L. Yaroslavsky. Fundamentals of Digital Image Processing. Course 0555.3230

Lecture 10 Principles of image restoration

10.1 Mathematical models of imaging systems

Imaging systems always have certain technical limitations in their design and implementations and generate images that are not as perfect as they would be if there were no implementation limitations. Deviations of real images from perfect ones may be treated as distortions introduced by imaging systems to hypothetical perfect, or “ ideal” signals. Correction of these distortions is the primary goal of image processing.

Methods for distortion correction are based on the canonical model of imaging systems shown in Fig. 10.1.

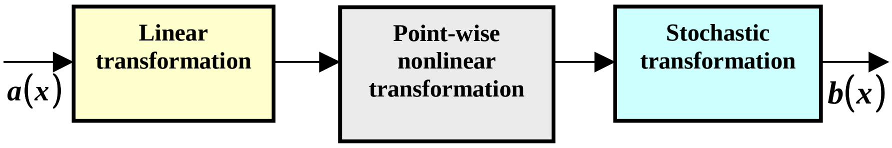

Fig. 10.1 A canonical model of imaging systems

The model represents image formation as a combination of signal linear transformations, point-wise nonlinear transformations and stochastic transformations that are applied to a hypothetical perfect signal $a(x)$ and jointly determine the system’s output signal $b(x)$.

Linear transformation are specified in terms of the system point spread function or frequency response. Frequency response of the ideal imaging system is assumed to be uniform for all frequencies in a certain base band. Frequency responses of real imaging usually more or less rapidly decay on high frequencies. This results in image distortions such as image blur.

Point-wise nonlinear are specified in terms of the system transfer functions. It is assumed that the ideal system has a linear transfer function. Deviations of system transfer functions form linear one cause distortions of gray scale nonlinear distortions.

Stochastic transformations model signal random distortions and cause random interferences, or noise in output images. Stochastic transformations are specified by statistical noise models discussed in Lect. 9.

The processing goal is estimating the perfect signal $a(x)$ given distorted signal $b(x)$ produced by the system. This problem is frequently referred to as the ***inverse problem***. Signal transformations that are applied to system’s output signals $\{b(x)\}$ to produce an estimate $\hat{a}(x)$ of the “ideal” signal are called ***signal recovery*** or, in application to image processing, ***image restoration***. Similar problem is ***image reconstruction***. This term is usually refers to image formation in transform domain imaging such as tomography and holography.

If the system does not introduce any random distortions and system’s parameters such as point spread function and transfer function are known, the inverse problem has a trivial solution: signal $b(x)$ has to be subjected to transformations inverse to those introduced by the system. However, one can not, in general, neglect random or similar uncontrolled distortions such as round-off errors in digital processing. In reality, applying inverse transformation may result in artifacts that may even be useless. To solving the inverse problem in such cases, the statistical approach that explicitly accounts for signal random distortions appears to be the most appropriate.

Generally, signal recovery, image restoration and reconstruction is indivisible procedure that should account for and correct all distortions. However, in practice this process is divided into separate steps carried out in the order inverse to that distortion factors have in the model.

10.2 Transform domain MSE optimal scalar Wiener Filters

In this section we consider image restoration for a reduced imaging system model that disregards point-wise nonlinear transformation in imaging systems and treats image distortions as a combination of distortions caused by linear filtering and of those caused by action of random interferences.

The design of the optimal restoration procedure requires specifying a criterion for evaluating the restoration quality. Let $\{b_k\}$ be a set of N signal samples ($k = 0,1,..., N-1$) at the output of the imaging system, $\{a_k\}$ be a set of the system’s input signal samples that model perfect, or “ideal” signal and $\{\hat{a}_k\}$ be a set of restored signal samples. For the sake of generality, we will consider the set $\{a_k\}$ as a realization taken from a statistical ensemble $\mathbf{\Omega_A}$ or data base of input signals and the set $\{b_k\}$ as a realization of a signal ensemble generated by ensembles $\mathbf{\Omega_A}$ and $\mathbf{\Omega_N}$ of input signals and of random interferences.

Define the restoration procedure performance measure as a squared difference between restored and perfect signals averaged over the available set of signal samples and over statistical ensembles $\mathbf{\Omega_A}$ and $\mathbf{\Omega_N}$. We will call this measure ***mean squared restoration error*** (MSE). The restoration procedure $\mathbf{R}\{b_k\} = \{\hat{a}_k\}$ that minimizes this difference:

$$\{\hat{a}_k\} = \mathbf{\arg\min}_{\mathbf{R}\{b_k\}=\{\hat{a}_k\}} \left\{ \mathbf{AV}_{\mathbf{\Omega_A}} \mathbf{AV}_{\mathbf{\Omega_N}} \left(\sum_{k=0}^{N-1} \left| a_k - \hat{a}_k \right|^2 \right) \right\}. \qquad (10.2.1)$$

will be referred to as to ***MSE-optimal filtering***.

For the implementation of the MSE-optimal filtering, we will restrict ourselves to linear filtering. In general, linear filtering of a discrete signal may be described as multiplication of a vector of input signal samples $\mathbf{B} = \{b_k\}$ by a filter matrix $\mathbf{H}$:

$$\hat{\mathbf{A}} = \mathbf{H} \cdot \mathbf{B}, \qquad (10.2.2)$$

where $\hat{\mathbf{A}} = \{\hat{a}_k\}$ is a vector of filter output signal samples.

