Lecture 2. Imaging transforms in digital computers (6 hours).

2.1 Principles of discrete representation of continuous signal transformation

The consistency principle and the mutual correspondence principle between continuous and digital transformations.

Main assumption: signal discrete representation through shift basis functions

$$\varphi_k^{(s)}(x)=\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right] \text{ and } \varphi_k^{(r)}(x)=\varphi^{(r)}\left[x-\left(k+u^{(r)}\right)\Delta x\right],$$

$$\tilde{a}\left(x^{(r)}\right)=\sum_k a_k\varphi^{(r)}\left(x^{(r)}-k\Delta x\right);\quad a_k=\int a(x)\varphi^{(s)}\left(x^{(s)}-\tilde{k}\Delta x\right)dx;$$

2.2 Convolution integral and digital filters.

Convolution integral: $b(x)=\int_{-\infty}^{\infty}a(\xi)h(x-\xi)d\xi \quad\Rightarrow\quad b_k=\sum_{n=0}^{N_h-1}h_n a_{k-n}$

$\{h_n\}$- Discrete PSF of the digital filter;

Overall PSF of the digital filter:

$$h_{ovall}(x,\xi)=\sum_{k=0}^{N-1}\sum_{n=0}^{N_h-1}h_n\varphi^{(r)}\left(x-\tilde{k}\Delta x\right)\varphi^{(s)}\left[\xi-\left(\tilde{k}-\tilde{n}\right)\Delta x\right],\ \tilde{k}=k+u^{(r)} \text{ and } \tilde{n}=n+u^{(s)}$$

Overall Freq. Response of the digital filter: $H_{ovall}(f,p)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{ovall}(x,\xi)\exp[i2\pi(fx-p\xi)]dxd\xi$

$$H_{ovall}(f,p)=CFR\_DF(p)\cdot\Phi^{(r)}(f)\cdot\Phi^{(s)}(-p)\cdot SV(f,p)$$

Continuous frequency response of the digital filter: $CFR\_DF(p)=\sum_{n=0}^{N_h-1}h_n\exp\left[i2\pi p\left(n+u^r-u^s\right)\Delta x\right]$

$$\Phi^{(r)}(f)=\int_{-\infty}^{\infty}\varphi^{(r)}(x)\exp(i2\pi fx)dx;\quad \Phi^{(s)}(-p)=\int_{-\infty}^{\infty}\varphi^{(s)}(x)\exp(-i2\pi px)dx$$

$$SV(f,p)=\sum_{k=0}^{N-1}\exp\left[i2\pi(f-p)\left(k+u^{(r)}\right)\Delta x\right]=\frac{\sin[\pi(f-p)N\Delta x]}{\sin[\pi(f-p)\Delta x]}=N\,\mathrm{sincd}[N;\pi(f-p)N\Delta x];$$

(for $u^{(r)}=-(N-1)/2$)

Theorem 1. *Given signal sampling and reconstruction devices and the number of signal samples, overall frequency response of the digital filter is completely determined by its continuous frequency response*

$$CFR\_DF(p)=\sum_{r=0}^{N-1}\eta_r\frac{\sin\left[\pi N\left(p\Delta x-\frac{r}{N}\right)\right]}{\sqrt{N}\sin\left[\pi\left(p\Delta x-\frac{r}{N}\right)\right]}\exp\left\{i2\pi\left[\left(\frac{N-1}{2}+u^{(r)}-u^{(s)}\right)p\Delta x\right]-\left(u+\frac{N-1}{2}\right)\frac{r}{N}\right\}$$

where $\{\eta_r\}$ are coefficients of the digital filter Discrete Frequency Response DFrR (SDFT of the filter DPSF):

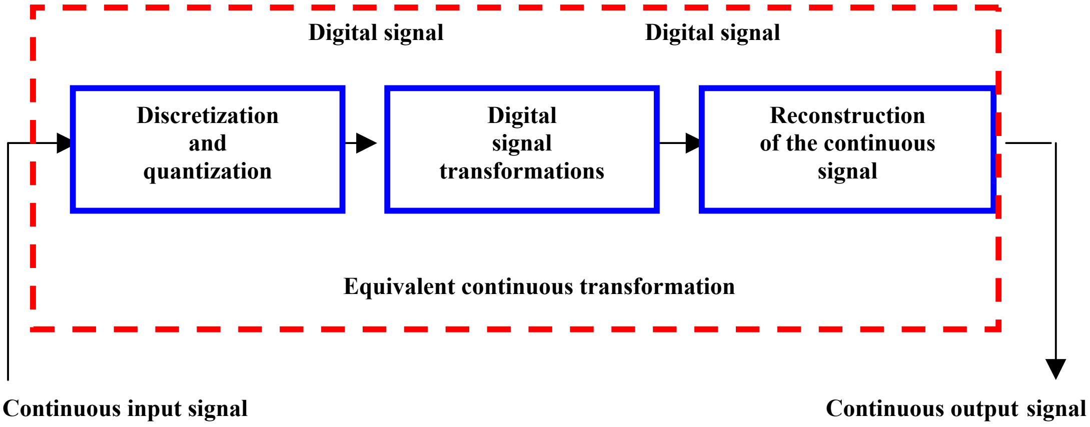
$$h_n=\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_r\exp\left[-i2\pi\frac{(n+u)r}{N}\right]$$

With settings $u=-(N-1)/2$; $u^{(s)}=0$, $CFR\_DF(p)=\sqrt{N}\sum_{r=0}^{N-1}\eta_r\,\mathrm{sincd}\left(N;p\Delta x-\frac{r}{N}\right)$

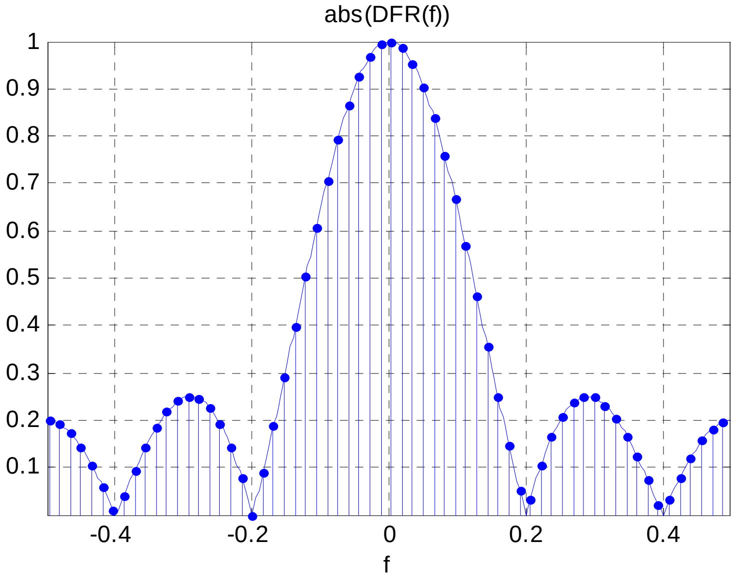
Theorem 2. *Discrete Frequency Response coefficients of the digital filter are samples of its Continuous Frequency Response CFR_DF taken with a sampling interval* $1/N\Delta x$

and

Theorem 3 *CFR_DF of the digital filter is a discrete sinc-interpolated function of its samples*

