L. Yaroslavsky. ***Course 0510.7211 "Digital Image Processing: Applications"***

Lecture 1. Principles of Image Digitization

<u>Images as signals.</u> Mathematical models of signals. Types of signals: single component (scalar) and multi component (vectorial) signals; one-dimensional and multi-dimensional signals; continuous (analogue), discrete, quantized and digital signals. Signal space. Metrics of signal space and image processing quality criteria. Deterministic and statistical signal treatment. Local criteria

<u>Principles of image digitization</u>. Generalized digitization in signal space. Metrics in signal space and image digital representation quality. Euclidean metrics and its justification. Examples of other metrics. ε -net and ε -entropy (rate distortion function). Estimation of informational volume of signals. Two-step digitization: discretization + element-wise quantization

<u>Discretization: signal expansion over a set basis functions</u>

Superposition principle. Linear signal space. Basis functions. Linear independent and orthogonal bases.

$$a(x) = \tilde{a}(x) = \sum_{r=0}^{N-1} \alpha_r \varphi_{rest}(x,r); \;\; \alpha_r = \int_X a(x)\phi_{discr}(x,r)dx\,; \;\; \sum_r \varphi_r(x)\phi_r(\xi) = \delta(x,\xi)$$

Signal space dimensionality. Optimal bases. Karhunen-Loeve basis and signal statistical ensemble covariance function $R_a(x,\xi)$

$$\int_X R_a(x,\xi)\varphi_{KL}(\xi,r)d\xi = \lambda_r \varphi_{KL}(x,r)\;; \qquad \|\varepsilon\|^2 = \|a(x) - \tilde{a}(x)\|^2 = \sum_{r=N}^{\infty} \lambda_r$$

Hotelling and Singular Value Decomposition transforms.

<u>Classes of basis functions</u>. Requirements to and methods of generation of basis functions.

Shift (convolution) basis functions: $\varphi_r(x) = \varphi_0(x - r\Delta x)$;

Rectangular b.f.: $\varphi_0(x) = rect(x/\Delta x)$; $a(x) = \sum_k \alpha_k rect((x - k\Delta x)/\Delta x);\quad \alpha_k = \dfrac{1}{\Delta x}\displaystyle\int_{k\Delta x}^{(k+1)\Delta x} a(x)dx$;

Sinc-basis functions: $\varphi_0(x) = \mathrm{sinc}(\pi x/\Delta x)$; $\mathrm{sinc}(\pi x/\Delta x) = \dfrac{\sin(\pi x/\Delta x)}{\pi x/\Delta x} = \Delta x \displaystyle\int_{-1/2\Delta x}^{1/2\Delta x} \exp(i2\pi fx)df$

$$a(x) = \sum_k \alpha_k \mathrm{sinc}[\pi(x - k\Delta x)/\Delta x];\; \alpha_k = \frac{1}{\Delta x}\int_{-\infty}^{\infty} a(x)\mathrm{sinc}(\pi(x - k\Delta x)/\Delta x)dx =$$

$$= \int_{-\infty}^{\infty} \alpha(f)\mathrm{rect}((f + 1/2\Delta x)/\Delta x)exp(-i2\pi fk\Delta x)dx \Rightarrow \textbf{band limited signals}$$

R-order spline bases: R-th order convolution of rectangular basis functions. Cardinal splines.

Multiplicative basis functions:

Exponential basis functions: $four_r(x) = \exp(i2\pi rx/X)$; Fourier series as discretization:

$$a(x) = \sum_k \alpha_k \exp(-i2\pi kx/X); \qquad \alpha_k = \frac{1}{X}\int_{k-X/2}^{X/2} a(x)\exp(i2\pi kx/X)dx = \frac{1}{X}\alpha(k/X)$$

Walsh basis functions: $wal_r(x) = (-1)^{\sum_{m=0}^{\infty} r_m^{Gr}(x/X)_{m+1}} = \exp\left(i\pi \sum_{m=0}^{\infty} r_m^{Gr}(x/X)_{m+1}\right)$.

Sequency ordering.

