L. Yaroslavsky. Fundamentals of Digital Image Processing. Course 0555.3230

Lecture 2. Elements of the theory of 2D signal processing.

Mathematical models of image signals. Signals as mathematical functions. 1-D, 2-D, and multidimensional signals. Continuous signals, discrete signals, quantized signals, digital signals. Scalar and vectorial signals.

Linear signal space, metrics and bases.

The signal space is called linear if:

1. An operation of signal summation $(+)$ is defined such that, for any two signals $a^{(1)}$ and $a^{(2)}$, a third signal $a^{(3)} = a^{(1)} + a^{(2)}$ can be uniquely found.
2. The summation operation obeys the commutative and associative laws:
$$a^{(1)} + a^{(2)} = a^{(2)} + a^{(1)}$$
$$a^{(1)} + a^{(2)} + a^{(3)} = \left(a^{(1)} + a^{(2)}\right) + a^{(3)} = a^{(1)} + \left(a^{(2)} + a^{(3)}\right)$$
3. There exists a "zero-signal" $\varnothing$ such that $a + \varnothing = a$ for all signals of the space.
4. Each element a of the space may be put into the correspondence with a unique opposite signal $(-a)$ such that
$$a + (-a) = a - a = \varnothing .$$
5. An operation $(*)$ of multiplying signal by a scalar number is defined such that, for any number α and any signal a, a signal $\alpha * a$ can be uniquely found.
6. The following properties hold for the multiplication operation:
$$\alpha_1 \alpha_2 a = \alpha_1 (\alpha_2 a) = \alpha_2 (\alpha_1 a);$$
$$(\alpha_1 + \alpha_2) a = \alpha_1 a + \alpha_2 a; \quad \alpha (a_1 + a_2) = \alpha a_1 + \alpha a_2;$$
$$1 * a = a; \quad 0 * a = \varnothing;$$

In linear signal space, one can specify a set of N signals and use them to generate an arbitrary large set of other signals as a linear combination (weighted sum)

$$a = \sum_{k=0}^{N-1} \alpha_k \varphi_k$$

of the selected signals that are called basis signals (***basis functions)***. The set of basis functions should be selected from functions that can't be represented as a linear combination of the rest of the functions from the set, otherwise the set of basis functions will be redundant.

Each signal a in N-dimensional signal space corresponds to a unique linear combination of basis functions $\{\varphi_k\}$, i.e. is defined by a unique set of scalar coefficients $\{\alpha_k\}$. An (ordered) set of scalar coefficients of decomposition of a given signal over the given basis is the ***signal's representation*** with respect to this basis.

The "zero-signal" may be treated as a reference signal of the signal space. The distinction of signal a from the "zero-signal" displays the individual characteristic of this signal. Mathematically, this characteristic is described through the concept of the signal's norm $\|a\|$:

$$\|a\| = d(a, \varnothing)$$

The geometrical interpretation of the vector norm is its length.

Signal representation as an expansion over a set of basis functions is meaningful only when for each signal a, its representation $\{\alpha_k\}$ for the given basis φ_k can be found. To this goal, the concept of the ***scalar product*** (***inner product***, ***dot product***) (a_1, a_2) of two signals is introduced as a method for computing a (generally, complex) number that satisfies the following conditions:

$$(a_1, a_2) = (a_1, a_2)^*; \quad (a,a) \geq 0; \quad (a,a) = 0, \quad \textit{iff } a = \varnothing;$$
$$(\alpha_1 a_1 + \alpha_2 a_2, a_3) = \alpha_1 (a_1, a_3) + \alpha_2 (a_2, a_3).$$

Usually the scalar product of two signals is calculated as an integral over their product:

$$(a_1, a_2) = \int_X a_1(x) a_2^*(x) dx$$

Functions whose scalar product is zero are called mutually ***orthogonal functions***. If functions are mutually orthogonal, then they are linearly-independent may be used as bases of linear signal spaces.