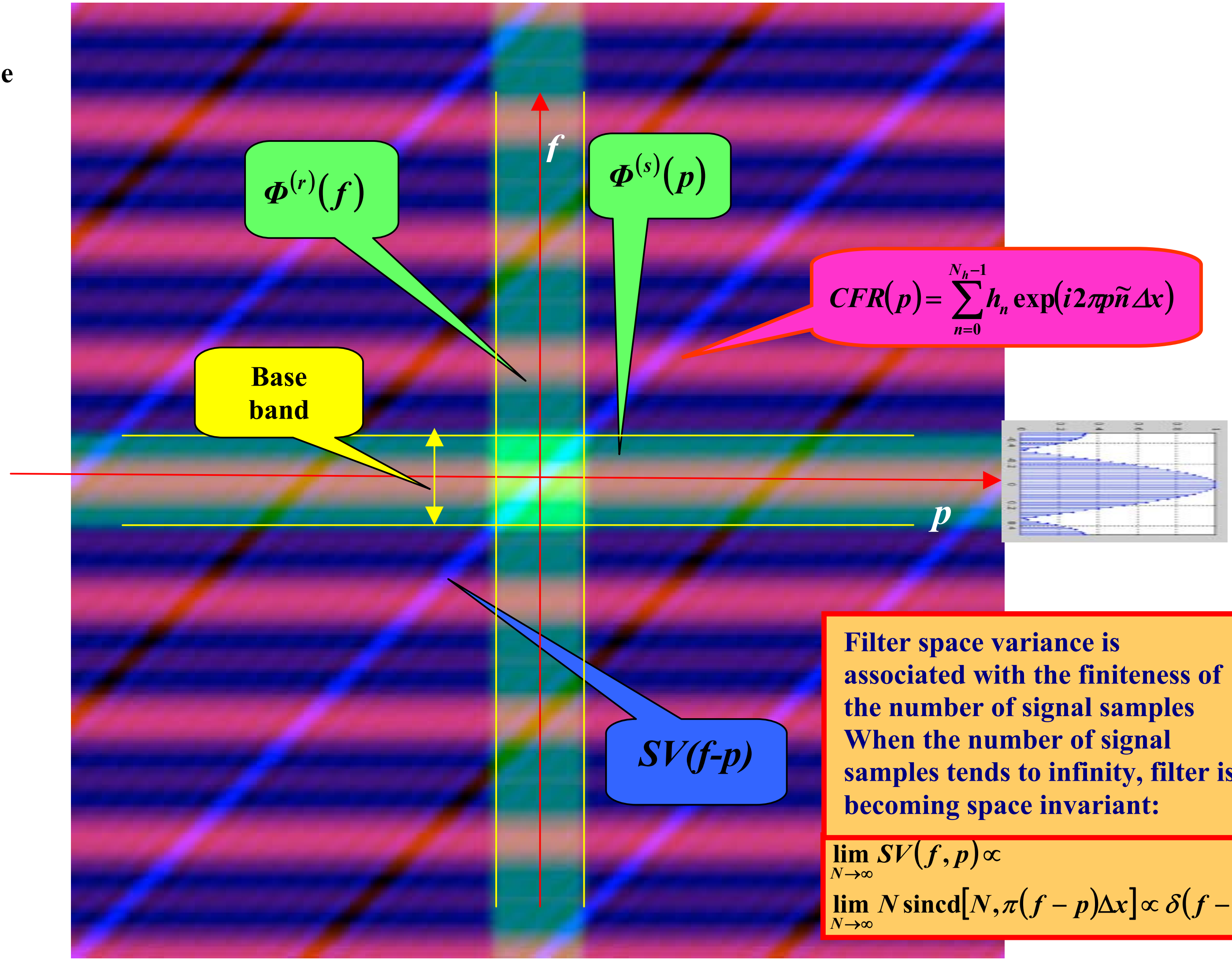


Mutual correspondence principle between continuous and digital signal transformations



Continuous frequency response and its samples (marked by circles) of a digital filter that computes signal local mean over 5 samples

Overall frequency response of a digital filter



$\Phi^{(r)}(f)$

$\Phi^{(s)}(p)$

$CFR(p) = \sum_{n=0}^{N_h-1} h_n \exp(i2\pi p\tilde{n}\Delta x)$

$SV(f\text{-}p)$

Filter space variance is associated with the finiteness of the number of signal samples When the number of signal samples tends to infinity, filter is becoming space invariant:

$$\lim_{N\to\infty} SV(f,p) \propto$$
$$\lim_{N\to\infty} N\,\mathrm{sincd}[N,\pi(f-p)\Delta x] \propto \delta(f-p)$$

2.3 Discrete Fourier Transforms

$$\alpha(f)=\int_{-\infty}^{\infty} a(x)\exp(i2\pi fx)dx \Rightarrow \alpha_r=\sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x\Delta f]\times$$

$$\int_{-\infty}^{\infty}\Phi^{(r)}[f+(r+v)\Delta f]\varphi^{(s)}(f)\exp[i2\pi f(k+u)\Delta x]df \propto \sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x\Delta f]$$

Cardinal sampling $\Delta x = 1/N\Delta f$; no sample grids shifts $\Rightarrow$

Canonical DFT and IDFT: $\alpha_r = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left(i2\pi\frac{kr}{N}\right)$; $a_k = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \alpha_r \exp\left(-i2\pi\frac{kr}{N}\right)$

Cardinal sampling $\Delta x = 1/N\Delta f$; sample grids shifts (u,v) $\Rightarrow$

Shifted DFTs

$$\alpha_r^{u,v} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+u)(r+v)}{N}\right] \quad a_k^{u,v} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \alpha_r^{u,v} \exp\left[-i2\pi\frac{(k+u)(r+v)}{N}\right]$$

A reduced version of SDFT: $\alpha_r^{u,v} = \frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k \exp\left(i2\pi\frac{kv}{N}\right)\right]\exp\left(i2\pi\frac{kr}{N}\right)\right\}\exp\left(i2\pi\frac{ur}{N}\right)$

Special cases of SDFT(1/2,0) for even-odd signals $\{a_k = \pm a_{2N-1-k}\}$:

DCT: $\alpha_r^{DCT} = \sum_{k=0}^{N-1} a_k \cos\left(\pi\frac{k+1/2}{N}r\right)$; DcST: $\alpha_r^{DcST} = \sum_{k=0}^{N-1} a_k \sin\left(\pi\frac{k+1/2}{N}r\right)$;