Mixed scale/shift basis functions: Haar, wavelets. Wavelet signal decomposition

Haar wavelets : $$har_r(x) = \frac{2^{m/2}}{\sqrt{X}} sign\left[\sin\left(2^{m+1}\pi\frac{x}{X}\right)\right] rect\left[2^m\frac{x}{X} - (r)_{\mathrm{mod}\,2^m}\right]$$

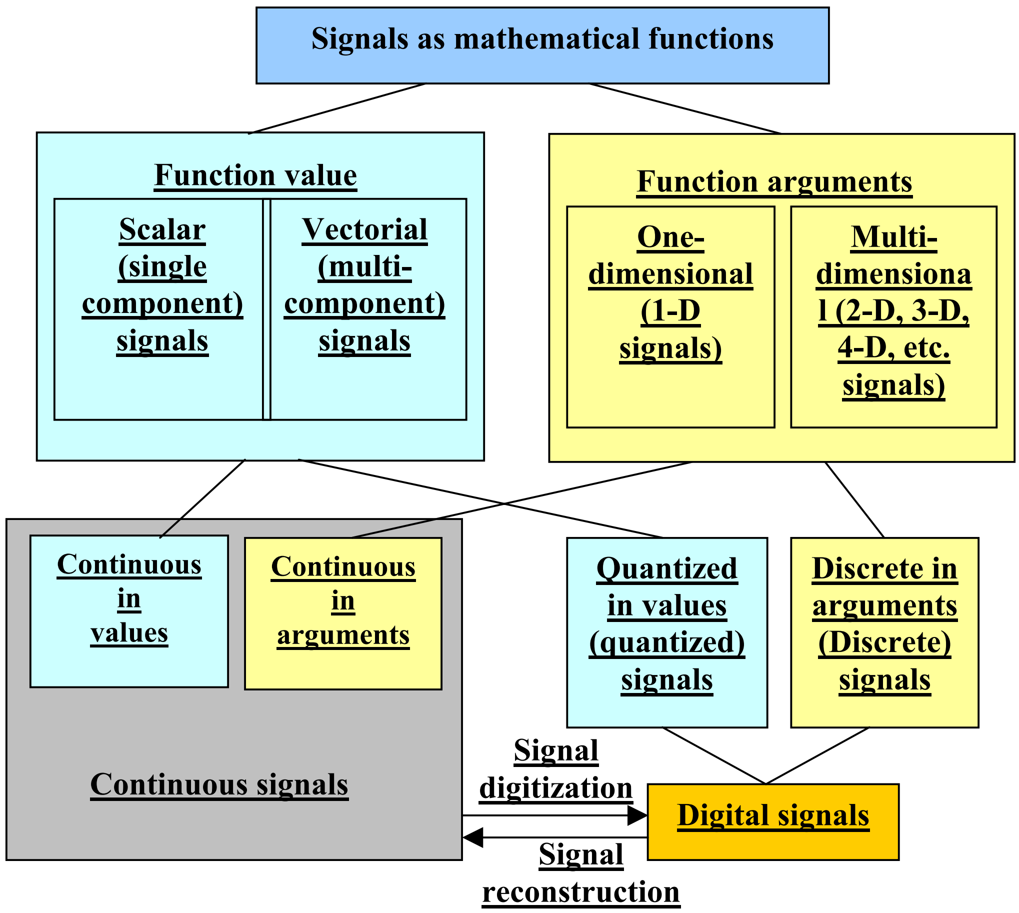
m - most significant non-zero bit in binary representation of r .

2-D bases function: Separability. Mixed 2-D bases.