For signals of N samples, a general vector filter matrix $\mathbf{H}$ has dimensions $N \times N$. Specification of such a filter requires determining N^2 filter coefficients, and the filtering itself requires performing N^2 operations per N signal samples. In image processing, the computational complexity of both determination of filter coefficients and of the filtering may be become too high because of high dimensionality of image arrays. Fast transforms that may be computed for $O(N \log N)$ operations allow to radically decrease the filter design and the implementation complexity. Therefore in what follows we will consider only scalar filtering in a domain of orthogonal transforms that can be computed with fast algorithms. This class of filters may be described by the equation:

$$\hat{\mathbf{A}} = \mathbf{T}^{-1} \cdot \mathbf{H_d} \cdot \mathbf{T} \cdot \mathbf{B}, \qquad (10.2.3)$$

where, $\mathbf{T}$ and $\mathbf{T}^{-1}$ are, correspondingly, direct and inverse orthogonal transforms, $\mathbf{H} = \mathbf{diag}\{\eta_r\}$, ($r = 0,1,\ldots,N-1$) is a diagonal filter matrix. Such a scalar filtering implies the following relationship between filter output and input signal samples $\{\hat{\alpha}_r\} = \mathbf{T} \cdot \hat{\mathbf{A}}$ and $\{\beta_r\} = \mathbf{T} \cdot \mathbf{B}$:

$$\hat{\alpha}_r = \eta_r \beta_r . \qquad (10.2.4)$$

In the assumption of orthogonality of the transform $\mathbf{T}$, one can, by virtue of the Parceval relationship (Eq. 2.1.28), modify the filter optimality condition defined by Eq. 10.2.1 in the following way:

$$\{\hat{\alpha}_r\} = \mathbf{\arg\min}_{\{\eta_r\}} \left\{ \mathbf{AV}_{\mathbf{\Omega_A}} \mathbf{AV}_{\mathbf{\Omega_N}} \left(\sum_{r=0}^{N-1} \left| \alpha_r - \hat{\alpha}_r \right|^2 \right) \right\} = \mathbf{\arg\min}_{\{\eta_r\}} \left\{ \mathbf{AV}_{\mathbf{\Omega_A}} \mathbf{AV}_{\mathbf{\Omega_N}} \left(\sum_{r=0}^{N-1} \left| \alpha_r - \eta_r \beta_r \right|^2 \right) \right\}. \qquad (10.2.5)$$

By computing derivatives over sought variables and equaling them to zero, one can obtain from Eq. 10.2.5 that optimal scalar filter coefficients may be found as cross-correlation coefficients between spectral coefficients β_r and α_r of the input and perfect signals:

$$\eta_r = \frac{\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\alpha_r \beta_r^*\right)}{\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\left|\beta_r\right|^2\right)}. \qquad (10.2.6)$$

We will refer to MSE optimal linear filters and, in particular, to MSE optimal scalar filters defined by Eq. 10.2.6 as to ***Wiener filters***. This name gives a credit to Norbert Wiener for his pioneer works in the theory of statistical methods of signal restoration [1].

In order to implement optimal scalar Wiener filter, one should therefore know cross-correlation $\left\{\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\alpha_r \beta_r^*\right)\right\}$ between filter input signal and perfect signal spectral coefficients and power spectrum $\left\{\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\left|\beta_r\right|^2\right)\right\}$ of the input signal in the selected basis. The statistical approach we adopted that assumes averaging of the restoration error over statistical ensembles of perfect signals and of filter input signals implies that these statistical parameters should be measured in advance for these ensembles or over the data bases.

8.2.1 Empirical Wiener filters for image denoising

Consider an additive signal independent noise model in which filter input signal samples $\{b_k\}$ are obtained as a sum of perfect signal samples $\{a_k\}$ and samples $\{n_k\}$ of signal independent zero mean random noise:

$$b_k = a_k + n_k . \qquad (10.2.7)$$

In spectral domain, the same relationship holds for signal and noise spectral coefficients:

$$\beta_r = \alpha_r + \nu_r , \qquad (10.2.8)$$

where $\{\nu_r\} = \mathbf{T}\{n_k\}$. For this model one can obtain that

$$\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\alpha_r \beta_r^*\right) = \mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left[\alpha_r\left(\alpha_r^* + \nu_r^*\right)\right] = \mathbf{AV}_{\Omega_A}\left(\left|\alpha_r\right|^2\right) \qquad (10.2.9)$$

and

$$\mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left(\left|\beta_r\right|^2\right) = \mathbf{AV}_{\Omega_A}\mathbf{AV}_{\Omega_N}\left[\left(\alpha_r + \nu_r\right)\left(\alpha_r^* + \nu_r^*\right)\right] = \\ \mathbf{AV}_{\Omega_A}\left(\left|\alpha_r\right|^2\right) + \mathbf{AV}_{\Omega_A}\left(\left|\nu_r\right|^2\right) \qquad (10.2.10)$$

because for zero mean noise $\mathbf{AV}_{\Omega_N}\left(\nu_r^*\right) = \mathbf{AV}_{\Omega_N}\left(\nu_r\right) = \mathbf{0}$. Therefore scalar Wiener filter for suppressing additive signal independent noise is defined through its coefficients $\{\eta_r\}$ as

$$\eta_r = \frac{\mathbf{AV}_{\Omega_A}\left(\left|\alpha_r\right|^2\right)}{\mathbf{AV}_{\Omega_A}\left(\left|\alpha_r\right|^2\right) + \mathbf{AV}_{\Omega_N}\left(\left|\nu_r\right|^2\right)}. \qquad (10.2.11)$$

One can give to this formula a clear physical interpretation. Define ***signal-to-noise ratio*** as:

$$SNR_r = \frac{AV_{\Omega_A}\left(\left|\alpha_r\right|^2\right)}{AV_{\Omega_N}\left(\left|\nu_r\right|^2\right)}. \qquad (10.2.12)$$

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[1] It will be just to give also a credit to Andrey N. Kolmogorov who developed the similar theory for discrete signals.