Other special cases of SDFTs: DCT(I-IV); DST(I-IV);

Sampling in σ -scaled coordinates (over/under sampling: $\Delta x = 1/\sigma N\Delta f$) ,no sampling grid shifts:

Scaled DFT: $\alpha_r^{\sigma} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+u)(r+v)}{\sigma N}\right]$;

Inverse ScDFT exists only if $\sigma N \in \mathbf{Z}$: $a_k^{\sigma} = \frac{1}{\sqrt{N}}\sum_{r=0}^{\sigma N-1} \alpha_r^{\sigma} \exp\left[-i2\pi\frac{(k+u)(r+v)}{\sigma N}\right]$

Computing ScDFT through the canonical DFT

$$\alpha_r^{\sigma} = IDFT\left\{DFT\left\{a_k \exp\left(i\pi\frac{k^2}{\sigma N}\right)\right\}\bullet DFT\left\{\exp\left(-i\pi\frac{k^2}{\sigma N}\right)\right\}\right\}\bullet \exp\left(i\pi\frac{r^2}{\sigma N}\right)$$

2-D DFTs:

Cardinal sampling, no sampling grid shifts:

Canonic separable 2-D DFT: $\alpha_{r,s} = \frac{1}{\sqrt{N_1 N_2}}\sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1} a_{k,l}\exp\left[i2\pi\left(\frac{kr}{N_1}+\frac{ls}{N_1}\right)\right]$

Sampling in affine transformed coordinate system: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$:

Affine DFT (AffDFT): $\alpha_{r,s} = \sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1} a_{k,l}\exp\left[i2\pi\left(\frac{rk}{\sigma_A N_1}+\frac{sk}{\sigma_C N_1}+\frac{rl}{\sigma_B N_2}+\frac{sl}{\sigma_D N_2}\right)\right]$;

$\sigma_A = 1/N_1 A\Delta\tilde{x}\Delta f_x$; $\sigma_B = 1/N_2 B\Delta\tilde{y}\Delta f_x$; $\sigma_C = 1/N_1 C\Delta\tilde{x}\Delta f_y$ $\sigma_D = 1/N_2 D\Delta\tilde{y}\Delta f_y$

Rotated DFT (RotDFT,):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \Rightarrow \alpha_{r,s} = \frac{1}{\sigma N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}\exp\left[i2\pi\left(\frac{r\cos\theta - s\sin\theta}{\sigma N}k + \frac{r\sin\theta + s\cos\theta}{\sigma N}l\right)\right]$$

Discrete Fourier Transforms

Transform	
Canonical Discrete Fourier Transform (DFT)	$\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N} \right)$
Shifted DFT	$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{N} \right]$
Discrete Cosine Transform (DCT)	$\alpha_r^{DCT} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r \right)$
Discrete Cosine-Sine Transform (DcST)	$\alpha_r^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left(\pi \frac{k+1/2}{N} r \right)$
Scaled DFT	$\alpha_r^{\sigma} = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{\sigma N} \right] = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{\sigma N} \right)$
Scaled DFT as a cyclic convolution	$\alpha_r^{\sigma} = \frac{\exp\left(i\pi \frac{\tilde{r}^2}{\sigma N} \right)}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} \left[a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N} \right) \right] \exp\left[-i\pi \frac{(\tilde{k}-\tilde{r})^2}{\sigma N} \right]$
Canonical 2-D DFT	$\alpha_{r,s} = \frac{1}{\sqrt{N_1 N_2}} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{kr}{N_1} + \frac{ls}{N_1} \right) \right]$
Affine DFT	$\alpha_{r,s} = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk}{\sigma_A N_1} + \frac{sk}{\sigma_C N_1} + \frac{rl}{\sigma_B N_2} + \frac{sl}{\sigma_D N_2} \right) \right]$
Rotated DFT (RotDFT)	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r\cos\theta - s\sin\theta}{N} k + \frac{r\sin\theta + s\cos\theta}{N} l \right) \right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk+sl}{N} \cos\theta - \frac{sk-rl}{N} \sin\theta \right) \right]$
Rotated Scaled DFT	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r\cos\theta - s\sin\theta}{\sigma N} k + \frac{r\sin\theta + s\cos\theta}{\sigma N} l \right) \right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk+sl}{\sigma N} \cos\theta - \frac{sk-rl}{\sigma N} \sin\theta \right) \right]$
Discrete Sinc-function	$\mathrm{sincd}(N, x) = \frac{\sin x}{N \sin(x/N)}$

Varieties of Discrete Cosine Transforms

DCT-1	$\alpha_r = \frac{a_0 - (-1)^r a_{N-1}}{2} + \sum_{k=1}^{N-2} a_k \cos\left(\pi \frac{kr}{N} \right)$
Canonical DCT (DCT-II)	$\alpha_r = \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r \right)$
DCT-III	$\alpha_r = \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{r+1/2}{N} k \right)$
DCT-IV	$\alpha_r = \sum_{k=0}^{N-1} a_k \cos\left[\pi \frac{(k+1/2)(r+1/2)}{N} \right]$

2.4 Point spread function and resolving power of discrete Fourier analysis.

$$\alpha_r^{\sigma,u,v} = \int_{-\infty}^{\infty} \alpha(f) PSF_{DFA}(r,f)\,df$$

$$\alpha_r^{\sigma,v_T,u_T} = \frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k \exp\left(i2\pi\frac{kv^{(T)}}{\sigma N}\right)\right]\exp\left(i2\pi\frac{kr}{\sigma N}\right)\right\}\exp\left(i2\pi\frac{ru^{(T)}}{\sigma N}\right) =$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty} a(x)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left(i2\pi\frac{kv^{(T)}}{\sigma N}\right)\exp\left(i2\pi\frac{kr}{\sigma N}\right)\right\}\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right) =$$

$$\frac{1}{\sqrt{N}}\int_{-\infty}^{\infty}\alpha(f)\left\{\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty}\exp(-i2\pi fx)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left[i2\pi\frac{k\left(r+v^{(T)}\right)}{\sigma N}\right]\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right)\right\}df$$

$$PSF_{DFA} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty}\exp(-i2\pi fx)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left[i2\pi\frac{k\left(r+v^{(T)}\right)}{\sigma N}\right]\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right)\Rightarrow$$

$$PSF_{DFA} = \sqrt{N}\,\text{sincd}\left[N;\pi\left(\frac{r+v^{(T)}}{\sigma}-fN\Delta x\right)\right]\times$$

$$\Phi^{(s)}(f)\exp\left\{i2\pi\left[\left(u^{(T)}+\frac{N-1}{2}\right)\frac{r}{\sigma N}-\left(u^{(s)}+\frac{N-1}{2}\right)f\Delta x+\frac{(N-1)v^{(T)}}{2\sigma N}\right]\right]\right\}$$