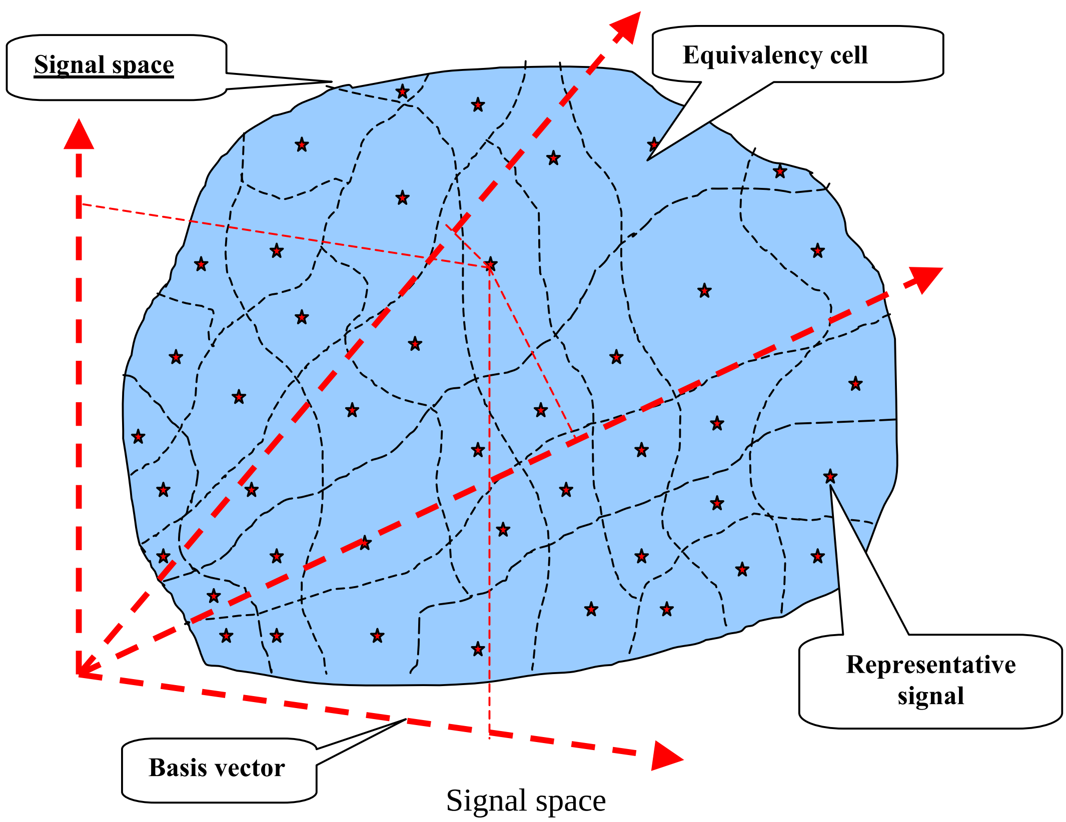
Unconventional discretization methods: Method of coded apertures; computed tomography.

Questions for self-examination

1. What is signal general digitisation and what is ε -entropy of signal space?
2. What is signal discretization and quantization; give geometrical interpretation.
3. Formulate requirements to the discretization/restoration basis functions. In what respect the design of multi-dimensional discretization/restoration bases differs from that of 1-D ones.
4. Describe basic classes of discretization basis functions. Give examples of devices that implement these functions. Give examples and describe unconventional discretization methods you know.
5. For what class of signals binary basis functions are appropriate? Shift functions are appropriate? Fourier series representation is appropriate?



Classification of signal and mathematical models



Signal space

Signal expansion over orthogonal bases.
Optimality of the Karhunen-Loeve transform

Let $\{a(x,\omega)\}$ be a signal from an ensemble Ω ($\omega \in \Omega$)to be represented by its approximation in form of a finite expansion:

$$a(x,\omega) \cong \tilde{a}(x,\omega) = \sum_{k=0}^{N-1} \alpha_k(\omega)\varphi_k(x) \tag{1}$$

over orthonormal basis functions $\{\varphi_k(x)\}$ such that

$$\int_X \varphi_k(x)\varphi_l^*(x)dx = \delta(k,l) = 0^{|k-l|}. \tag{2}$$

Mean square error (MSE) of the signal approximation is

$$|\varepsilon(\omega)|^2 = \int_X |a(x,\omega) - \tilde{a}(x,\omega)|^2 dx = \int_X \left| a(x,\omega) - \sum_{k=0}^{N-1} \alpha_k(\omega)\varphi_k(x) \right|^2 dx$$

Optimal values of the signal representation coefficients $\{\alpha_k(\omega)\}$ that minimize MSE can be found by equalling derivatives of Eq. (3) over $\{\alpha_k(\omega)\}$ to zero:

$$\frac{\partial}{\partial \alpha_k(\varpi)^*} \int_X \left| a(x,\omega) - \sum_{k=0}^{N-1} \alpha_k(\omega)\varphi_k(x) \right|^2 dx = 0 \;\Rightarrow\; \alpha_k(\omega) = \int_X a(x;\omega)\varphi_k^*(x)dx .$$

Minimal MSE is then equal to

$$|\varepsilon(\omega)|^2_{\min} = \int_X |a(x,\omega)|^2 dx - \sum_{k=0}^{N-1} |\alpha_k(\omega)|^2 . \tag{4}$$