Then obtain:

$$\eta_r = \frac{\boldsymbol{SNR_r}}{\mathbf{1} + \boldsymbol{SNR_r}} \tag{10.2.13}$$

which means that scalar Wiener filter weight coefficients are defined, for each signal spectral coefficient, by the signal-to-noise ratio for this coefficient. The lower is signal-to-noise ratio for a particular signal spectral component, the lower will be the contribution of this component to the filter output signal.

In order to implement scalar Wiener filter one have to measure in advance power spectra $\mathbf{AV}_{\mathbf{\Omega_A}}\left(|\alpha_r|^2\right)$ and $\mathbf{AV}_{\mathbf{\Omega_N}}\left(|\nu_r|^2\right)$ of perfect signals and of noise in the selected basis. Noise power spectrum may be known from the specification certificate of the imaging device. Otherwise it may be measured in noisy input signals using methods described in Sect 7.1.2. As for the prefect signal power spectrum, it is most frequently not known. However, one can, using Eq. 10.2.10, attempt to estimate it from the power spectrum $\mathbf{AV}_{\mathbf{\Omega_A}}\mathbf{AV}_{\mathbf{\Omega_N}}\left(|\beta_r|^2\right)$ of input noisy signals as

$$\mathbf{AV}_{\mathbf{\Omega_A}}\left(|\alpha_r|^2\right) = \mathbf{AV}_{\mathbf{\Omega_A}}\mathbf{AV}_{\mathbf{\Omega_N}}\left(|\beta_r|^2\right) - \mathbf{AV}_{\mathbf{\Omega_A}}\left(|\nu_r|^2\right). \tag{10.2.14}$$

The latter has to be estimated from the observed signal spectrum $|\beta_r|^2$. Denote this estimate as $\overline{|\beta_r|^2}$. This empirical estimate made by averaging over available realization of input images may, when used as a replacement for $\mathbf{AV}_{\mathbf{\Omega_A}}\mathbf{AV}_{\mathbf{\Omega_N}}\left(|\beta_r|^2\right)$, give negative values for some spectral coefficients because of the limited depth of the averaging. Since power spectra can not assume negative values, the following modified spectrum estimation may be adopted:

$$\mathbf{AV}_{\mathbf{\Omega_A}}\left(|\alpha_r|^2\right) = \mathbf{max}\left[\overline{|\beta_r|^2} - \mathbf{AV}_{\mathbf{\Omega_A}}\left(|\nu_r|^2\right);\quad \mathbf{0}\right]. \tag{10.2.15}$$

In this way we arrive at the filter:

$$\eta_r = \mathbf{max}\left[\frac{\overline{|\beta_r|^2} - \mathbf{AV}_{\mathbf{\Omega_A}}\left(|\nu_r|^2\right)}{\overline{|\beta_r|^2}};\quad \mathbf{0}\right]. \tag{10.2.16}$$

We will refer to this filter as to the ***empirical Wiener filter***.

If the imaging system noise is known to be white noise with variance σ_n^2, the empirical Wiener filter takes the form:

$$\eta_r = \mathbf{max}\left[\frac{\overline{|\beta_r|^2} - \sigma_n^2}{\overline{|\beta_r|^2}};\quad \mathbf{0}\right]. \tag{10.2.17}$$

As a zero order approximation to the input images power spectrum, power spectrum $|\beta_r|^2$ of a single input image subjected to filtering may be used. In this case empirical Wiener filter weight coefficients are found as:

$$\eta_r = \mathbf{max}\left[\frac{|\beta_r|^2 - \sigma_n^2}{|\beta_r|^2};\quad \mathbf{0}\right]. \tag{10.2.18}$$

Note that such an empirical Wiener filter is adaptive because its weight coefficients depend on the spectrum of the image to which it will be applied.

Weight coefficients of scalar Wiener filters assume values in the range between zero and one. A version of the empirical Wiener filter of Eq. 10.2.18 with binary weight coefficients:

$$\eta_r = \begin{cases} 1, \; if \; \left|\beta_r\right|^2 \geq Thr \\ 0, \quad otherwise \end{cases} \tag{10.2.19}$$

where Thr is a rejecting threshold is called the ***rejecting filter***. As it follows from Eq. 10.2.18, the rejecting threshold has a value of the order of magnitude of the noise variance σ_n^2. Rejecting filters eliminate from the input images spectra all components for which signal-to-noise ratio is lower then a certain threshold.

A version of the empirical Wiener filter and of the rejecting filters that are implemented with wavelet transform image decomposition is known as ***wavelet shrinkage*** filtering. Empirical Wiener filtering according to Eq. 10.2.18 an a wavelet jargon is called "***soft thresholding"***. Rejecting filtering according to Eq. 10.2.19 is called "***hard thresholding***". Fig. 10.2 shows flow diagram of signal denoising by the wavelet shrinkage