2.5 Boundary effect free convolution in DCT domain.

$$\tilde{a}_{(k)\bmod 2N} = \begin{cases} a_k, & k = 0,1,\ldots,N-1; \\ a_{2N-k-1}, & k = N, N = 1,\ldots,2N-1 \end{cases}$$

$$\tilde{\alpha}_r = \frac{1}{\sqrt{2N}}\sum_{k=0}^{2N-1}\tilde{a}_k \exp\left(i2\pi\frac{kr}{2N}\right) = \alpha_r^{(DCT)}\exp\left(-i\pi\frac{r}{2N}\right);$$

$$\tilde{h}_{(n)\bmod 2N} = \begin{cases} 0, & n = 0,\ldots,[N/2]-1 \\ h_{n-[N/2]}, & n = [N/2],\ldots,[N/2]+N-1 \\ 0, & n = [N/2]+N,\ldots,2N-1 \end{cases};\quad \left[\frac{N}{2}\right] = \begin{cases} N/2, & \text{for even } N \\ (N-1)/2, & \text{for odd } N \end{cases}$$

$$\eta_r = \frac{1}{\sqrt{2N}}\sum_{n=0}^{2N-1}\tilde{h}_n \exp\left(i2\pi\frac{nr}{2N}\right)$$

$$b_k = \frac{1}{\sqrt{2N}}\sum_{r=0}^{2N-1}\alpha_r^{(DCT)}\exp\left(-i\pi\frac{r}{2N}\right)\eta_r \exp\left(-i2\pi\frac{kr}{2N}\right) =$$

$$\frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\eta_0 + \sum_{r=1}^{N-1}\alpha_r^{(DCT)}\left[\eta_r \exp\left(-i2\pi\frac{k+1/2}{2N}r\right)+\eta_r^{*}\exp\left(i2\pi\frac{k+1/2}{2N}r\right)\right]\right]\right\}\right\}\Rightarrow$$

$$b_k = \frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\eta_0 + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\eta_r^{re}\cos\left(\pi\frac{k+1/2}{N}r\right) - 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\eta_r^{im}\sin\left(\pi\frac{k+1/2}{N}r\right)\right\}$$

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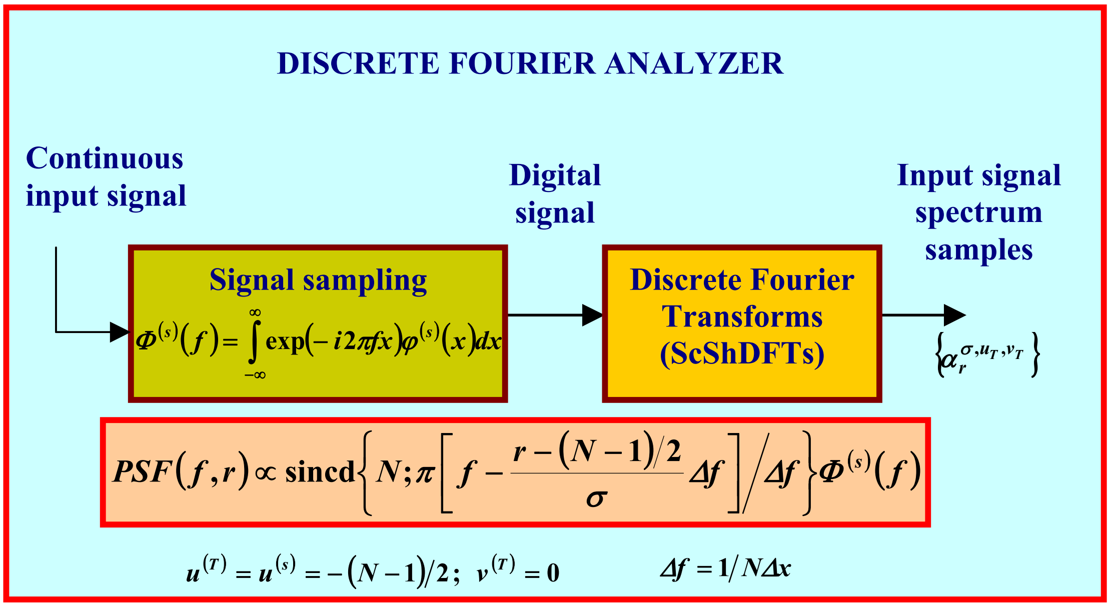


DISCRETE FOURIER ANALYZER

Continuous input signal

Signal sampling

$$\Phi^{(s)}(f)=\int_{-\infty}^{\infty}\exp(-i2\pi fx)\varphi^{(s)}(x)dx$$

Digital signal

Discrete Fourier Transforms (ScShDFTs)