By an appropriate selection of bases functions $\{\varphi_k(x)\}$ one can further minimize average MSE over the set Ω of signals as defined by parameter ω:

$$\{\varphi_k(x)\}_{opt} = \arg\min_{\{\varphi_k(x)\}} \left\{ AV_\Omega \left(\int_X |a(x,\omega)|^2 dx - \sum_{k=0}^{N-1} |\alpha_k(\omega)|^2 \right) \right\} =$$

$$\arg\min_{\{\varphi_k(x)\}} \left\{ AV_\Omega \left(\int_X |a(x,\omega)|^2 dx \right) - AV_\Omega \left(\sum_{k=0}^{N-1} |\alpha_k(\omega)|^2 \right) \right\} =$$

$$\arg\max_{\{\varphi_k(x)\}} \left\{ AV_\Omega \left(\sum_{k=0}^{N-1} |\alpha_k(\omega)|^2 \right) \right\} = \arg\max_{\{\varphi_k(x)\}} \left\{ AV_\Omega \left(\sum_{k=0}^{N-1} \iint_X a(x,\omega)a^*(y,\omega)\varphi_k^*(x)\varphi_k(y)dxdy \right) \right\} =$$

$$\arg\max_{\{\varphi_k(x)\}} \left\{ \sum_{k=0}^{N-1} \iint_X AV_\Omega \left(a(x,\omega)a^*(y,\omega) \right) \varphi_k^*(x)\varphi_k(y)dxdy \right\} =$$

$$\arg\max_{\{\varphi_k(x)\}} \left\{ \sum_{k=0}^{N-1} \iint_X R_a(x,y)\varphi_k^*(x)\varphi_k(y)dxdy \right\}, \tag{5}$$

where

$$R_a(x,y) = AV_\Omega \{a(x,\omega)a^*(y,\omega)\}. \tag{6}$$

is called “correlation function” of the signals $\{a(x,\omega)\}$

Denote

$$\mathbf{R\Phi}(x) = \int_X R_a(x,y)\varphi_k(y)dy . \tag{7}$$

Then optimal bases functions $\{\varphi_k(x)\}$ are found from

$$\{\varphi_k(x)\} = \arg\max_{\{\varphi_k(x)\}} \left\{ \sum_{k=0}^{N-1} \int_X \mathbf{R\Phi}(x)\varphi_k^*(x)dx \right\} . \tag{8}$$

From Schwarz inequality $\int_X f_1(x)f_2^*(x)dx \le \left(\int_X |f_1(x)|^2 dx \right)^{1/2} \left(\int_X |f_2(x)|^2 dx \right)^{1/2}$,it follows that solution of Eq. (8) is as

$$\mathbf{R\Phi}(x) = \int_X R_a(x,y)\varphi_k(y)dy = \lambda_k \varphi_k(x), \tag{9}$$

that is optimal bases functions are eigen functions of integral equation Eq.(8) with kernel defined by the correlation function (7) of the set of signals.

Signal space metrics and processing quality criteria

Discrete signals			
$L_1 : \sum_{k=0}^{N-1} \lvert a_k - b_k \rvert$	$L_2 : \left(\sum_{k=0}^{N-1} \lvert a_k - b_k \rvert^2 \right)^{1/2}$	$L_p : \left(\sum_{k=0}^{N-1} \lvert a_k - b_k \rvert^p \right)^{1/p}$	$M : \max\{\lvert a_k - b_k \rvert\}$
Continuous signals			
$L_1 : \int_X \lvert a(x) - b(x) \rvert dx$	$L_2 : \left(\int_X \lvert a(x) - b(x) \rvert^2 dx \right)^{1/2}$	$L_P : \left(\int_X \lvert a(x) - b(x) \rvert^P dx \right)^{1/P}$	$L_1 : \sup_X(\lvert a(x) - b(x) \rvert);$

Statistical treatment of signals: spatial/time averaging is supplemented with or replaced by averaging over a statistical ensemble or a data base.

