Lecture 1. Convolution integral and digital filters

1.1. Imaging transforms and principles of their discrete representation

The consistency principle and the mutual correspondence principle between continuous and digital transformations.

Main assumption: signal discrete representation through shift sampling $\left\{\varphi_k^{(s)}(x) = \varphi^{(s)}\left[x - \left(k + u^{(s)}\right)\Delta x\right]\right\}$ and reconstruction $\left\{\varphi_k^{(r)}(x) = \varphi^{(r)}\left[x - \left(k + u^{(r)}\right)\Delta x\right]\right\}$ basis functions

$$\tilde{a}\left(x^{(r)}\right) = \sum_k a_k \varphi^{(r)}\left(x^{(r)} - \tilde{k}^{(r)}\Delta x\right); \quad a_k = \int a(x)\varphi^{(s)}\left(x - \tilde{k}^{(s)}\Delta x\right)dx; \quad \tilde{k}^{(r)} = k + u^{(r)}; \quad \tilde{k}^{(s)} = k + u^{(s)}$$

1.2. Convolution integral and digital convolution

Convolution integral: $b(x) = \int_{-\infty}^{\infty} a(\xi)h(x-\xi)d\xi \quad \Rightarrow \quad b_k = \sum_{n=0}^{N_h-1} h_n a_{k-n}$

$\{h_n\}$- Discrete PSF of the digital filter;

Overall PSF of the digital filter:

$$h_{ovall}(x,\xi) = \sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n \varphi^{(s)}\left[\xi - \left(k - n + u^{(s)}\right)\Delta x\right]\varphi^{(r)}\left(x - k\Delta x - u^{(r)}\Delta x\right), \text{ and}$$

Overall Freq. Response of the digital filter: $H_{ovall}(f,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{ovall}(x,\xi)\exp[i2\pi(fx - p\xi)]dxd\xi$

$$H_{ovall}(f,p) = CFR\_DF(p)\cdot\Phi^{(r)}(f)\cdot\Phi^{(s)}(-p)\cdot SV(f,p)$$

Continuous frequency response of the digital filter: $CFR\_DF(p) = \sum_{n=0}^{N_h-1} h_n \exp(i2\pi pn\Delta x)$

$$\Phi^r(f) = \exp\left(i2\pi f u^r \Delta x\right)\int_{-\infty}^{\infty}\varphi^{(r)}(x)\exp(i2\pi fx)dx; \quad \Phi^s(-p) = \exp\left(-i2\pi p u^{(s)}\Delta x\right)\int_{-\infty}^{\infty}\varphi^{(s)}\left(\bar{\xi}\right)\exp\left(-i2\pi p\bar{\xi}\right)d\bar{\xi}$$

$$SV(f,p) = N_b\,\text{sincd}\left[\pi N_b(f-p)\Delta x\right]\exp\left[\pi(N_b-1)(f-p)\Delta x\right];$$

Theorem 1. *Given signal sampling and reconstruction devices and the number of signal samples, overall frequency response of the digital filter*

$$CFR\_DF(p) = \frac{\exp\left(i2\pi\frac{up}{N}\right)}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r \frac{\sin\left[\pi\frac{N_h(r-p\Delta x)}{N}\right]}{\sin\left[\pi\frac{(r-p\Delta x)}{N}\right]}$$

is completely determined by coefficients $\{\eta_r\}$ of the digital filter Discrete Frequency Response DFrR (SDFT of the filter DPSF):

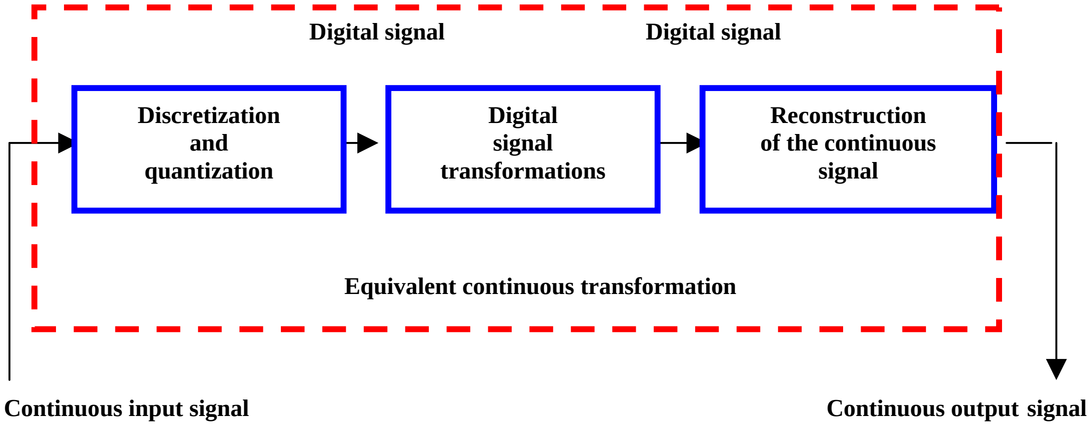
$$h_n = \frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r \exp\left[-i2\pi\frac{(n+u)r}{N}\right]$$

or by its continuous frequency response

Theorem 2. *Coefficients of Discrete Frequency Response of the digital filter are samples of its Continuous Frequency Response CFR_DF taken with a sampling interval* $1/N\Delta x$