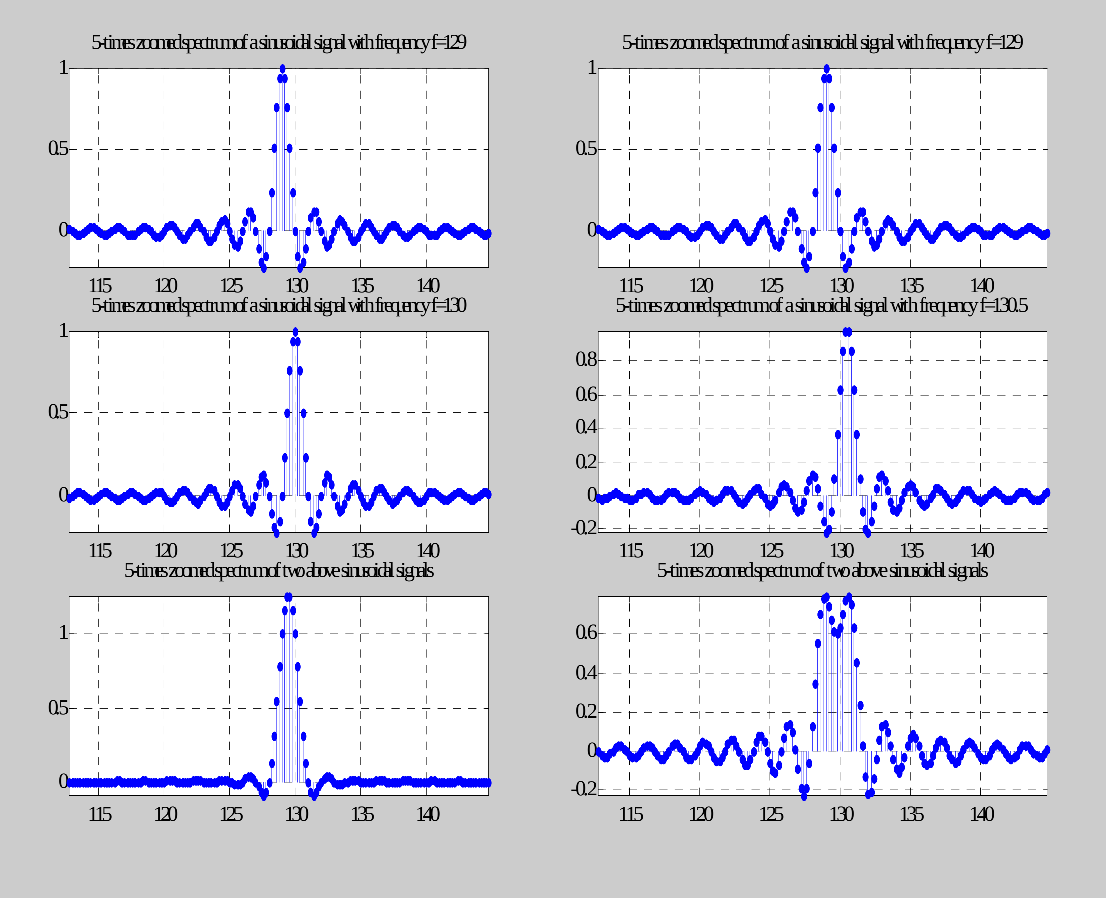
Input signal spectrum samples

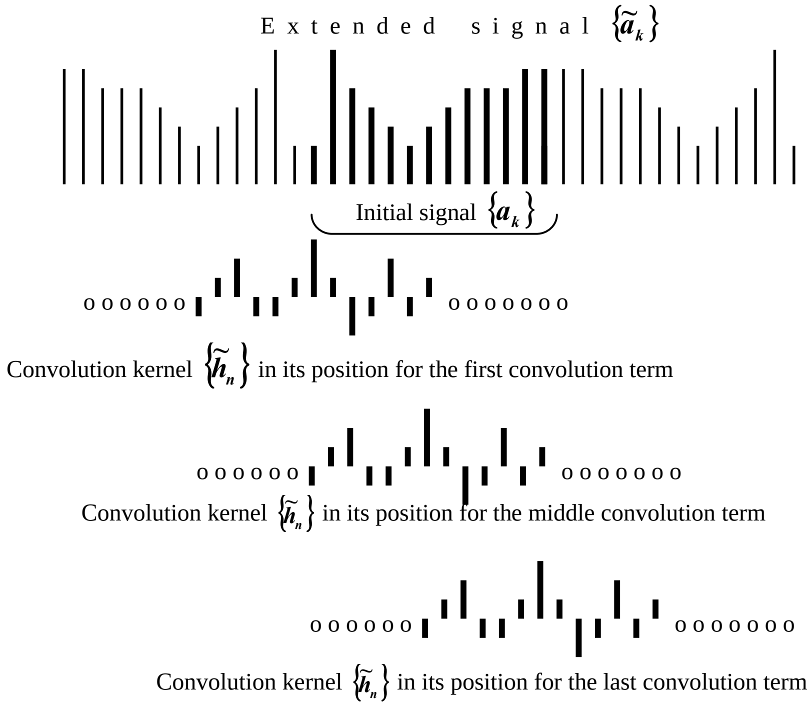
$$\left\{\alpha_r^{\sigma,u_T,v_T}\right\}$$

$$PSF(f,r)\propto \mathrm{sincd}\left\{N;\pi\left[f-\frac{r-(N-1)/2}{\sigma}\Delta f\right]\Big/\Delta f\right\}\Phi^{(s)}(f)$$

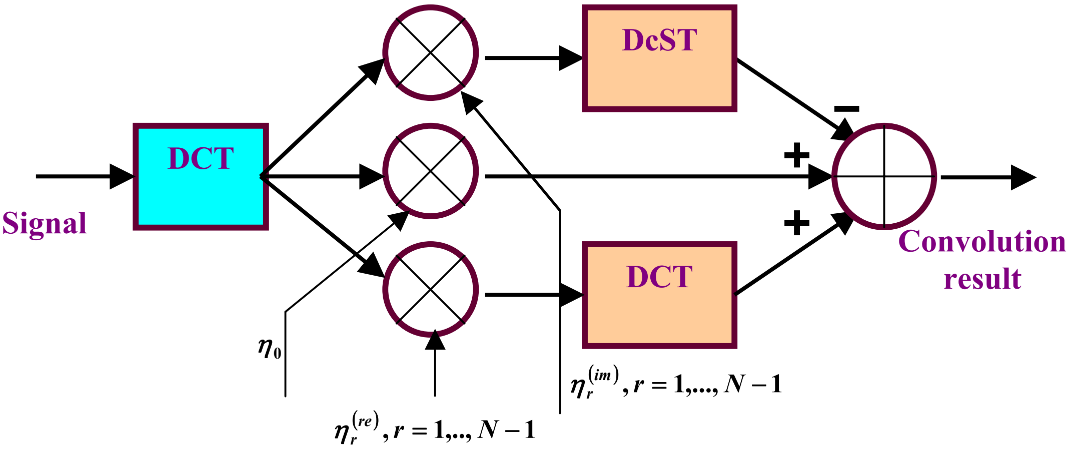
$$u^{(T)}=u^{(s)}=-(N-1)/2;\; v^{(T)}=0 \qquad \Delta f=1/N\Delta x$$

Resolving power of discrete Fourier analysis





Cyclic convolution of a signal extended by its mirror reflection from its borders



$$\eta_r = \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} \tilde{h}_n \exp\left(i2\pi \frac{nr}{2N} \right)$$

Flow chart of an algorithm for signal convolution in DCT domain

2.6 Discrete Fresnel Transform.

$$\alpha(f)=\int_{-\infty}^{\infty} a(x)\exp\left[-i\pi\frac{(f-x)^2}{\lambda D}\right]dx \Rightarrow \alpha_r=\sum_{k=0}^{N-1} a_k \exp\left[-i\pi\left(k\Delta x-r\Delta f+u\Delta x-v\Delta f\right)^2/\lambda D\right]\times$$

$$\int_{-\infty}^{\infty}\left[\exp\left(-i\pi x^2/\lambda D\right)\varphi^{(r)}\left(x/\sqrt{\lambda D}\right)\right]\exp\left[-i2\pi x\left(k\Delta x-r\Delta f+u\Delta x-v\Delta f\right)/\lambda D\right]dx\times$$

$$\int_{-\infty}^{\infty}\left[\exp\left(-i\pi f^2/\lambda D\right)\varphi^{(s)}\left(f/\sqrt{\lambda D}\right)\right]\exp\left[i2\pi f\left(x+k\Delta x-r\Delta f+u\Delta x-v\Delta f\right)/\lambda D\right]df$$

For discrete representation of Fresnel Transform, two last terms are neglected.
For cardinal sampling ($\Delta x=\lambda D/N\Delta f$), no sampling grid shifts:

Canonical DFrT $\alpha_r=\sum_{k=0}^{N-1} a_k \exp\left[-i\pi\frac{(k\mu-r/\mu)^2}{N}\right]$; $\mu^2=\lambda D/N\Delta f^2$