Local criteria: signal local approximation

$$AVLOSS(k) = AV_\Omega \left\{ \sum_m LOC(k; a(k)) LOSS(a(m), \hat{a}(m)) \right\}$$

Euclidean metrics and additive image formation model

Let image signal space is generated from a signal $\{a_k\}$ by adding to it a “random” signal $\{n_k\}$:

$$\{b_k = a_k + n_k\}$$

Then distance between signals $\{b_k\}$ and $\{a_k\}$ is fully determined by the “noise” $\{n_k\}$. Since “noise” is regarded “random”, it is specified in terms of its probability distribution density $p(\{n_k\})$. Assume that $\{n_k\}$ are statistically independent:

$$p(\{n_k\}) = \prod_k p(n_k)$$

and that $\{n_k\}$ have normal distribution density with zero mean and standard deviation σ_n:

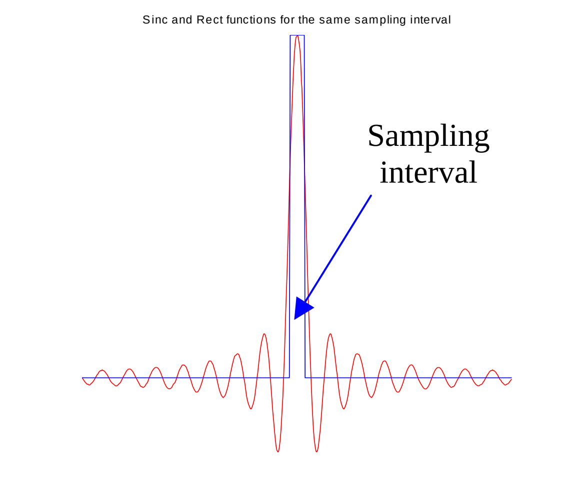
$$p(n_k) \propto \exp\left(-\frac{n_k^2}{2\sigma_n^2} \right).$$

Then obtain

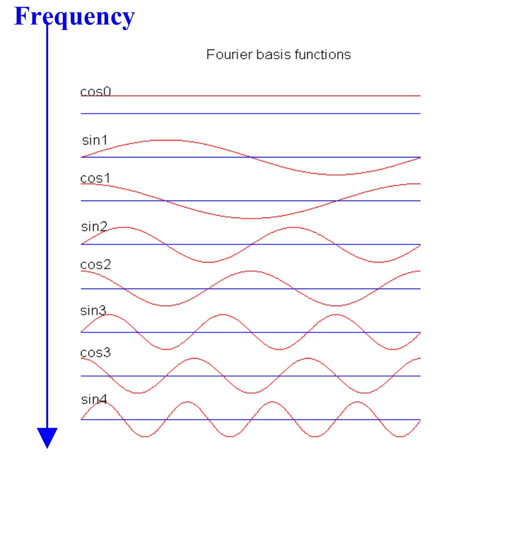
$$p(\{n_k\}) = \prod_k p(n_k) \propto \prod_k \exp\left(-\frac{n_k^2}{2\sigma_n^2} \right) = \exp\left(-\frac{1}{2\sigma_n^2} \sum_k \lvert b_k - a_k \rvert^2 \right)$$

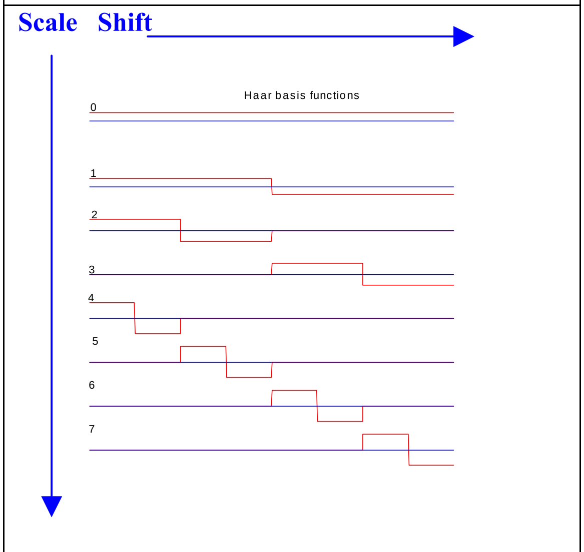
which is a monotonic function of $\sum_k \lvert b_k - a_k \rvert^2$, a Euclidean distance.