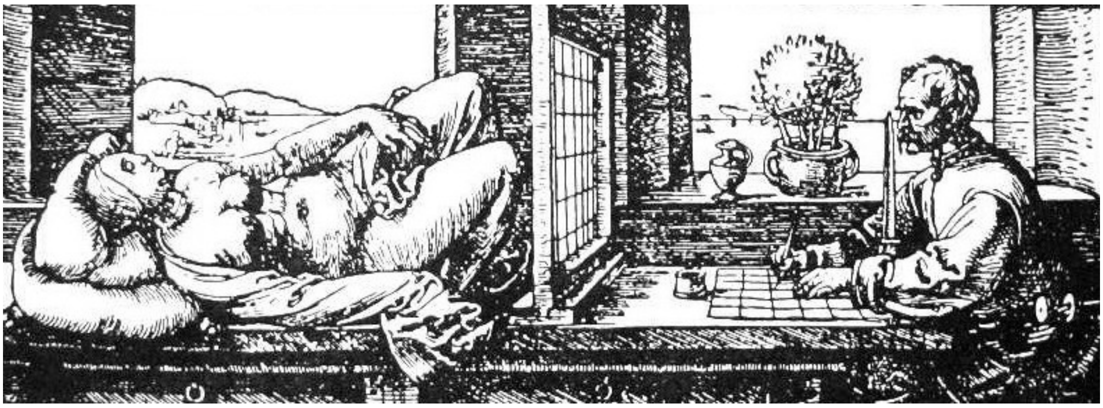
and

Theorem 3 *CFR_DF of the digital filter is a discrete sinc-interpolated function of its samples*

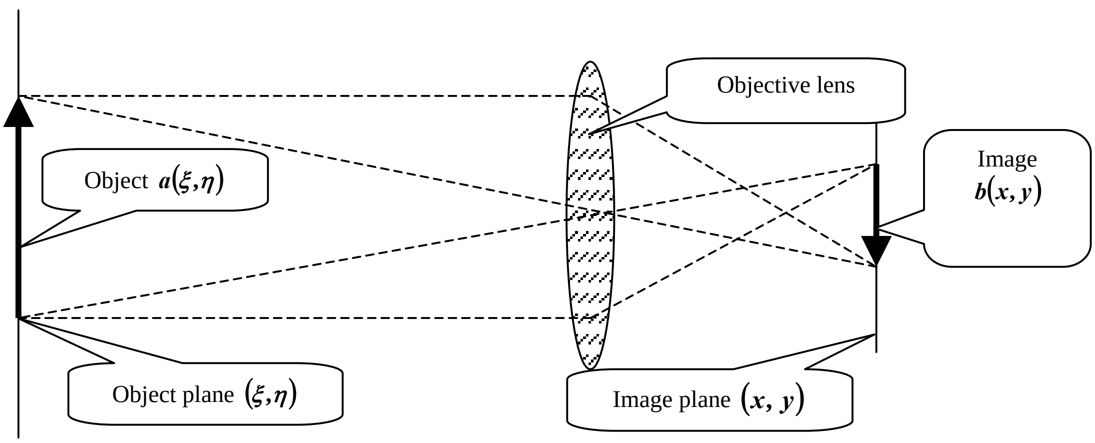


Consistency and mutual correspondence principle between continuous and digital signal transformations

Direct imaging: man made devices



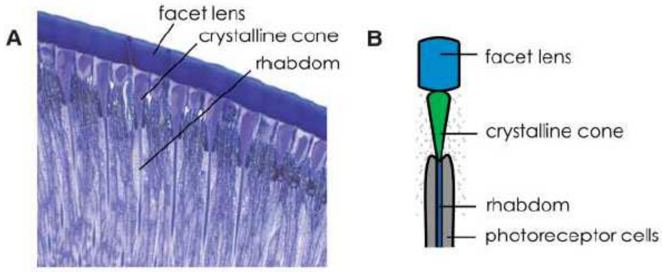
A woodcut by Albrecht Dürer showing the relationship between the light distribution on an object and image plane (adopted form R. Bracewell, Two-dimensional imaging, Prentice Hall Int. 1995)