DFrT via DFT: $\alpha_r=\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k\exp\left(i\pi\frac{k^2}{\mu^2 N}\right)\right]\exp\left(-i2\pi\frac{kr}{N}\right)\right\}\exp\left(i\pi\frac{r^2\mu^2}{N}\right)$

Sampling in σ-scaled coordinates : $\Delta x=\lambda D/\sigma N\Delta f$, with shifts $\{u\Delta x, v\Delta f\}$ in coordinate systems collinear with those of signal and its transform:

Shifted Scaled DFrT: $\alpha_r=\sum_{k=0}^{N-1} a_k\exp\left[-i\pi\frac{(k\mu-r/\mu+w)^2}{\sigma N}\right]$, where; $w=u\mu-v/\mu$

Shifted Scaled Partial ShScPDFrT (transform domain chirp-function is ignored):

$$\alpha_r=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left[a_k\exp\left(i\pi\frac{k^2}{\mu^2\sigma N}\right)\right]\exp\left[-i2\pi\frac{k(r-w\mu)}{\sigma N}\right]$$

Cardinal sampling with shifts $w=u\mu-v/\mu=N/2\mu$ in coordinate systems collinear with those of signal and its transform:

Focal plane invariant DFrT: $\alpha_r=\sum_{k=0}^{N-1} a_k\exp\left\{-i\pi\frac{\left[k\mu-(r-N/2)/\mu\right]^2}{N}\right\}$

Invertibility of DFrT and discrete frinc-function:

$$a_k^{(\mu^\pm, w^\pm)}=\frac{1}{N}\exp\left[-i\pi\frac{(k\mu_-+w_-)^2}{N}\right]\sum_{n=0}^{N-1} a_n\exp\left[i\pi\frac{(n\mu_++w_+)^2}{N}\right]\mathbf{frincd}\left(N;q;n-k+\overline{w}_\pm+qN/2\right)$$

$$\mathbf{frincd}(N;q;x)=\frac{1}{N}\sum_{r=0}^{N-1}\exp\left(i\pi\frac{qr^2}{N}\right)\exp\left(-i2\pi\frac{xr}{N}\right);\ q=1/\mu_+^2-1/\mu_-^2;\ \overline{w}_\pm=\overline{w}_+/\mu_+-\overline{w}_-/\mu_-$$

In DFrT $\mathbf{frincd}(N;q;x)$ plays a role that $\mathbf{sincd}(N,x)$ plays in DFT.

$\mathbf{frincd}(N;0;x)\propto\mathbf{sincd}(N,x)$

2.7 Convolutional Discrete Fresnel Transform.

$$\int_{-\infty}^{\infty} a(x)\exp\left[-i\pi\frac{(x-f)^2}{\lambda D}\right]dx \propto \int_{-\infty}^{\infty}\left\{\left[\int_{-\infty}^{\infty} a(x)\exp\left(i2\pi\frac{px}{\lambda D}\right)dx\right]\exp\left(-i\pi\frac{p^2}{\lambda D}\right)\right\}\exp\left(-i2\pi\frac{pf}{\lambda D}\right)dp$$

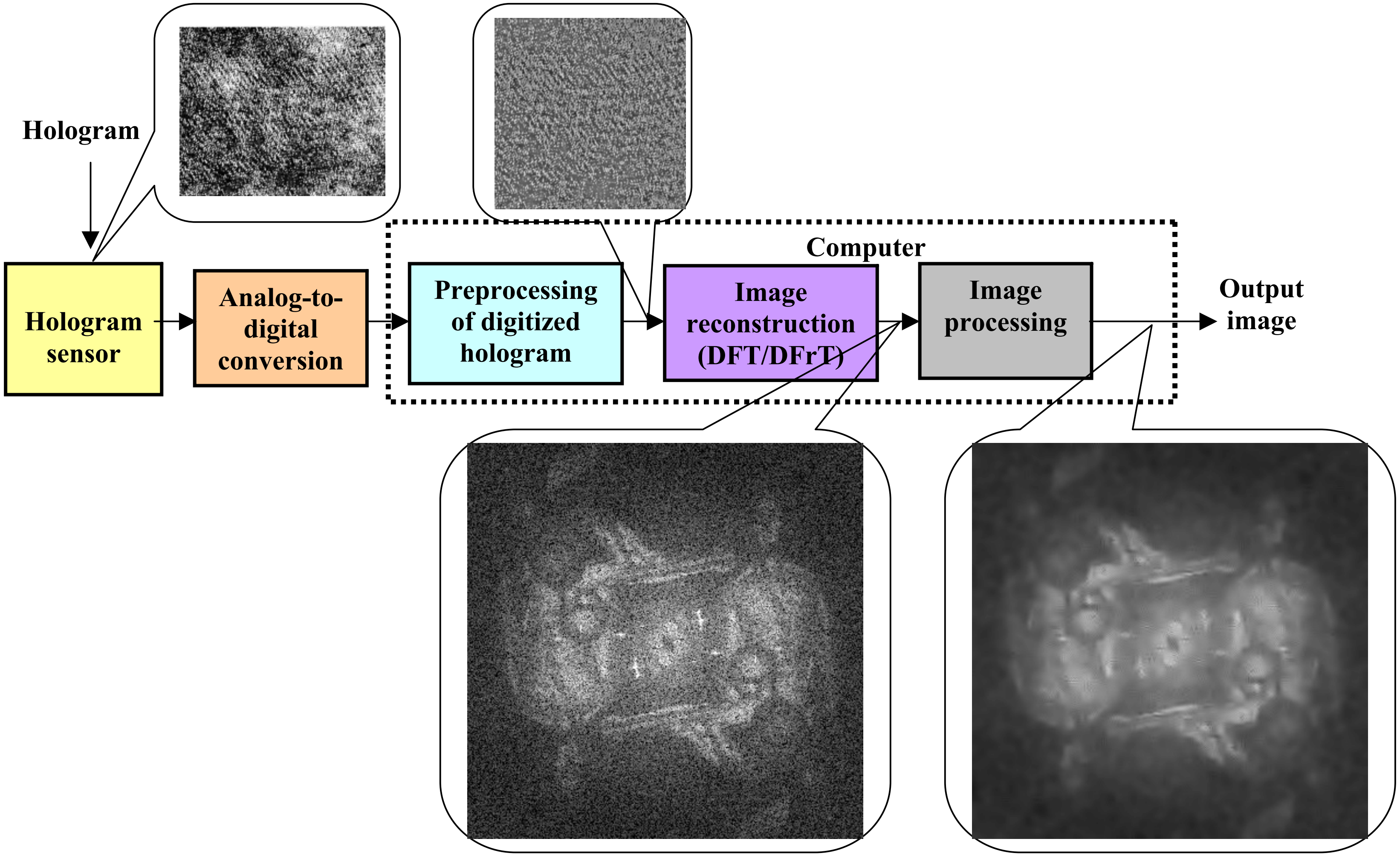
Assuming sampling grid shifts $\{u\Delta x, v\Delta f\}$ and the same sampling intervals in signal and transform domains, $\Delta x=\Delta f \Rightarrow$

