from exactly K of their arbitrarily taken samples.*
Analysis of transforms: DCT

DCT is an orthogonal transform with very good energy compaction properties. It is well suited for compressed representation of many types of signals.

$N$-point Discrete Cosine Transform of a signal is equivalent to $2N$-point Shifted Discrete Fourier Transform (SDFT) with shift parameters $(1/2, 0)$ of the $2N$-sample signal obtained from the initial one by its mirror reflection from its borders.

$KofN$-trimmed matrix of $SDFT(1/2, 0)$

\[
KofN_{SDFT} = \left\{ \exp\left( i 2\pi \frac{(\tilde{k} + 1/2)\tilde{r}}{2N} \right) \right\} = \left\{ \exp\left( i 2\pi \frac{\tilde{k}\tilde{r}}{2N} \right) \exp\left( i \pi \frac{\tilde{r}}{2N} \right) \delta(k - r) \right\} = KofN_{DFT} \left\{ \exp\left( i \pi \frac{\tilde{r}}{2N} \right) \delta(k - r) \right\}
\]

Therefore, for DCT theorems similar to those for DFT hold.
Analysis of transforms: Discrete Fresnel transform

Canonical Discrete Fresnel Transform (DFrT) is defined as

\[ a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp \left[ -i\pi \left( \frac{k\mu - r}{\mu} \right)^2 \right] \]

where \( \mu \) is a distance parameter.

DFrT can easily be expressed via DFT:

\[ \alpha_r = \frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[ a_k \exp \left( i\pi \frac{k^2\mu^2}{N} \right) \right] \exp \left( -i2\pi \frac{kr}{N} \right) \right\} \exp \left( i\pi \frac{r^2}{\mu^2N} \right) \]

In a matrix form, it can be represented as a matrix product of diagonal matrices and the matrix of Discrete Fourier Transform. Therefore for Discrete Fresnel Transform formulation of band-limitation and requirements to positions of sparse samples are the similar to those for DFT.
2-D transforms:
• Separable band-limitation
• Inseparable band-limitation

Other transforms:
• Walsh transform
• Haar transform
• Wavelet transforms
Algorithms:

1. Direct matrix inversion

\[
\hat{A}_{N}^{BL} = \left\{ \hat{a}_k = \sum_{\tilde{r} \in \tilde{R}} [K_{N}^{1} \cdot \tilde{A}_{K}] \varphi_{\tilde{r}}(k) \right\}
\]

An open question: do fast algorithms for matrix inversion exist?

For DFT, DCT, Walsh, Haar and other transforms that feature FFT-type algorithms, pruned fast algorithms might be used.

2. Iterative Gershberg-Papoulis-type algorithm
Initial guess: available signal samples on a dense sampling grid defined by the accuracy of measuring sample coordinates, supplemented with a guess of the rest of the samples, for which zeros, signal mean value or random numbers can be used.
Experiments with DFT

Restoration of a DFT low pass band-limited signal by matrix inversion for the cases of random (a), upper) and compactly placed signal samples (a), bottom) and by the iterative algorithm (b). Bottom right plot shows standard deviation of signal restoration error as a function of the number of iterations. The experiment was conducted for a test signal length 64 samples; bandwidth 13 frequency samples (~1/5 of the signal base band)
Image recovery from sparse samples: DCT with separable band limitation

Recovery of an image band limited in DCT domain by a square: a) – initial image with 3136 “randomly” place samples (shown by white dots); b) – the shape of the image spectrum in DCT domain; c) – image restored by the iterative algorithm after 100000 iterations with restoration PSNR (peak signal-to-error standard deviation) 4230; d) image restored by B-spline interpolation with restoration PSNR 966; e) iterative algorithm restoration error (white – large errors; black – small errors); f) – restoration error standard deviation versus the number of iterations for the iterative algorithm and that for the B-spline interpolation