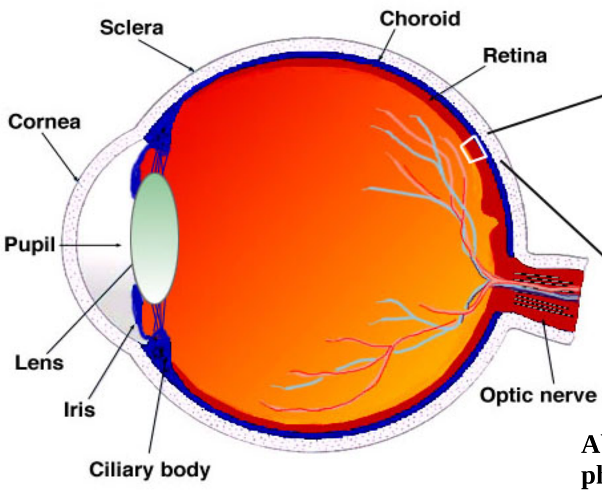
$$b(x,y) = \int_{-\infty}^{\infty} a(\xi,\eta) PSF(x-\xi, y-\eta) d\xi d\eta$$

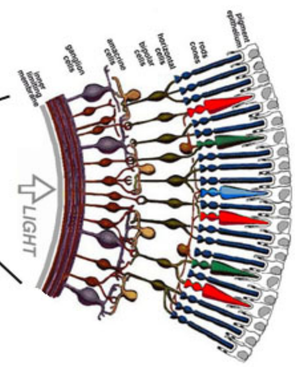
Schematic diagram of optics of photographic and TV cameras

Direct imaging in animal kingdom



Compound eye of insects



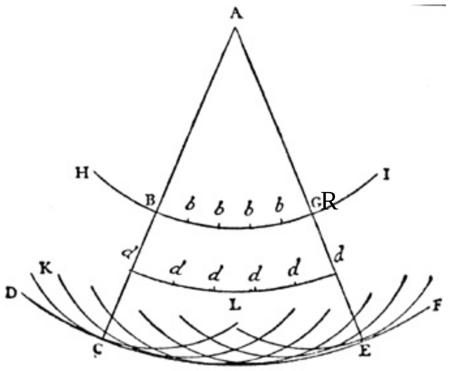


About 100,000,000 photoreceptors (6% cones, 94% rods)

Human eye

Transform imaging and wave propagation

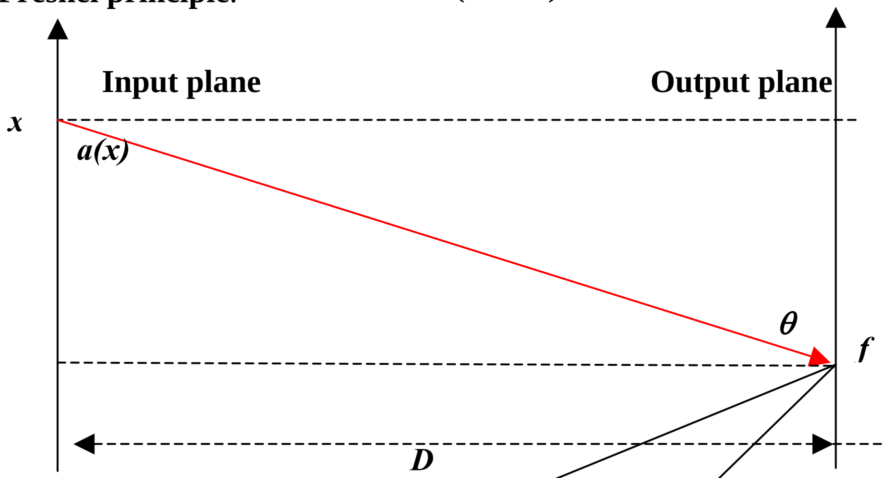
C. *Huygens* (1629-95), Dutch physicist



$$\frac{A}{R}\exp\left(i2\pi\frac{R}{\lambda}\right)$$

A. J. Fresnel (1788-1827), French physicist

Huygens-Fresnel principle.



$$\frac{a(x)\exp\left[i2\pi\left(\frac{\sqrt{D^2+(x-f)^2}}{\lambda}\right)\right]}{\sqrt{D^2+(x-f)^2}}\cos\theta = a(x)\frac{\exp\left[i2\pi\left(\frac{\sqrt{D^2+(x-f)^2}}{\lambda}\right)\right]}{\sqrt{D^2+(x-f)^2}}\frac{D}{\sqrt{D^2+(x-f)^2}}$$