Convolutional Discrete Fresnel Transform:

$$\alpha_r=\frac{1}{N}\sum_{s=0}^{N-1}\left[\sum_{k=0}^{N-1} a_k\exp\left(i2\pi\frac{k-r-w}{N}s\right)\right]\exp\left(-i\pi\frac{\mu^2 s^2}{N}\right)=\sum_{k=0}^{N-1} a_k\,\mathbf{frincd}\left(N;\mu^2;r+w-k\right)$$

$w=u\mu-v/\mu$

Principle of numerical reconstruction of digitally recorded holograms



Hologram

Hologram sensor

Analog-to-digital conversion

Computer

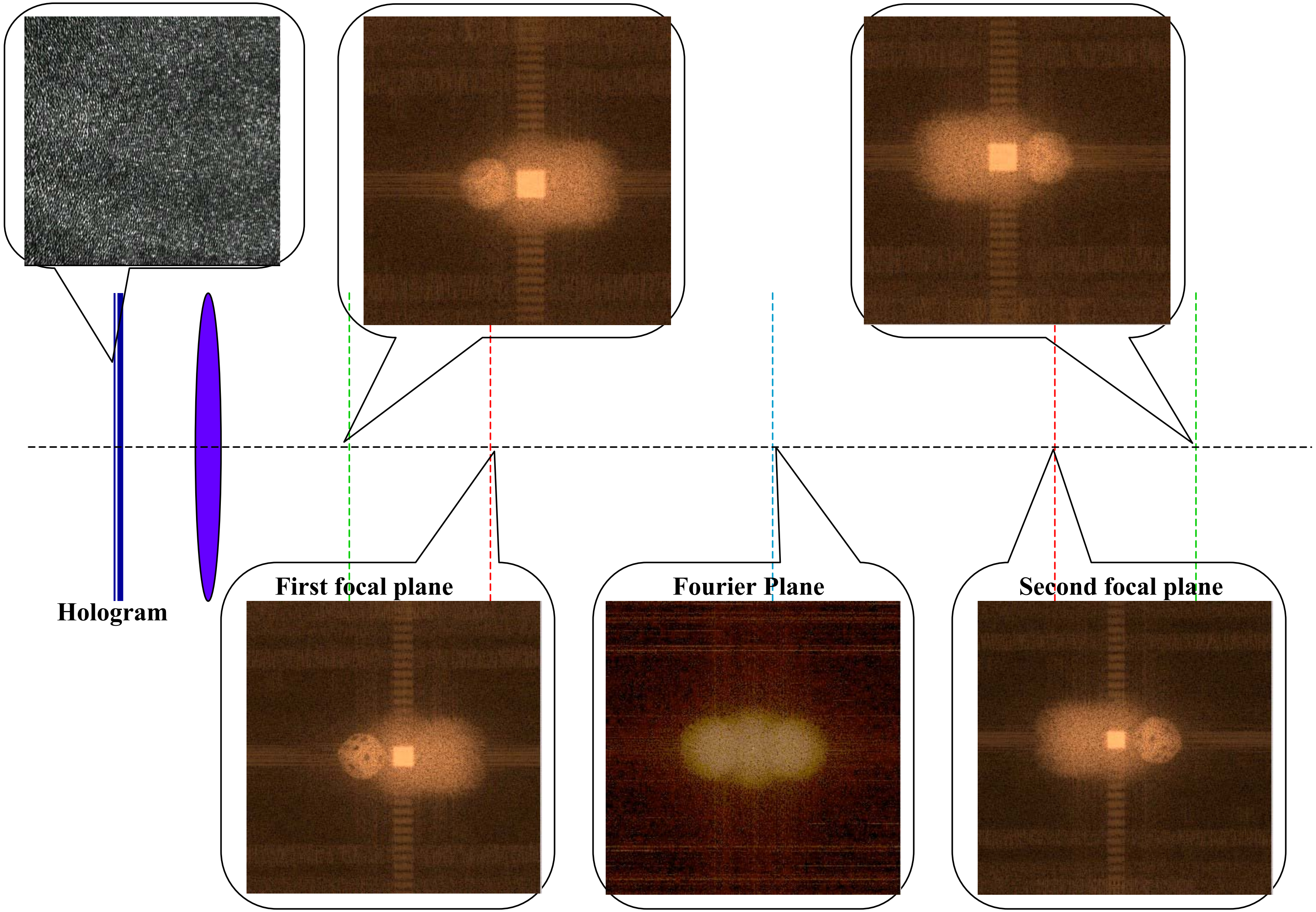
Preprocessing of digitized hologram

Image reconstruction (DFT/DFrT)

Image processing

Output image

Reconstruction of hologram recoreded in near diffraction zone (Fresnel holograms): equivalent optical setup