[OUTIMG,t,t_r,StdErr,OUTIMG_spline, StdErr_spline, msk]=map_reconstr_test_sinc_spline(64,0.05);
Image recovery from sparse samples: 
DCT with non-separable band limitation

Recovery of an image band limited in DCT domain by a circle sector: a) – initial image with 3964 “randomly” placed samples (shown by white dots); b) – the shape of the image spectrum in DCT domain; c) – image restored by the iterative algorithm after 100000 iterations with restoration PSNR (peak signal-to-error standard deviation) 21.5; d) image restored by B-spline interpolation with restoration PSNR 7.42; e) iterative algorithm restoration error (white – large errors; black – small errors); f) – the restoration error standard deviation versus the number of iterations of the iterative algorithm and that for the B-spline interpolation.
Image recovery from level lines: DCT with non-separable band limitation

Recovery of an image band limited in DCT domain by a circle sector from its level lines: a) – initial image with level lines (shown by white dots); b) – image restored by the iterative algorithm after 1000 iterations with restoration PSNR 3.5x10^4 (note that the restoration error is concentrated in a small area of the image); c) image restored by B-spline interpolation with restoration PSNR 29.4; d) iterative algorithm restoration error (white – large errors; black – small errors); e) – the restoration error standard deviation versus the number of iterations of the iterative algorithm for the iterative algorithm and that for the B-spline interpolation.
Signal recovery from sparse or non-uniformly sampled data as an approximation task

Signal recovery from sparse or non-uniformly sampled data can be treated as finding best signal band-limited approximation. For this, the above theory and algorithms can be applied as following:

1. Given a certain number of available signal samples, specify the signal dense sampling grid and the required number of samples to be recovered.

2. Select a transform with presumably better energy compaction capability for the signal at hand and specify the signal band limitation in the domain of this transform.

3. Place available signal samples on the signal dense sampling grid and run the direct matrix inversion or the iterative reconstruction algorithm.
Energy compaction capability of transforms

Test image 1

Test image 2

Test image 3

Test image 1: Fraction of signal energy

Test image 2: Fraction of signal energy

Test image 3: Fraction of signal energy
Errors due to image band limitation

DCT_LP image; BW=0.3 ; StdErr=16.3692

DCT_LP image; BW=0.3 ; StdErr=2.9807

Walsh_LP image; BW=0.3 ; StdErr=22.1208

Walsh_LP image; BW=0.3 ; StdErr=4.1801

Haar_LP image; BW=0.3 ; StdErr=22.3649

Haar_LP image; BW=0.3 ; StdErr=4.2585
Applications examples:

- Image super-resolution from turbulent videos (shown above)

- Image super-resolution in computed tomography
Super-resolution in computed tomography
Image recovery from sparse samples: the “Compressed sensing” approach
Described methods for image recovery from sparse samples by means of their band-limited approximation in certain transform domain require explicit formulation of the desired band limitation in the selected transform domain.

This a priori knowledge that one has to invest is quite natural to assume. If one selects a transform according to its energy compaction capability, one may know how this capability works, i.e. what transform coefficients are expected to be zero or non-zero.

If, however, this is not known or not certain a priori, image recovery can be achieved using an approach, which obtained the name, in fact, quite confusing, “compressed sensing” (Donoho, D., “Compressed sensing”(2006), IEEE Trans. On Information Theory, v. 52(4), pp. 1289-1306).
The dogma of signal processing maintains that a signal must be sampled at a rate at least twice its highest frequency in order to be represented without error. However, in practice, we often compress the data soon after sensing, trading off signal representation complexity (bits) for some error (consider JPEG image compression in digital cameras, for example). Clearly, this is wasteful of valuable sensing resources. Over the past few years, a new theory of "compressive sensing" has begun to emerge, in which the signal is sampled (and simultaneously compressed) at a greatly reduced rate. Compressive sensing is also referred to in the literature by the terms: compressed sensing, compressive sampling, and sketching/heavy-hitters.
Fourier spectra of images are usually quite rapidly decaying with frequency $f$. However high frequency spectral components carry highly important information for image analysis, object detection and recognition that can’t be neglected in spite of the fact that their contribution to signal energy $\int_{-\infty}^{\infty} |a(x)|^2 \, dx = \int_{-\infty}^{\infty} |\alpha(f)|^2 \, dx$ is relatively small. For this reason, sampling interval $\Delta x$ must be taken sufficiently small in order to preserve image essential high frequencies. As a consequence, image representation by samples is frequently very redundant because samples are highly correlated.