Point spread function of free space wave propagation	$h(x,y;f_x,f_y)=\dfrac{\exp\left[i2\pi\left(\dfrac{\sqrt{D^2+(x-f_x)^2+(y-f_y)^2}}{\lambda}\right)\right]}{D^2+(x-f_x)^2+(y-f_y)^2}$
Kirchhoff-Rayleigh-Sommerfeld integral transform	$\alpha(f)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}a(x,y)\dfrac{\exp\left(i2\pi\dfrac{D\sqrt{1+(x-f_x)^2/D^2+(y-f_y)^2/D^2}}{\lambda}\right)}{1+(x-f_x)^2/D^2+(y-f_y)^2/D^2}dxdy$
"Near zone" approximation: $D^2>>(x-f_x)^2+(y-f_y)^2$, **Fresnel integral transform:**	$\alpha(f_x,f_y)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}a(x,y)\exp\left[i\pi\dfrac{(x-f_x)^2+(y-f_y)^2}{\lambda D}\right]dxdy$

Discrete representation of convolution integrals

Digital filter that corresponds to a convolution integral:

$$b(x)=\int_{-\infty}^{\infty}a(\xi)h(x-\xi)d\xi$$

$$a(\xi)=\sum_{n}a_{n^{(r)}}\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right);\ \tilde{n}^{(r)}=n+u^{(r)};$$

$$b(x)=\int_{-\infty}^{\infty}\sum_{n}a_{n^{(r)}}\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right)h(x-\xi)d\xi=\sum_{n}a_{n^{(r)}}\int_{-\infty}^{\infty}\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right)h(x-\xi)d\xi$$

$$b_{k^{(s)}}=\int_{-\infty}^{\infty}b(x)\varphi^{(s)}\left(x-\tilde{k}^{(s)}\Delta x\right)dx=\int_{-\infty}^{\infty}\sum_{n}a_{n^{(r)}}\int_{-\infty}^{\infty}\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right)h(x-\xi)\varphi^{(s)}\left(x-\tilde{k}^{(s)}\Delta x\right)d\xi dx=$$

$$\int_{-\infty}^{\infty}\sum_{n}a_{n^{(r)}}\int_{-\infty}^{\infty}\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right)h(x-\xi)\varphi^{(s)}\left(x-\tilde{k}^{(s)}\Delta x\right)d\xi dx=$$

$$\sum_{n}a_{n^{(r)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x-\xi)\varphi^{(r)}\left(\xi-\tilde{n}^{(r)}\Delta x\right)\varphi^{(s)}\left(x-\tilde{k}^{(s)}\Delta x\right)d\xi dx=$$

$$\sum_{n}a_{n^{(r)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h\left[x-\xi-\left(\tilde{k}^{(s)}-\tilde{n}^{(r)}\right)\Delta x\right]\varphi^{(r)}(\xi)\varphi^{(s)}(x)d\xi dx$$

$$h_{k-n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h\left[x-\xi-\left(\tilde{k}^{(s)}-\tilde{n}^{(r)}\right)\Delta x\right]\varphi^{(r)}(\xi)\varphi^{(s)}(x)d\xi dx=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h\left[x-\xi-(k-n)\Delta x+\left(u^{(s)}-u^{(r)}\right)\right]\varphi^{(r)}(\xi)\varphi^{(s)}(x)d\xi dx$$

$$b_k=\sum_{n}a_n h_{k-n};$$

Assume $n\in[-\infty,\infty]$:

$$b_k=\sum_{n=-\infty}^{\infty}a_n h_{k-n}=\sum_{n=-\infty}^{\infty}h_n a_{k-n}$$

<u>Overall continuous PSF of the digital filter</u>:

$$b_k=\sum_{n=0}^{N_h-1}h_n a_{k-n}$$

$$b(x)=\sum_{k=0}^{N_b-1}b_k\varphi^{(r)}\left(x-\tilde{k}^{(r)}\Delta x\right)=\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1}h_n a_{k-n}\varphi^{(r)}\left(x-\tilde{k}^{(r)}\Delta x\right)$$

$$a_{k-n}=\int_{-\infty}^{\infty}a(\xi)\varphi^{(s)}\left[\xi-(k-n)\Delta x+u^{(s)}\Delta x\right]d\xi$$

$$b(x)=\sum_{k=0}^{N_b-1}b_k\varphi^{(r)}\left(x-\tilde{k}^{(r)}\Delta x\right)=\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1}h_n a_{k-n}\varphi^{(r)}\left(x-\tilde{k}^{(r)}\Delta x\right)=$$

$$\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1}h_n\left\{\int_{-\infty}^{\infty}a(\xi)\varphi^{(s)}\left[\xi-\left(k-n+u^{(s)}\right)\Delta x\right]d\xi\right\}\varphi^{(r)}\left(x-k\Delta x-u^{(r)}\Delta x\right)=$$

$$\int_{-\infty}^{\infty} a(\xi)d\xi \sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n\varphi^{(s)}\left[\xi-\left(k-n+u^{(s)}\right)\Delta x\right]\varphi^{(r)}\left(x-k\Delta x-u^{(r)}\Delta x\right)$$

$$h_{ovall}(x,\xi)=\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n\varphi^{(s)}\left[\xi-\left(k-n+u^{(s)}\right)\Delta x\right]\varphi^{(r)}\left(x-k\Delta x-u^{(r)}\Delta x\right)$$