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hologr_reconstr_fastmovie(N);
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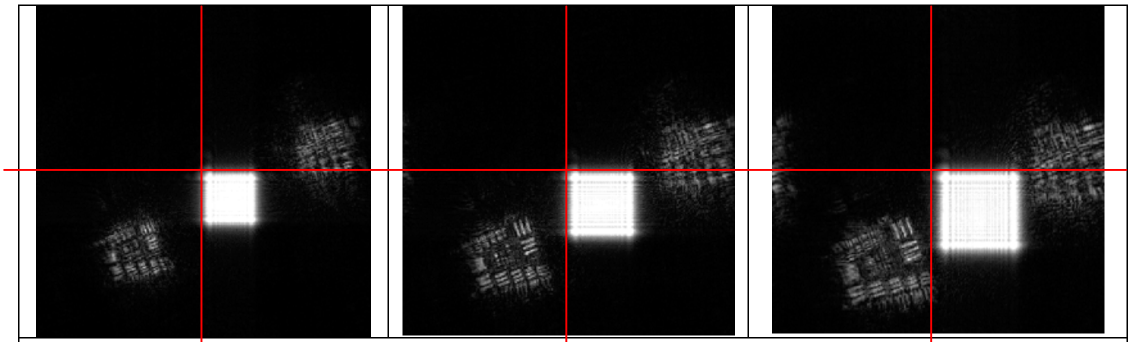
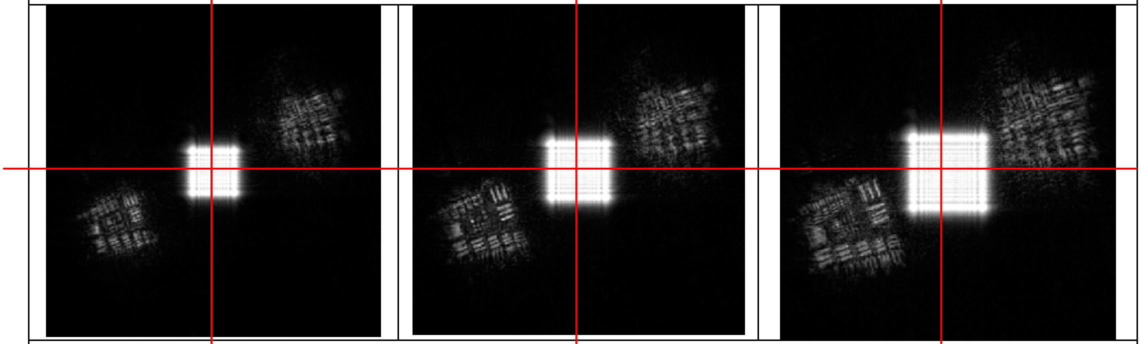
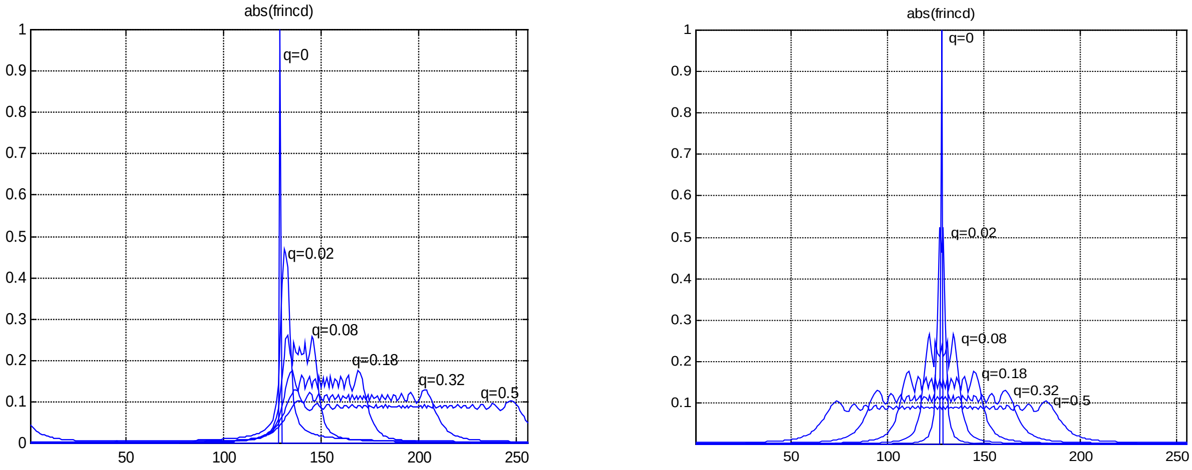


Numerical reconstruction of images on different distances from a hologram using canonic DFrT



Numerical reconstruction of images on different distances from a hologram using Focal plane invariant DFrT



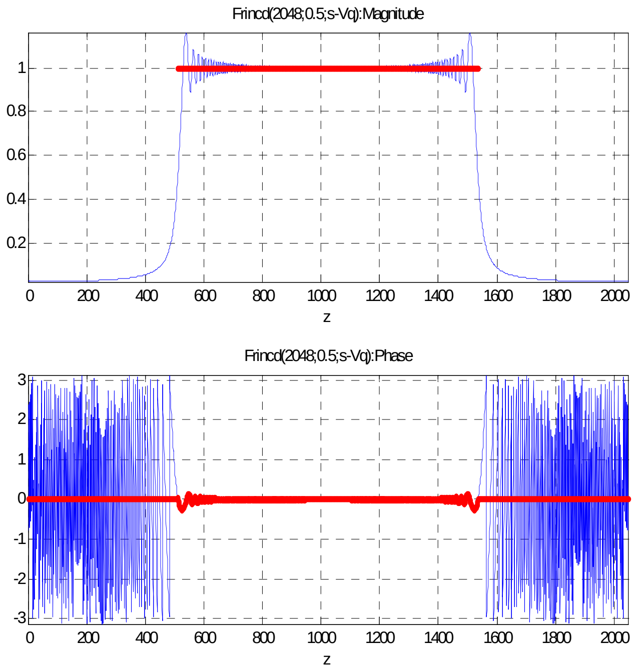
a) b)

Focal plane variant and focal plane invariant discrete frinc-function:

Approximation of the discrete frinc-function

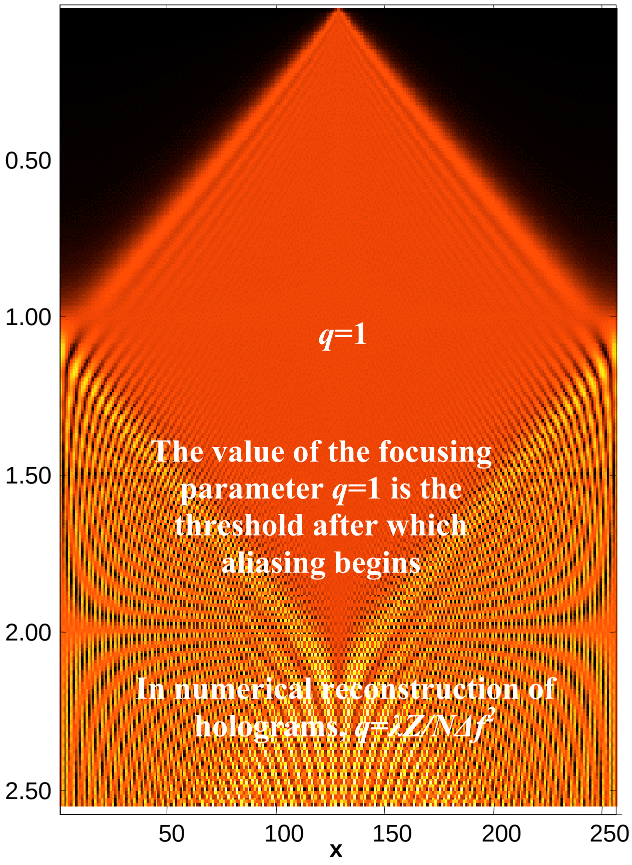
$$\mathbf{frincd}(N;q;x)=\frac{1}{N}\sum_{r=0}^{N-1}\exp\left(i\pi\frac{qr^2}{N}\right)\exp\left(-i2\pi\frac{xr}{N}\right)\cong\sqrt{\frac{i}{Nq}}\exp\left(-i\pi\frac{r^2}{qN}\right)\mathrm{rect}\left[\frac{r}{q(N-1)}\right]$$

For integer r, $\mathbf{frincd}(N;1;r)=\sqrt{\frac{i}{N}}\exp\left(-i\pi\frac{r^2}{N}\right)$


Frincd(2048;0.5;s-Vq):Magnitude

Frincd(2048;0.5;s-Vq):Phase

Frincd(256,q,x) for q=0:0.01:2.56


q=1

The value of the focusing parameter q=1 is the threshold after which aliasing begins

In numerical reconstruction of holograms, $q=\lambda Z/N\Delta f^2$