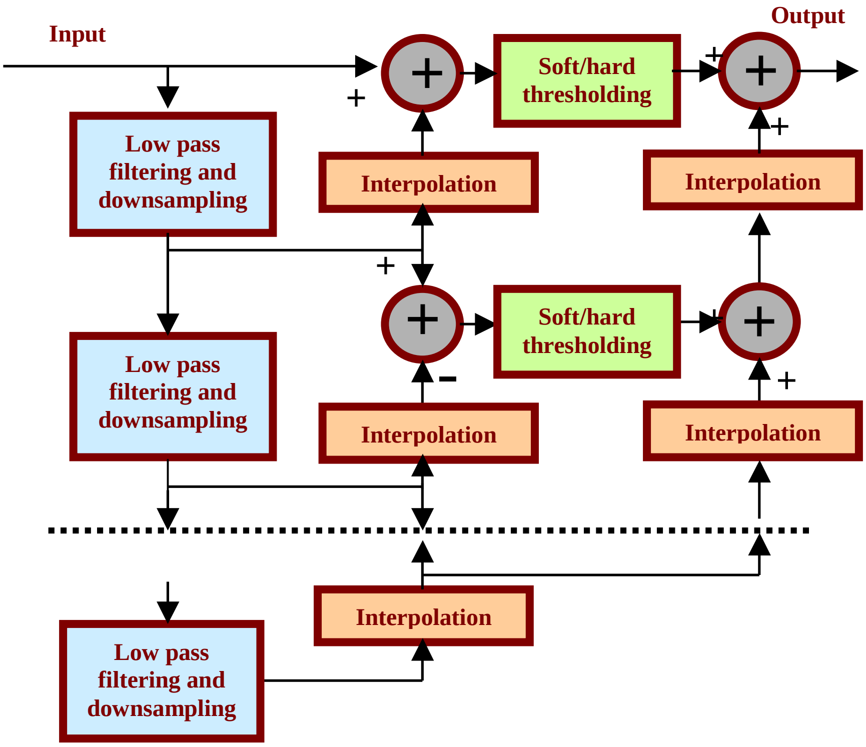


Fig. 10.2. Wavelet shrinkage: signal denoising in wavelet transform domain

Described empirical Wiener filters for signal denoising are particularly very efficient if signal and/or noise spectra are well concentrated and are separated in the transform domain. A typical example of such a situation is filtering of narrow band noise, whose spectrum has only a few components in the transform domain. Figs. 10.3 through 10.5 illustrate examples of such a narrow band noise filtering. Fig. 10.3 demonstrates filtering periodical noise pattern in an image. Such interferences frequently appear in images digitized by frame grabbers from analog video. Left column in Fig. 10.3 shows input and filtered images. Right column shows averaged spectra of input and output image rows. One can clearly see anomalous peaks of noise spectrum in input image spectrum that are eliminated in the output image spectrum after applying empirical Wiener filtering. Note that the filtering is carried out in this example as 1-D row-wise filtering in DFT domain. This type of interferences can also be successfully filtered in Walsh transform domain .