Scalar product can used to establish a correspondence between signals and their representation for the given basis. For $\{\varphi_k\}$ the basis functions and $\{\phi_l\}$ the reciprocal set of functions orthogonal to $\{\varphi_k\}$ normalized such that

$$(\varphi_k, \phi_l) = \delta_{k,l} = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases}$$

(symbol $\delta_{k,l}$ is called the ***Kronecker delta***)., signal representation coefficients $\{\alpha_k\}$ over basis $\{\varphi_k\}$ can be found as

$$\alpha_k = (a, \phi_k) = \sum_{l=0}^{N-1} \alpha_l (\varphi_l, \phi_k) = \sum_{l=0}^{N-1} \alpha_l \delta_{k,l} \ .$$

If basis $\{\varphi_k\}$ is composed of mutually orthogonal functions of unit norm: $(\varphi_k, \varphi_l) = \delta_{k,l}$, it is reciprocal to itself and is called ***orthonormal basis***. In such a basis signals are representable as

$$a = \sum_{k=0}^{N-1} (a, \varphi_k) \varphi_k \text{ , where } (a, \varphi_k) = \alpha_k = \int_X a(x) \varphi_k^*(x) dx \ .$$

Knowing the signal representation over an orthonormal basis, one can calculate signal scalar product

$$(a,b) = \left(\sum_{k=0}^{N-1} \alpha_k \varphi_k, \sum_{k=0}^{N-1} \beta_k \varphi_k \right) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} (\alpha_k \varphi_k, \beta_l \varphi_l) = \sum_{k=0}^{N-1} \alpha_k \beta_k^*$$

Signal norm is defined as scalar product of the signal with its complex conjugate:

$$\|a\| = \sum_{k=0}^{N-1} |\alpha_k|^2$$

Representations $\{\alpha_k^{(1)}\}$, $\{\beta_k^{(1)}\}$ and , $\{\alpha_k^{(2)}\}$, $\{\beta_k^{(2)}\}$ of signals a and b over two orthonormal bases $\{\varphi_k^{(1)}\}$ and $\{\varphi_k^{(2)}\}$ are related by means of the ***Parseval's relationship***

$$\sum_{k=0}^{N-1} \alpha_k^{(1)} \left(\beta_k^{(1)}\right)^* = \sum_{k=0}^{N-1} \alpha_k^{(2)} \left(\beta_k^{(2)}\right)^* .$$

Signal norm is invariant to the orthonormal bases.

Integral representation of signals

The discrete representation of signals can be extended to a continuous representation

$$a(x) = \int_F \alpha(f) \varphi(x, f) df$$

The function $\alpha(f)$ is called the ***integral transform*** of signal $a(x)$, or its ***spectrum***, over continuous basis $\{\phi(f, x)\}$ called ***transform kernel***. The transform is reversible if a reciprocal kernel $\phi(f, x)$ exist such that

$$\alpha(f) = \int a(x) \phi(f, x) dx$$

The reciprocity condition for functions $\varphi(x, f)$ and $\phi(f, x)$:

$$a(x) = \int_F \left[\int_X a(\xi) \phi(f, \xi) d\xi \right] \varphi(x, f) df = \int_X a(\xi) \delta(x, \xi) d\xi \text{ , where } \delta(x, \xi) = \int_F \varphi(x, f) \phi(f, \xi) df \text{ . is}$$

called the ***delta-function*** (Dirac delta). A basis satisfying the relationship

$$\int_F \varphi(x,f)\varphi^*(\xi,f)\,df = \delta(x,\xi)$$

is called ***self-conjugate***. For self-conjugate bases, signal scalar product is invariant to the bases

$$\int_X a_1(x)a_2^*(x)\,dx = \int_F \alpha_1(x)\alpha_2^*(f)\,df \; . \; \int_X |a_1(x)|^2\,dx = \int_F |\alpha_1(x)|^2\,df$$

This is the Parseval's relation for continuous signals.