EXAMPLES OF BASIS FUNCTIONS:

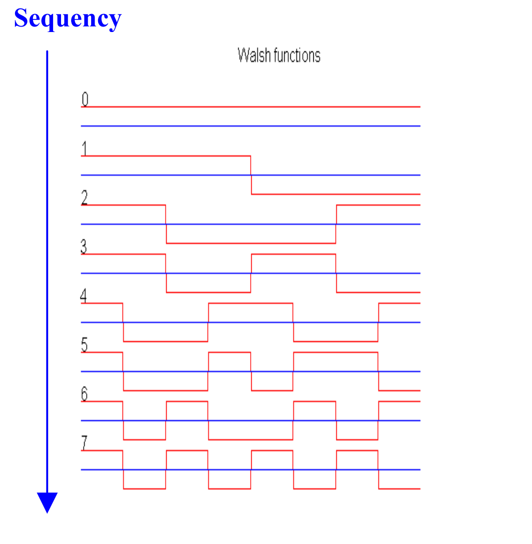


Shift basis functions





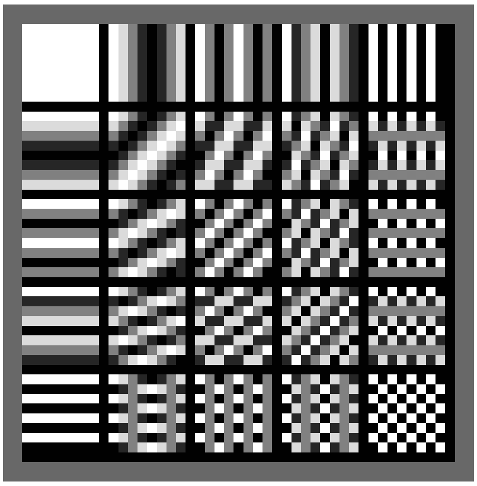
Shift/scale basis functions (wavelets)



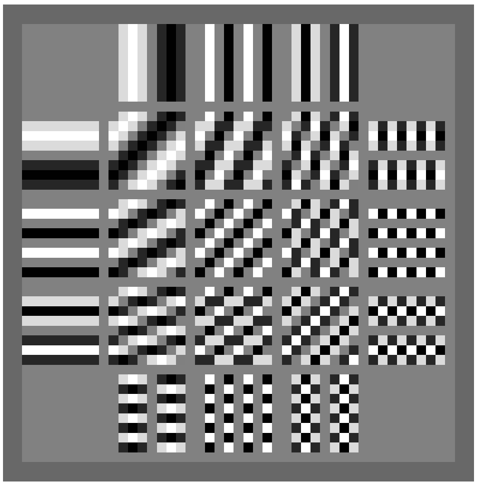
Multiplicative (scale) basis functions

2-D Transform Basis Functions

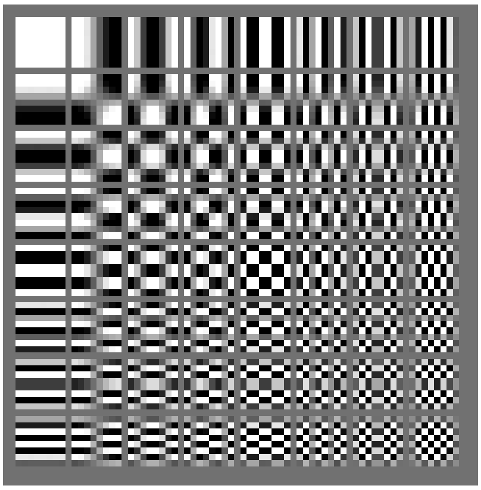
DFT (real)