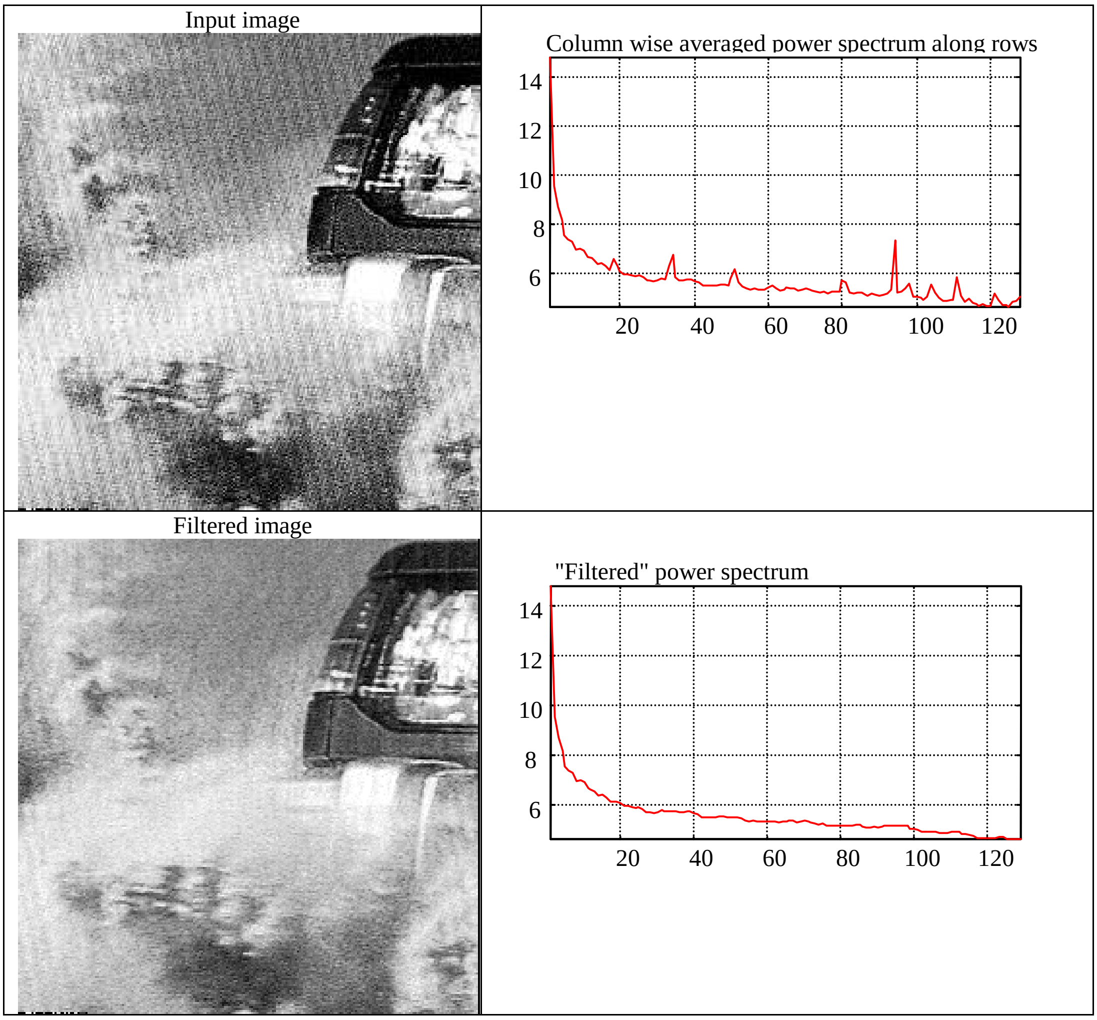
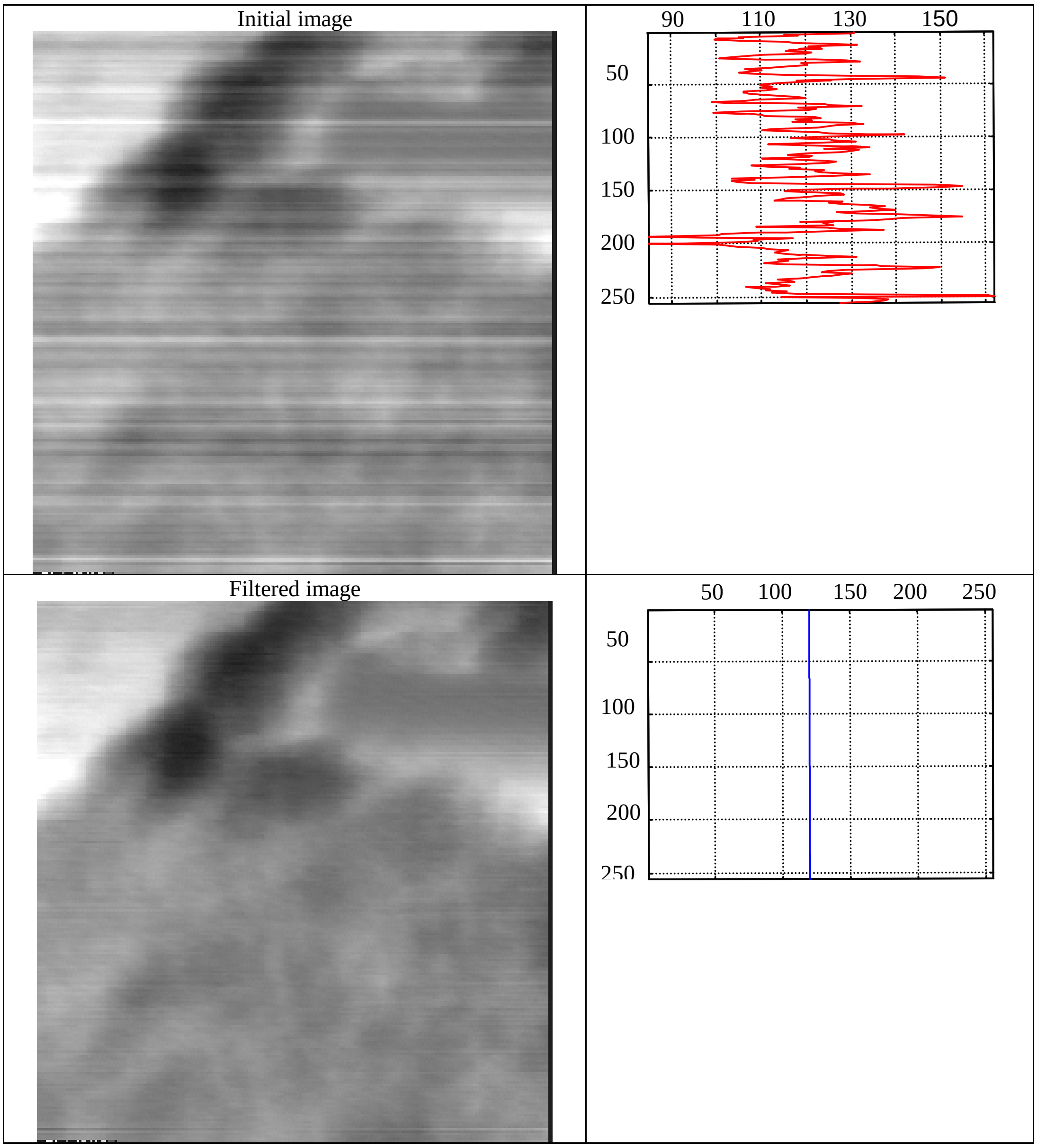


Fig. 10.3 Filtering periodic interferences

Fig. 10.4 shows yet another example of empirical Wiener filtering narrow-band interferences. Banding noise in the initial image shown in Fig. 10.4 is characteristic for imaging systems with mechanical scanning. This particular image was produced by an atomic force microscope. Banding noise randomly changes image dc component (row-wise mean value) in the direction of scanning. Upper right plot in Fig. 10.4 shows row-wise mean values, or row-wise image Radon transform, (horizontal coordinate) as a function of the row number (vertical coordinate). Taking as an estimate of perfect mean row-wise values a mean value over all rows (shown with a straight line in the bottom right plot in Fig. 10.4), one can estimate banding interference on every row by subtracting this estimate from the observed mean values. Subtracting the found values from all pixels in the corresponding row eliminates the noise.

Wiener filtering of wide band noise and especially white noise is less efficient. When filtering of white noise, Wiener filter tends to weaken low energy signal spectral components. However usually these components are exactly the components that are the most important because they carry information about signal changes such as at edges in images. Moreover, Wiener filtering converts input white noise into output correlated noise though with a reduced variance. As one can see from Eq. 10.2.11, in case of the intensive input white noise, power spectrum of the residual noise is roughly proportional to the signal power spectrum which means that the residual noise becomes, statistically, signal alike. This may hamper subsequent image analysis. In particular, it is well known that human vision is more sensitive to correlated noise that to white noise of the same intensity. Therefore Wiener filtering for image denoising may even worsen images.