Overall continuous frequency response of digital filter:

$$H_{ovall}(f,p)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{ovall}(x,\xi)\exp[i2\pi(fx-p\xi)]dxd\xi =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n\varphi^{(s)}\left[\xi-\left(k-n+u^{(s)}\right)\Delta x\right]\varphi^{(r)}\left(x-k\Delta x-u^{(r)}\Delta x\right)\right\}\exp[i2\pi(fx-p\xi)]dxd\xi =$$

$$\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(s)}\left[\xi-\left(k-n+u^{(s)}\right)\Delta x\right]\varphi^{(r)}\left(x-k\Delta x-u^{(r)}\Delta x\right)\exp[i2\pi(fx-p\xi)]dxd\xi =$$

$$\sum_{k=0}^{N_b-1}\sum_{n=0}^{N_h-1} h_n\exp\left\{i2\pi\left[f\left(k\Delta x+u^r\Delta x\right)-p\left[(k-n)\Delta x+u^{(s)}\Delta x\right]\right]\right\}\times$$

$$\int_{-\infty}^{\infty}\varphi^{(s)}\left(\bar{\xi}\right)\exp\left(-i2\pi p\bar{\xi}\right)d\bar{\xi}\int_{-\infty}^{\infty}\varphi^{(r)}(x)\exp(i2\pi fx)dx =$$

$$\sum_{k=0}^{N_b-1}\exp[i2\pi(f-p)k\Delta x]\sum_{n=0}^{N_h-1} h_n\exp(i2\pi pn\Delta x)\times$$

$$\left\{\exp\left(-i2\pi pu^{(s)}\Delta x\right)\int_{-\infty}^{\infty}\varphi^{(s)}\left(\bar{\xi}\right)\exp\left(-i2\pi p\bar{\xi}\right)d\bar{\xi}\right\}\left\{\exp\left(i2\pi fu^r\Delta x\right)\int_{-\infty}^{\infty}\varphi^{(r)}(x)\exp(i2\pi fx)dx\right\}=$$

$$SV(f,p)CFR\_DF(p)\Phi^s(-p)\Phi^r(f);$$

where $\Phi^r(f)$ is frequency response of the signal reconstruction device:

$$\Phi^r(f)=\exp\left(i2\pi fu^r\Delta x\right)\int_{-\infty}^{\infty}\varphi^{(r)}(x)\exp(i2\pi fx)dx$$

$\Phi^s(-p)$ is frequency response of the signal sampling device:

$$\Phi^s(-p)=\exp\left(-i2\pi pu^{(s)}\Delta x\right)\int_{-\infty}^{\infty}\varphi^{(s)}\left(\bar{\xi}\right)\exp\left(-i2\pi p\bar{\xi}\right)d\bar{\xi}$$

$CFR\_DF(p)$ is continuous frequency response of the digital filter:

$$CFR\_DF(p)=\sum_{n=0}^{N_h-1} h_n\exp(i2\pi pn\Delta x)$$

$SV(f,p)=\sum_{k=0}^{N_b-1}\exp[i2\pi(f-p)k\Delta x]$ is a “Filter space-variance” term.

Continuous frequency response of the digital filter:

$$CFR\_DF(p)=\sum_{n=0}^{N_h-1} h_n\exp(i2\pi pn\Delta x);$$

Let discrete signal has N samples and let $\{\eta_r\}$ be Discrete Fourier Transform coefficients of discrete PSF $\{h_n\}$ of the digital filter:

$$h_n=\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\exp\left[-i2\pi\frac{(n+u)r}{N}\right]$$

Then

$$CFR\_DF(p)=\sum_{n=0}^{N_h-1} h_n \exp(i2\pi pn\Delta x)=\sum_{n=0}^{N_h-1}\left[\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r \exp\left[-i2\pi\frac{(n+u)r}{N}\right]\right]\exp(i2\pi pn\Delta x)=$$

$$\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\left[\sum_{n=0}^{N_h-1}\exp\left(-i2\pi\frac{n(r-p\Delta x)}{N}\right)\right]\exp\left(-i2\pi\frac{ur}{N}\right)=$$

$$\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\frac{\exp\left(-i2\pi\frac{N_h(r-p\Delta x)}{N}\right)-1}{\exp\left(-i2\pi\frac{(r-p\Delta x)}{N}\right)-1}\exp\left(-i2\pi\frac{ur}{N}\right)=$$

$$\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\frac{\sin\left[\pi\frac{N_h(r-p\Delta x)}{N}\right]}{\sin\left[\pi\frac{(r-p\Delta x)}{N}\right]}\exp\left[-i\pi\frac{(N_h-1)(r-p\Delta x)}{N}\right]\exp\left(-i2\pi\frac{ur}{N}\right)=$$

$$\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\frac{\sin\left[\pi\frac{N_h(r-p\Delta x)}{N}\right]}{\sin\left[\pi\frac{(r-p\Delta x)}{N}\right]}\exp\left(-i\pi\frac{[(N_h-1)+2u]}{N}r\right)\exp\left(i2\pi\frac{up}{N}\right);$$