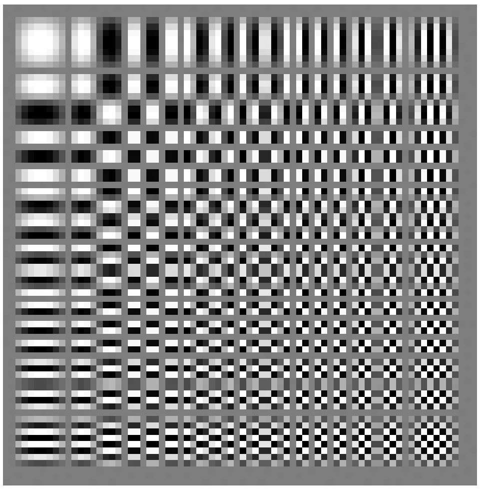
DFT (imaginary



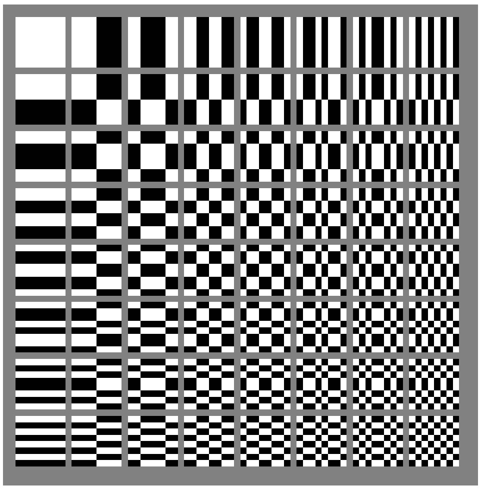
DCT



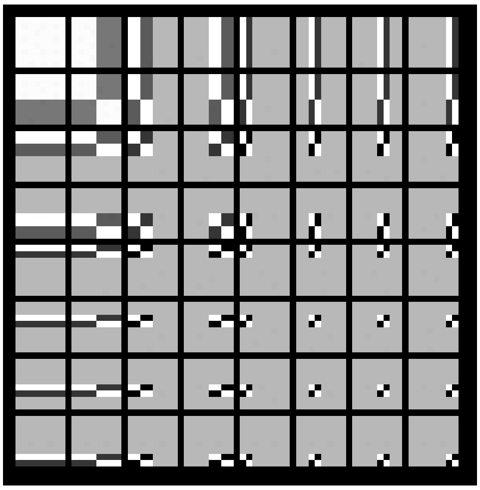
DST

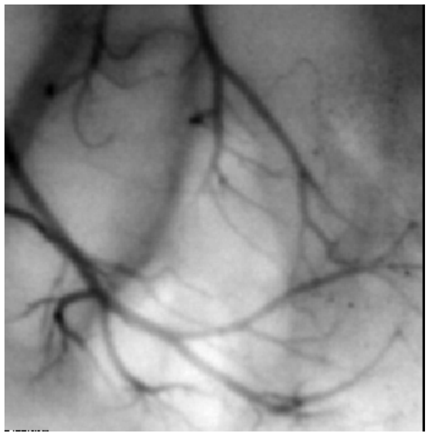


Walsh

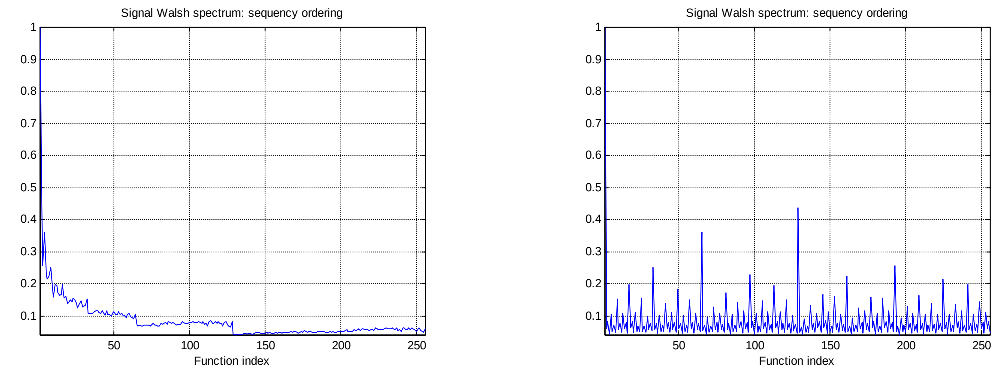


Haar





An image

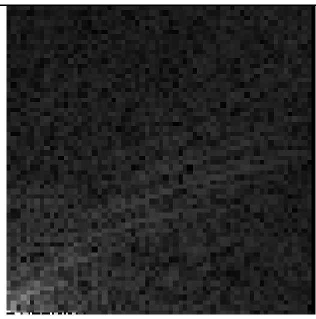


Comparison of image Walsh function spectra in Walsh (left) and Hadamard (right) ordering

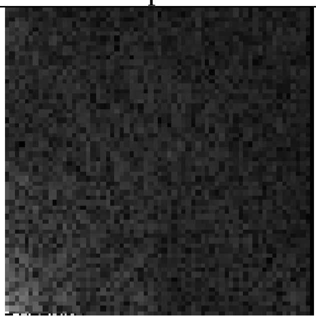
Comparison of image spectra in different bases

Image

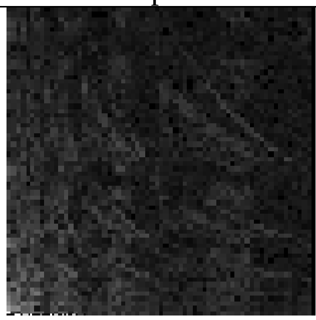
DFT spectrum



DCT spectrum



Walsh spectrum



Haar spectrum