Initial image

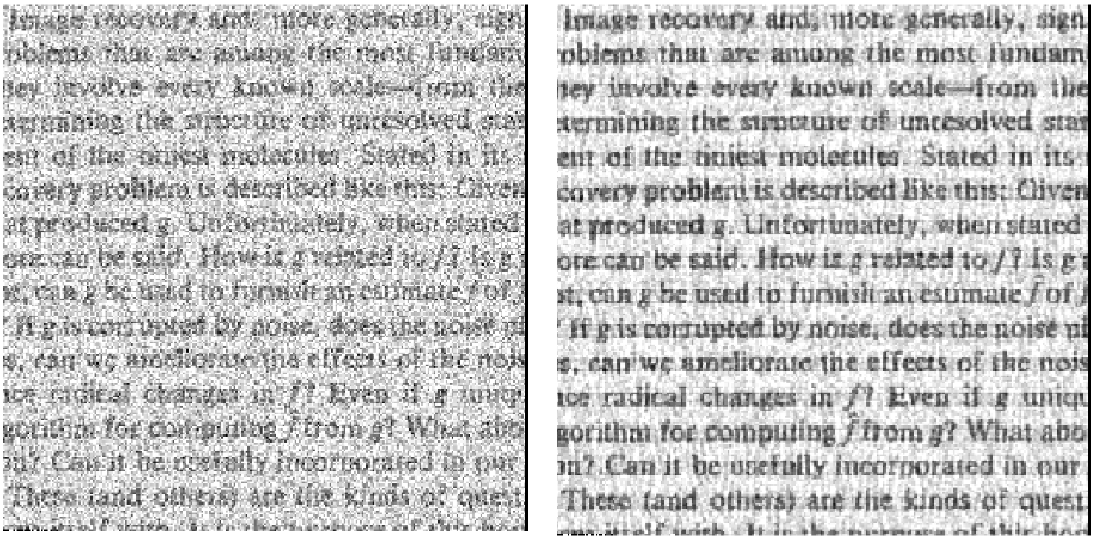
Filtered image

Fig. 10.4. Filtering banding noise

Fig. 10.5 gives an example of computer simulation of Wiener filtering of a test image with additive white noise. In the simulation, one can implement the ideal Wiener filter because both signal and noise spectra are known. The empirical Wiener filter was implemented in this example according to Eq. 10.2.18. As one can see from the figure, ideal Wiener filtering does improve image quality. For the empirical Wiener filter, improvement is much less appreciable. The difference image between initial noisy image and the result of the empirical Wiener filtering (restoration error) shows that the filtering, along with noise suppression, destroys image edges.

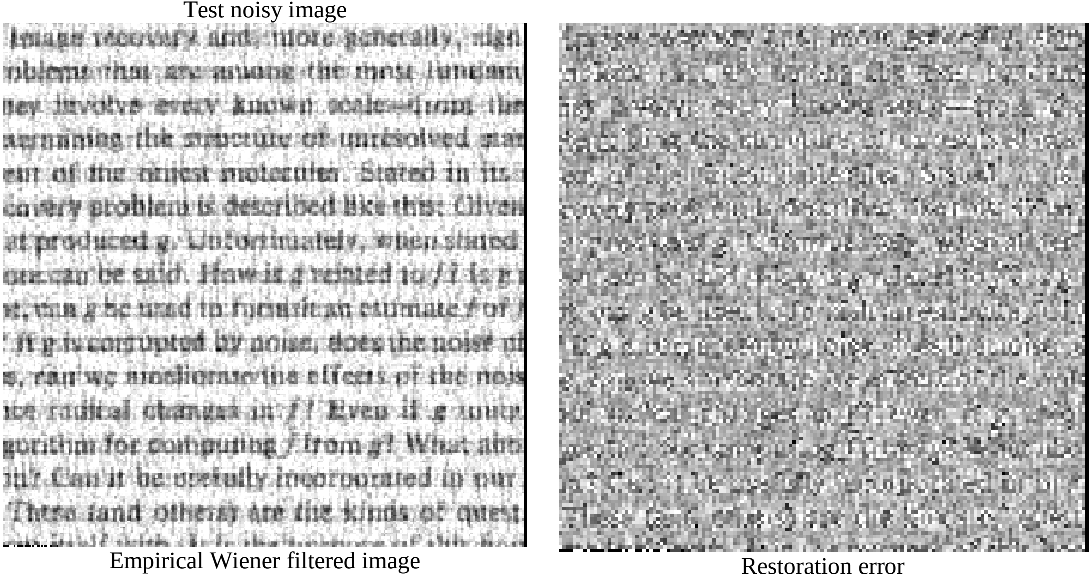
8.2.2 Image deblurring, inverse filters and aperture correction

Consider now an imaging system model that accounts also for signal linear transformations in imaging systems. Suppose that the linear transformation unit can be modeled as a scalar filter with filter coefficients $\{\lambda_r\}$ in a selected basis. For DFT basis, this assumption is just, to the accuracy of boundary effects, for shift invariant linear filtering (see Ch. 5). In this case coefficients $\{\lambda_r\}$ are samples of imaging system frequency response. In ideal imaging systems, they should all be equal to unity. In reality, they decay with the frequency index r, which results, in particular, in image blur. Processing aimed at correcting this type of distortions is frequently referred to as ***image deblurring***.



Ideal Wiener filtered image

Test noisy image



Empirical Wiener filtered image

Restoration error

Fig. 10.5 Ideal Wiener and Empirical Wiener filtering for image denoising

For such systems, we have in the transform domain:

$$\beta_r = \lambda_r \alpha_r + \nu_r \, , \tag{10.2.20}$$

Using Eqs. 10.2.6, one can obtain that the scalar Wiener image restoration filter is defined in this case by the equation:

$$\eta_r = \frac{1}{\lambda_r} \frac{SNR_r}{1 + SNR_r} \, , \tag{10.2.21}$$

where SNR is signal-to-noise ratio at the output of the linear filter unit of the imaging system model:

$$SNR_r = \frac{\left|\lambda_r\right|^2 \mathbf{AV}_{\Omega_A}\left(\left|\alpha_r\right|^2\right)}{\mathbf{AV}_{\Omega_N}\left(\left|\nu_r\right|^2\right)} \, . \tag{10.2.22}$$