This means, that, in principle, much less data would be sufficient for image reconstruction if the reconstruction is done in a more sophisticated way than by means of conventional weighted summation of regularly placed samples according to the sampling theorem.
The compressive sensing approach to signal reconstruction from sparse data

The “compressive sensing” approach also assumes obtaining band-limited, in certain selected transform domain, approximation of images but does not require explicit formulation of the band-limitation.

According to this approach, from available $M < N$ signal samples $\{a_{\tilde{m}}\}$, a signal $\{a_k\}$ of $N$ samples is recovered that provides minimum to $L_1$ norm $\|\alpha\|_{L_1} = \sum_{r=0}^{N-1} |\alpha_r|$ of signal transform coefficients $\{\alpha_r\}$ for the selected transform.

The basis of this approach is the observation, that minimization of signal $L_1$ norm “almost always” in transform domain leads to minimization of $L_0$ – norm $\|\alpha\|_{L_0} = \sum_{r=0}^{K} |\alpha_r|^0$, that is to the minimization of the number of non-zero signal transform coefficients.

The price for the uncertainty regarding band limitation is that the number of required signal samples $M$ must be in this case redundant with respect to the given number $K$ non-zero spectral coefficients: $M/K = O(\log N)$.
Sinc-lets and other discrete signals sharply limited both in signal and DFT/DCT domains
The uncertainty principle:

Continuous signals cannot be both finite and sharply band-limited:

\[ X_{\varepsilon S} \times F_{\varepsilon B} > 1 \]

where \( X_{\varepsilon S} \) is interval in signal domain that contains \( \varepsilon S \) - fraction of its entire energy, \( F_{\varepsilon B} \) is interval in Fourier spectral domain that contains \( \varepsilon B \) - fraction of signal energy and both \( \varepsilon S \) and \( \varepsilon B \) are sufficiently small.
How the uncertainty principle can be translated to discrete signals?

- Discrete signals that represent continuous signals through their samples are always finite as they contain a finite number of samples.

- Discrete signals can be sharply "band-limited" in any transform, including DFT and DCT.

- For some transforms, such as, for instance, Haar transform and Radon Transform, discrete signals that are sharply limited both in signal and transform domain obviously exist.

Do exist discrete signals that are sharply limited both in signal and DFT or/and DCT, domains?
Space-limited & Band-limited discrete signals do exist

They are fixed points of the iterative algorithm:

\[
\hat{A}_{N}^{BL}|_{t} = K_{N \text{SL}}^{SL} T_{N}^{-1} \left( K_{N \text{BL}}^{BL} T_{N} (\hat{A}_{N}^{BL}|_{t-1}) \right)
\]
The discrete uncertainty principle

The (continuous) uncertainty principle

Signal sampling interval

The number of signal non-zero samples

Cardinal sampling relationship

The number of signal samples

The discrete uncertainty relationship
Sinc-lets:

sharply band limited basis functions with sharply limited support

\[ SL_{N}^{BL} \big|_{t=0} = \delta(k - k_0) \]
Cross-correlations of shifted sinc-lets

Signal; Nit=100; N=512; Slim=103

Spectrum; Nit=100; BlimDFT=103

Signal; Nit=100; N=612; Slim=103

Spectrum; Nit=100; BlimDFT=31

Cross-correlations of shifted sinc-lets
2D Sinc-lets

Sinclets and Spectra; ResidStDev=0.00447; N=256; SzLim=128 ;Blim=51
Imaging without optics: “Optics-Less Smart Sensors”
Conventional optical imaging systems use photo-sensitive plane arrays of sub-sensors coupled with focused optics that form a map of the environment onto this image plane. The optics carry out all the information processing needed to form this mapping in parallel and at the speed of light, but comes with some disadvantages.