Selection $u=-(N_h-1)/2$ gives:

$$CFR\_DF(p)=\frac{\exp\left(i2\pi\frac{up}{N}\right)}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\frac{\sin\left[\pi\frac{N_h(r-p\Delta x)}{N}\right]}{\sin\left[\pi\frac{(r-p\Delta x)}{N}\right]};$$

<u>“Filter space-variance” term</u>:

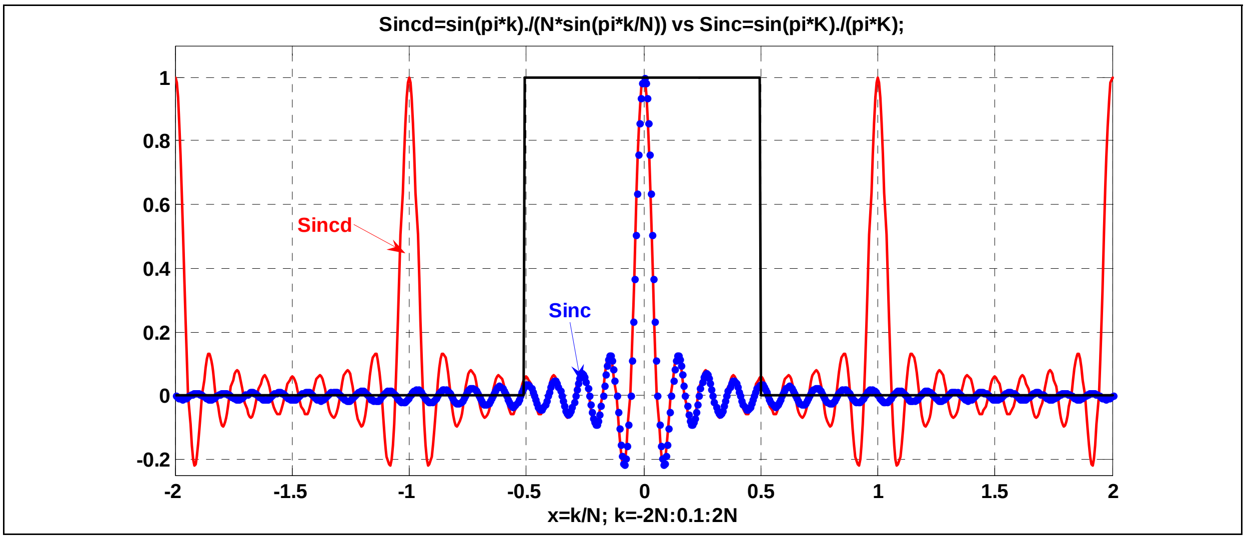
$$SV(f,p)=\sum_{k=0}^{N_b-1}\exp[i2\pi(f-p)k\Delta x]=\frac{\exp[i2\pi N_b(f-p)\Delta x]-1}{\exp[i2\pi(f-p)\Delta x]-1}=$$

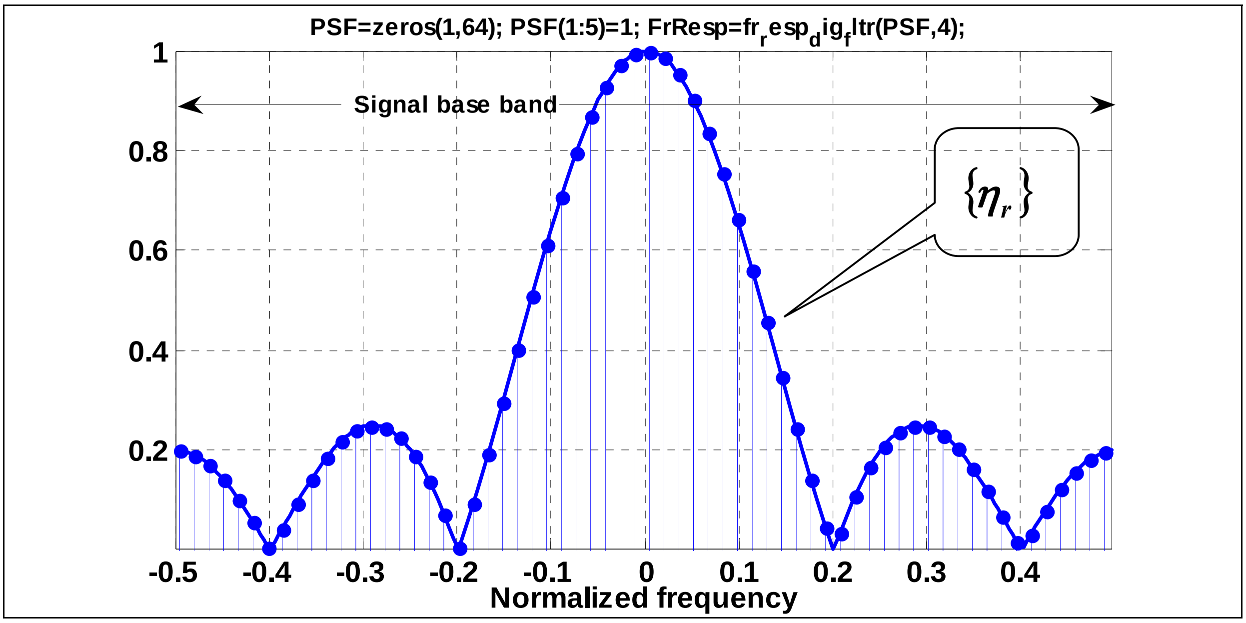
$$N_b\frac{\sin[\pi N_b(f-p)\Delta x]}{N_b\sin[\pi(f-p)\Delta x]}\exp[\pi(N_b-1)(f-p)\Delta x]=$$