Correspondingly, the general empirical Wiener filter, empirical Wiener filter with zero order approximation to perfect signal spectrum and the rejecting filter for aperture correcting and image deblurring will be in this case as follows:

$$\eta_r = \max\left[\frac{1}{\lambda_r}\frac{\overline{|\beta_r|^2} - \mathrm{AV}_{\Omega_A}\left(|\nu_r|^2\right)}{\overline{|\beta_r|^2}};\ \ 0\right]; \tag{10.2.23}$$

$$\eta_r = \max\left[\frac{1}{\lambda_r}\frac{\overline{|\beta_r|^2} - \sigma_n^2}{\overline{|\beta_r|^2}};\ \ 0\right] \tag{10.2.24}$$

and

$$\eta_r = \begin{cases} \dfrac{1}{\lambda_r}, if\ |\beta_r|^2 \ge Thr \\ 0, \qquad otherwise \end{cases}. \tag{10.2.25}$$

All these filters may be treated as two filters in cascade: the filter with coefficients

$$\eta_r^{inv} = \frac{1}{\lambda_r} \tag{10.2.26}$$

usually called the ***inverse filter*** and signal denoising filters described by Eqs. 10.2.17 - 19. Inverse filters compensate weakening signal frequency components in the imaging system while denoising filters prevent from excessive amplification of noise and perform what is called "***regularization***" of inverse filters. As one can see from Eq. 10.2.22, weight coefficients of denoising filters for small $\{\lambda_r\}$ fall faster than weight coefficients of the inverse filter grow.

One of the most immediate applications of inverse filters is correcting distortions caused by finite size of apertures of image sensors, image discretization devices and image displays. We will refer to this processing as to ***aperture correction***.

Let an image sensor and discretization device is an array of light sensitive elements with a square aperture of size $d^{(d)} \times d^{(d)}$ (Fig. 8-7). Then frequency response of the individual sensor elements is

$$H(f_x, f_y) = \int_{-d^{(d)}/2}^{d^{(d)}/2} \exp(i2\pi f_x x)dx \int_{-d^{(d)}/2}^{d^{(d)}/2} \exp(i2\pi f_y y)dx =$$

$$\frac{\sin\left(\pi f_x d^{(d)}/2\right)}{\pi f_x d^{(d)}/2}\frac{\sin\left(\pi f_y d^{(d)}\right)}{\pi f_y d^{(d)}} = \mathrm{sinc}\left(\pi f_x d^{(d)}\right)\mathrm{sinc}\left(\pi f_x d^{(d)}\right), \tag{10.2.27}$$

or, in dimensionless coordinates $\{\bar{f}_x = f_x\Delta x, \bar{f}_y = f_y\Delta x\}$, where Δx is the discretization interval in both coordinates,

$$H(\bar{f}_x, \bar{f}_y) = \mathrm{sinc}\left(\pi\bar{f}_x d^{(d)}/\Delta x\right)\mathrm{sinc}\left(\pi\bar{f}_x d^{(d)}/\Delta x\right). \tag{10.2.28}$$

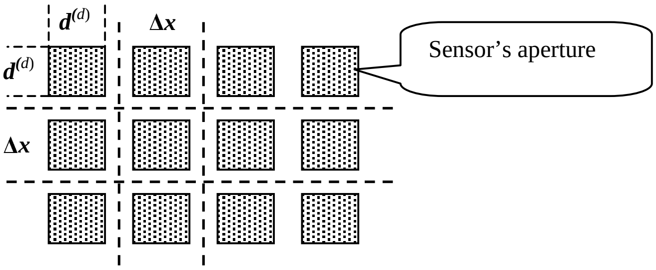


Fig.10.6. Arrangement of light sensitive elements in image sensor arrays

The discrete frequency response is then:

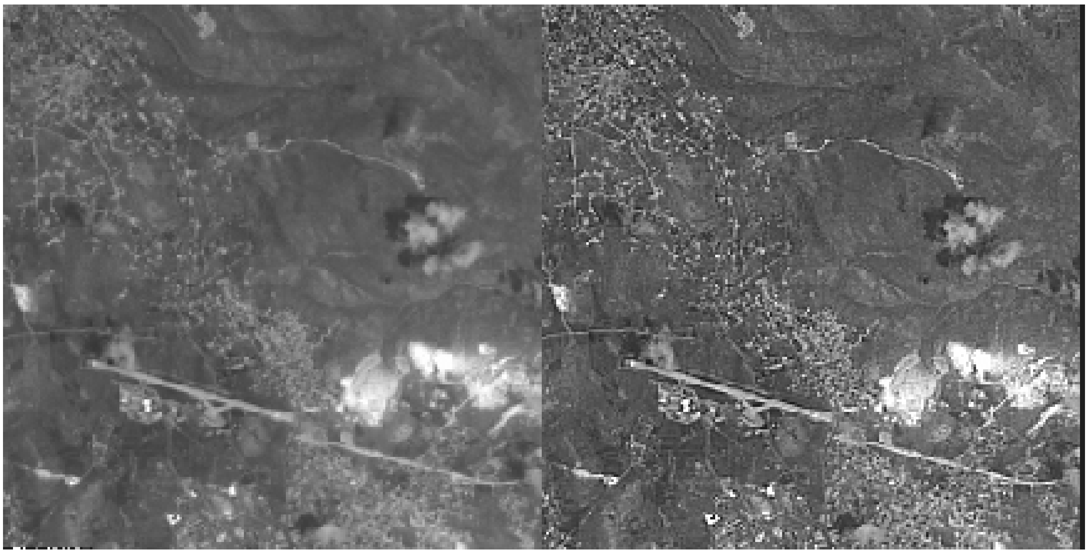
$$\lambda_{r,s} = \lambda_r \lambda_s = \mathbf{sinc}\left(\pi \bar{d}^{(d)} \lambda_r / N_x\right) \cdot \mathbf{sinc}\left(\pi \bar{d}^{(d)} \lambda_s / N_y\right), \qquad (10.2.29)$$

where $\bar{d}^{(d)} = d^{(d)} / \Delta x$.