- Because of the law of diffraction, accurate mapping requires large lens sizes and complex optical systems.
- Lenses limit the field of view and are only available within a limited range of the electromagnetic spectrum.
Conventional optical imaging systems use photo-sensitive planar arrays of detectors coupled with focusing optics that form a map of the environment on the image plane. The optics carry this out at the speed of light, but lenses come with some disadvantages:

• Accurate mapping requires large lens sizes and complex optical systems.
• Lenses limit the field of view and
• Lenses are only available within a limited range of the electromagnetic spectrum.

The ever-decreasing cost of computing makes it possible to make imaging devices smaller and less expensive by replacing optical and mechanical components with computation.
We treat images as data that indicate locations in space and intensities of sources of radiation and show that imaging tasks can be performed by means of optics less imaging devices consisting of set of bare radiation detectors arranged on flat or curved surface and supplemented with signal processing units that use detector outputs to compute optimal statistical estimations of sources’ intensities and coordinates.

We call this class of sensors “Optics-Less Smart” (OLS) sensors.
Optics-less radiation sensors: arrays of radiation detectors with natural cosine-low or alike angular sensitivity arranged on flat or curved surfaces and supplemented with a signal processing unit.

Examples of the physical design and models used in experiments:
An outline of the basic idea:
Locating a single distant radiation source (planar model)

- **First detector** with a cosine-law angular sensitivity
- **Second detector** with a cosine-law angular sensitivity
- **Signal processing unit**

Direction to the distant light source:

1. **Maximum Likelihood estimates:**
   \[ \hat{A} = \sqrt{s_1^2 + s_2^2 + 2s_1s_2 \cos 2\phi} \]
   \[ \tan \hat{\theta} = \frac{s_1 - s_2}{s_1 + s_2} \tan \phi \]

\((A, \theta)\)
Sensor’s operation principle:
generating, using signals from all elemental detectors, optimal statistical estimates of the radiation source intensity and coordinates or directional angles

A Maximum Likelihood model of sensing distant radiation sources

\[ s[n] = \sum_{k=1}^{K} I[k] \text{AngSens} \left( \theta_{SRC}[k] + \theta_{SENS}[n] \right) + v[n] \]

\[ \{ \hat{I}[k], \hat{\theta}_{SRC}[k] \} = \arg \min_{\{I[k], \theta_{SRC}[k]\}} \left\{ \sum_{n=1}^{N} \left( s[n] - \sum_{k=1}^{K} I[k] \text{AngSens} \left( \theta_{SRC}[k] + \theta_{SENS}[n] \right) \right)^2 \right\} \]
Performance evaluation: theoretical lower bounds. Spherical sensor, single distant source

• OLS sensors are essentially nonlinear devices that can't be described in terms of point-spread functions. Their performance can be characterized by the probability distribution function of source parameter estimation errors.

• Statistical theory of parameter estimation shows that, for parameter estimation from data subjected to sufficiently small independent Gaussian additive noise, estimation errors have a normal distribution with mean of zero and standard deviation given by the Cramer-Rao lower bound (CRLB).