$$N_b\,\mathrm{sincd}[\pi N_b(f-p)\Delta x]\exp[\pi(N_b-1)(f-p)\Delta x],$$

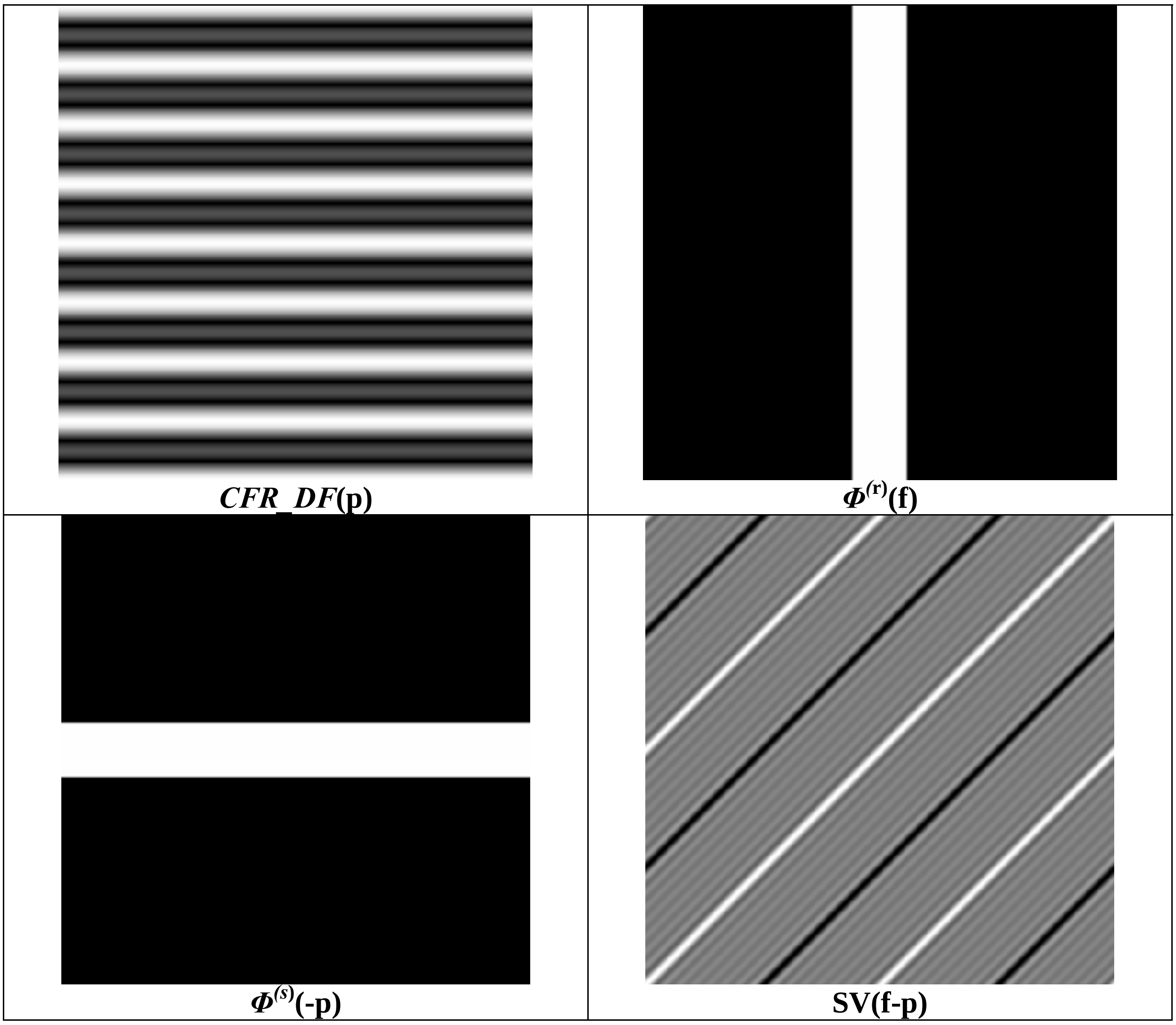
where

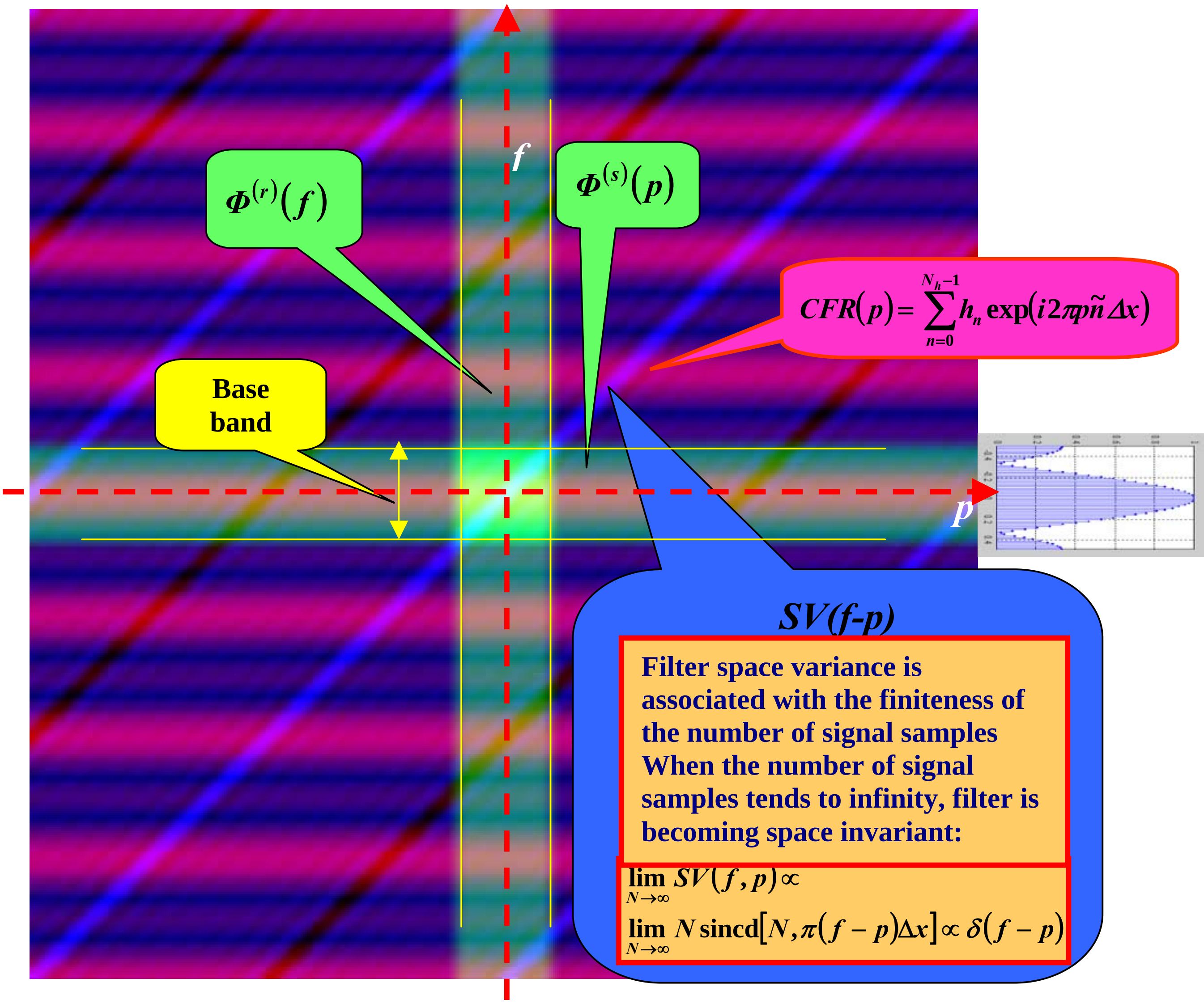
$$\mathrm{sincd}(N,x)=\frac{\sin x}{N\sin x/N}$$





Continuous frequency response and its samples (marked by circles) of a digital filter that computes signal local mean over 5/64 of signal size





Overall frequency response
of a digital filter