Metrics of signal space and signal norm

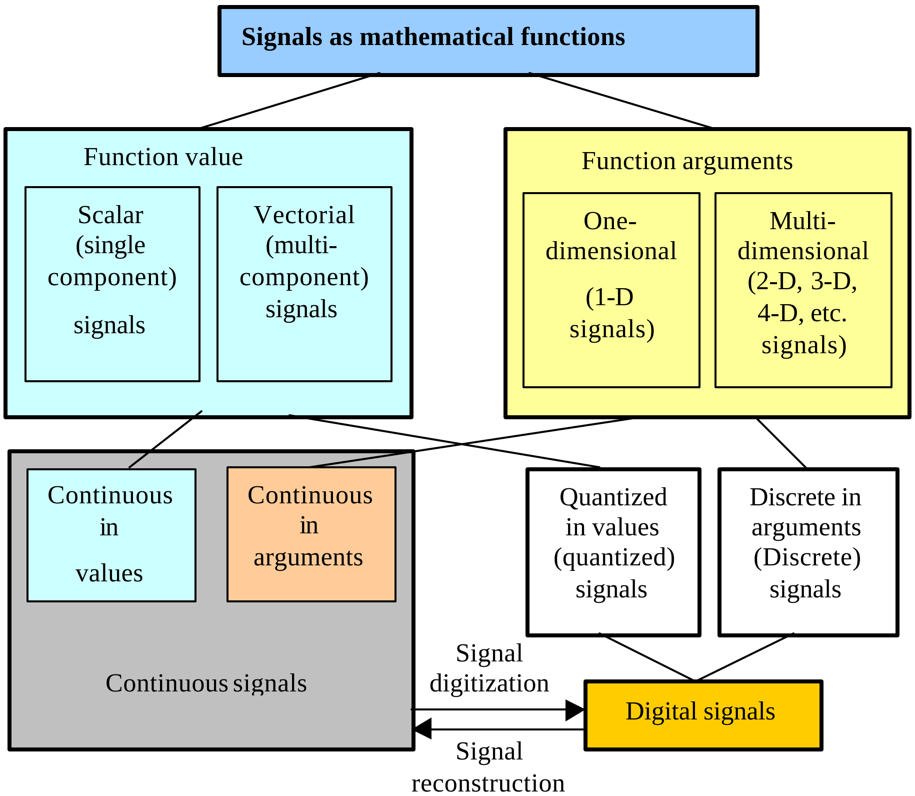
The ***signal space metrics*** associates with each pair of signals in the space, say, $a^{(1)}$ and $a^{(2)}$, a nonnegative real number $d\left(a^{(1)},a^{(2)}\right)$. The method for obtaining this number normally has the following properties:

$$d(a,a)=0\,;\; d\left(a^{(1)},a^{(2)}\right)\geq 0\,;$$
$$d\left(a^{(1)},a^{(2)}\right)=d\left(a^{(2)},a^{(1)}\right);\; d\left(a^{(1)},a^{(3)}\right)\leq d\left(a^{(1)},a^{(2)}\right)+d\left(a^{(1)},a^{(3)}\right)$$

Typical signal space metrics

Signal type	Designation	Deterministic definition	Statistical definition AV is a statistical averaging operator
Discrete signals	L_N	$\sum_{k=0}^{N-1}\left\|a_k^{(1)}-a_k^{(2)}\right\|$	$AV\left\{\left\|a_k^{(1)}-a_k^{(2)}\right\|\right\}$
	L_N^2	$\sum_{k=0}^{N-1}\left\|a_k^{(1)}-a_k^{(2)}\right\|^2$	$AV\left(\left\|a_k^{(1)}-a_k^{(2)}\right\|^2\right)$
	L_N^P	$\sum_{k=0}^{N-1}\left\|a_k^{(1)}-a_k^{(2)}\right\|^P$	$AV\left(\left\|a_k^{(1)}-a_k^{(2)}\right\|^P\right)$
	LMh_N	$\sum_{k=0}^{N-1}W_k^2\left\|a_k^{(1)}-a_k^{(2)}\right\|^2$ ($\{W_k^2\}$ are scalar weight coefficients)	-
	M_N	$\max_k\left\|a_k^{(1)}-a_k^{(2)}\right\|$	-
Continuous signals	L_x	$\int_X \left\|a_1(x)-a_2(x)\right\|dx$	$AV\left\{\left\|a_1(x)-a_2(x)\right\|\right\}$
	L_X^2	$\int_X \left\|a_1(x)-a_2(x)\right\|^2 dx$	$AV\left(\left\|a_1(x)-a_2(x)\right\|^2\right)$
	L_X^P	$\int_X \left\|a_1(x)-a_2(x)\right\|^P dx$	$AV\left(\left\|a_1(x)-a_2(x)\right\|^P\right)$
	M_X	$\sup_X\left(\left\|a_1(x)-a_2(x)\right\|\right)$, where $\sup(a(x))$ - minimal value that is not exceeded by $a(x)$	-

Statistical treatment of signals: averaging metrics measures over a set or an ensemble of signals



Signal space