Assuming the simplest Lambertian cosine law angular sensitivity function of the detectors

\[ \text{AngSens}(\vartheta) = \begin{cases} \cos \vartheta, & |\vartheta| < \pi/2 \\ 0, & |\vartheta| \geq \pi/2 \end{cases} \]

CRLBs are found, for a single source and for the spherical model, to be

\[ \sqrt{\text{var}\{\hat{\theta}_{\text{SRC}}\}} \geq \frac{2\sigma}{\sqrt{NI}}; \quad \sqrt{\text{var}\{\hat{I}\}} \geq \frac{2\sigma}{\sqrt{N}} \]

where \( \sigma \) is standard deviation of detector’s noise; \( N \) is the number of detectors, \( I \) is the source intensity.
Performance evaluation: theoretical lower bounds. Spherical sensor, two and more sources:

For the case of two sources, CRLBs are found to be

$$\sqrt{\text{var}\{\hat{\theta}_{SRC}[k]\}} = \frac{2\sigma}{I[k]\sqrt{\Delta \theta_{SRC} / \Delta \theta_{SENS}}}; \quad \text{given} \quad \Delta \theta_{SRC} > \Delta \theta_{SENS} ; k = 1,2$$

where $\Delta \theta_{SRS}$ is the angular difference between sources and $\Delta \theta_{SENS} = \pi/N$ is the angle between neighboring detectors.

Numerical results for cases with more than two sources show that regardless of the number of equally-spaced sources, the average estimation error for all the sources is equal to the error predicted for the 2-source problem.

**Resolving power of the sensor:**
If the angular separation between sources is smaller than the angular separation between neighboring detectors ($\Delta \theta_{SENS} > \Delta \theta_{SRS}$), the estimator’s performance rapidly worsens and becomes no better than that of random guessing.
Optics Less sensor basic operation modes

- "General localization" mode: localization and intensity estimation of a given number of radiation sources.

- “Constellation localization” mode: estimation of intensities and locations of “constellations” of radiation sources, which consist of a known number of point sources of known configuration and relative distribution of intensities.

- "Imaging" mode: estimation of intensities of a given number of radiation sources in the given locations, for instance, on a regular grid.
Computer model:

Spherical and planar models of optics less radiations sensors were tested in the localization and imaging modes by numerical simulation using, for generating Maximum Likelihood estimates of sources’ intensities and locations, the multi-start global optimization method with pseudo-random initial guesses and Matlab’s quasi-Newton method for finding local optima.

In order to improve reliability of global maximum location and accelerate the search, input data were subjected to decorrelation preprocessing by means of the “whitening” algorithm that proved to be optimal preprocessing algorithm for point target location in clutter. It is interesting to note that a similar data decorrelation is known in vision science as “lateral inhibition”
Experiment: Spherical sensor in the imaging mode

- Pattern of 19x16 sources
- Pattern of detectors’ outputs
  (spherical array of 16x20 = 320 detectors;
   detector’s noise
   standard deviation 0.01)
- Reconstructed image
  Standard deviation of estimation errors 0.064
Planar sensor in the localization mode

Spread of “hits” (color dots) of estimation of a single radiation source locations in different positions (marked by blue circles) with respect to the sensors, consisting of 11 detectors (yellow boxes)
Planar sensor in the localization mode: estimation errors of position and intensity of a single radiation source placed in different positions in front of the sensor.

Maps of standard deviations of estimation errors of X-Y coordinates (a, b) and of intensity (c) of a radiation source as a function of the source position with respect to the surface of the line array of 25 detectors with detector noise StDev=0.01. Darker areas correspond to larger errors. Plot d) shows standard deviations of X, Y and intensity estimation errors as function of the distance from the sensor along the sensor ‘optical axis’ (central sections of Figs. a)-c))
Planar sensor in the imaging mode

<table>
<thead>
<tr>
<th>Distance</th>
<th>Detector readings</th>
<th>Estimated source intensities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z=1</td>
<td>![Image]</td>
<td>![Image] Error standard deviation 5.6850e-04</td>
</tr>
<tr>
<td>Z=2</td>
<td>![Image]</td>
<td>![Image] Error standard deviation 0.0105</td>
</tr>
<tr>
<td>Z=4</td>
<td>![Image]</td>
<td>![Image] Error standard deviation 0.0590</td>
</tr>
<tr>
<td>Z=8</td>
<td>![Image]</td>
<td>![Image] Error standard deviation 0.1197</td>
</tr>
</tbody>
</table>