If the image display device has also a square aperture of size $d^{(r)} \times d^{(r)}$, the overall imaging discrete system frequency response is:

$$\lambda_{r,s} = \lambda_r \lambda_s = \mathbf{sinc}\left(\pi \bar{d}^{(d)} \lambda_r / N_x\right) \cdot \mathbf{sinc}\left(\pi \bar{d}^{(r)} \lambda_r / N_x\right) \times \mathbf{sinc}\left(\pi \bar{d}^{(d)} \lambda_s / N_y\right) \mathbf{sinc}\left(\pi \bar{d}^{(r)} \lambda_s / N_y\right). \qquad (10.2.30)$$

Parameters $d^{(d)}$, $d^{(r)}$ and Δx are imaging system design parameters that may be know system's certificate. They can be used for correcting image distortions by processing images in computer.
Fig. 8-8 illustrates an example of the aperture correction of an air photograph. Right image in this figure is obtained by applying to the left image an inverse filter for the system's frequency response defined by Eq. 10.2.20 with $\bar{d}^{(d)} = \bar{d}^{(r)} = 1$.



Initial and aperture corrected images

Fig. 10.7. Aperture correction: initial (left) and aperture corrected (right) images

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Lecture 10 Principles of image restoration

Summary

Canonical model of imaging systems consists of a linear filter and point-wise nonlinearity in cascade and a generator of random interferences.
Image restoration assumes a virtual ideal imaging system and is aimed at converting images generated by a real imaging system into an image that is as close, in terms of a certain quality criterion, to the virtual “ideal image” as possible.

Transform domain scalar filtering General linear filtering of a digital signal can be described as a matrix multiplication of a signal vector by a filter matrix. Computational complexity of the general filtering per output pixel is $O(N)$, where N is the size of the image array. Scalar filtering is implemented as point-wise modification of signal spectral coefficients in a basis of a certain orthogonal transform. Provided the transform with fast transform algorithm is used, computational complexity of the scalar filtering is $O(\log N)$ per output pixel or lower.

MSE optimal Wiener filtering:

$$\{\hat{a}_k\} = \operatorname*{arg\,min}_{\mathrm{R}\{b_k\}=\{\hat{a}_k\}} \left\{ \mathbf{AV}_{\Omega_A} \mathbf{AV}_{\Omega_N} \left(\sum_{k=0}^{N-1} |a_k - \hat{a}_k|^2 \right) \right\}.$$

MSE optimal scalar Wiener filtering:

$$\{\bar{\alpha}_r\} = \operatorname*{arg\,min}_{\{\eta_r\}} \left\{ \mathbf{AV}_{\Omega_A} \mathbf{AV}_{\Omega_N} \left(\sum_{r=0}^{N-1} |\alpha_r - \eta_r \beta_r|^2 \right) \right\}.$$

$$\eta_r = \frac{\mathbf{AV}_{\Omega_A} \mathbf{AV}_{\Omega_N} \left(\alpha_r \beta_r^* \right)}{\mathbf{AV}_{\Omega_A} \mathbf{AV}_{\Omega_N} \left(|\beta_r|^2 \right)}.$$

MSE optimal Wiener filter for image denoising:

$$\eta_r = \frac{\mathbf{AV}_{\Omega_A}\left(|\alpha_r|^2\right)}{\mathbf{AV}_{\Omega_A}\left(|\alpha_r|^2\right) + \mathbf{AV}_{\Omega_N}\left(|\nu_r|^2\right)} = \frac{SNR_r}{1 + SNR_r}$$

Empirical Wiener filters for image denoising (white additive noise model):

$$\eta_r = \max\left[\frac{\overline{|\beta_r|^2} - \sigma_n^2}{\overline{|\beta_r|^2}}; \quad 0 \right];$$

Soft thresholding:
$$\eta_r = \max\left[\frac{|\beta_r|^2 - \sigma_n^2}{|\beta_r|^2}; \quad 0 \right]$$

Rejecting filter (hard thresholding):
$$\eta_r = \begin{cases} 1, \ if \ |\beta_r|^2 \geq Thr \\ 0, \quad otherwise \end{cases}.$$

MSE optimal scalar Wiener filter for image deblurring:

$$\eta_r = \frac{1}{\lambda_r} \frac{SNR_r}{1 + SNR_r}.$$

Empirical Wiener filters for image deblurring:

Soft thresholding:
$$\eta_r = \max\left[\frac{1}{\lambda_r} \frac{\overline{|\beta_r|^2} - \sigma_n^2}{\overline{|\beta_r|^2}}; \quad 0 \right]$$

Rejecting filter:
$$\eta_r = \begin{cases} \dfrac{1}{\lambda_r}, if \ |\beta_r|^2 \geq Thr \\ 0, \quad otherwise \end{cases}.$$

Questions for self-testing

1. Describe the canonic model of imaging systems
2. Formulate the image restoration problem and explain why it is usually implemented as a multi-stage process
3. What is transform domain scalar filtering?
4. Formulate the principle of MSE optimal filtering
5. Derive optimal scalar Wiener filters for image denoising
6. Derive optimal scalar Wiener filters for image deblurring and show its relation to inverse filtering
7. Explain what is empirical Wiener filter and give examples of implementations of the empirical Wiener filter.
8. When Wiener filtering can be efficiently used for image restoration and when it fails?
9. What is image sensor aperture correction and how can it be implemented?