Sensing of 8x16 radiation sources arranged on a plane in form of characters “SV” by a 3-D model of a flat OLS sensor of 8x16 elementary detectors in the “imaging” mode for distances of sources from the sensor Z=1 to 8 (in units of inter-detector distance). SNR was kept constant at 100 by making the source amplitude proportional to the distance between the source plane and sensor plane. Detector noise StDev=0.01.
Sensors on convex surfaces in the localization mode

Sensors on bent convex surfaces (1D model, 11 detectors, noise standard deviation 0.01): map of standard deviations of estimation errors of source intensity (left column), that of source direction angle (central column) and that of the distance to the source (right column) as functions of the source position with respect to the sensor’s surface. Darker areas correspond to larger errors.
Sensors on concave surfaces in the localization mode
Optics-less “smart” sensors: advantages and limitations

Advantages
• No optics are needed, making this type of sensor applicable to virtually any type of radiation and to any wavelength
• The angle of view (of spherical sensors) is unlimited
• The resolving power is determined ultimately by the sub-sensor size, and not by diffraction-related limits
• Sensors without optics can be made more compact and robust than traditional optical sensors

Limitations:
High computational complexity, especially when good imaging properties for multiple sources are required.

However, the inexorable march of Moore’s law makes such problems more feasible each year. Furthermore, the computations lend themselves to high-concurrency computation, so the computational aspects are not expected to hinder usage of OLS sensors.
A little imagination: a flying sighted brain
COMPUTATIONAL IMAGING AND EVOLUTION OF VISION IN THE NATURE
Optics-less extra ocular cutaneous (skin) vision in Nature

- Heliotropism of some plants
- Eye spots (patches of photosensitive cells on the skin), cup eyes, and pit eyes
- Cutaneous photoreception in reptiles
- Infra-red radiation sensitive “pit organs” of vipers
- The pressure sensitive “lateral line system” of fish, which they use to localize sources of vibration located within approximately one body length
- Electric field sensitive receptors in sharks and in some types of fish, which allow animals to sense electrical field variations in their surroundings within approximately one body length

There are also a number of reports on the phenomenon of primitive cutaneous vision in humans.
Presented simulation results

• Show that reasonably good directional vision without optics is possible even using the simplest possible detectors whose angular sensitivity is defined only by the surface absorptivity.

• Are in a good correlation with published observations in studies of cutaneous vision

• Allow suggesting that the operational principle and capabilities of OLS sensors can be used to model operational principles and capabilities of cutaneous vision and its neural circuitry.

• Motivate advancing a hypothesis that evolution of vision started from formation, around primordial light sensitive cells, of neural circuitry for implementing imaging algorithms similar to those in our model of the flat OLS sensor, including, at one of the first step, the lateral inhibition.
The reported OLS sensor models naturally suggests also, that flat primordial eyespots may have evolved, through bending of the sensor’s surface to convex or concave spherical forms, to the compound facet eye or camera-like vision, correspondingly.
TWO BRANCHES OF EVOLUTION OF VISION

Subcutaneous neural net

Transparent protective medium with refracting index >1

Subcutaneous neural net

Central nervous system

BRAIN

Central nervous system

Compound apposition or superposition eyes of insects

Camera like eye of vertebrates

Cup eyes of mollusks
In both cases, the evolution of eye optics had to be paralleled by the evolution of eye neural circuitry as an inseparable part of animal brains.

As it follows from the theory, detection and localization of targets does not necessarily require formation of sharp images and can be carried out directly on not sharply focused images. Image sharpness affects the reliability of detection and becomes important only for low signal-to-noise ratio at detectors.

Therefore, gradual improvements of eye optics in course of evolution of eye optics may have translated into improved target detection reliability and allowed transferring, in course of evolution, the higher and higher fraction of eye neural circuitry and brain resources from image formation to image understanding. In a certain sense one can suggest that animals’ brain is a result of evolution of vision.
References


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