

Lect. 2. Fourier integral and Discrete Fourier Transforms**2.1 Fourier integral and its discrete representation**

$$\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx) dx \Rightarrow \alpha_r = \sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x \Delta f] \times$$

$$\int_{-\infty}^{\infty} \Phi^{(r)}[f+(r+v)\Delta f] \varphi^{(s)}(f) \exp[i2\pi f(k+u)\Delta x] df \propto \sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x \Delta f]$$

Cardinal sampling $\Delta x = 1/N\Delta f$; no sample grids shifts \Rightarrow

$$\text{Canonical DFT and IDFT: } \alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N}\right); a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{N}\right)$$

Cardinal sampling $\Delta x = 1/N\Delta f$; sample grids shifts $(u, v) \Rightarrow$

$$\text{Shifted DFTs } \alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{N}\right] a_k^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_r^{u,v} \exp\left[-i2\pi \frac{(k+u)(r+v)}{N}\right]$$

$$\text{A reduced version of SDFT: } \alpha_r^{u,v} = \frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[a_k \exp\left(i2\pi \frac{kv}{N}\right) \right] \exp\left(i2\pi \frac{kr}{N}\right) \right\} \exp\left(i2\pi \frac{ur}{N}\right)$$

Special cases of SDFT(1/2,0) for even-odd signals $\{a_k = \pm a_{2N-1-k}\}$:

$$\text{DCT: } \alpha_r^{DCT} = \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r\right); \text{DcST: } \alpha_r^{DcST} = \sum_{k=0}^{N-1} a_k \sin\left(\pi \frac{k+1/2}{N} r\right);$$

Other special cases of SDFTs: DCT(I-IV); DST(I-IV);

Sampling in σ -scaled coordinates (over/under sampling: $\Delta x = 1/\sigma N\Delta f$), no sampling grid shifts:

$$\text{Scaled DFT: } \alpha_r^\sigma = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{\sigma N}\right];$$

$$\text{Inverse ScDFT exists only if } \sigma N \in \mathbb{Z}: a_k^\sigma = \frac{1}{\sqrt{N}} \sum_{r=0}^{\sigma N-1} \alpha_r^\sigma \exp\left[-i2\pi \frac{(k+u)(r+v)}{\sigma N}\right]$$

Computing ScDFT through the canonical DFT

$$\alpha_r^\sigma = \text{IDFT} \left\{ \text{DFT} \left\{ a_k \exp\left(i\pi \frac{k^2}{\sigma N}\right) \right\} \bullet \text{DFT} \left\{ \exp\left(-i\pi \frac{k^2}{\sigma N}\right) \right\} \right\} \bullet \exp\left(i\pi \frac{r^2}{\sigma N}\right)$$

2-D DFTs:

Cardinal sampling, no sampling grid shifts:

$$\text{Canonic separable 2-D DFT: } \alpha_{r,s} = \frac{1}{\sqrt{N_1 N_2}} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{kr}{N_1} + \frac{ls}{N_2}\right)\right]$$

$$\text{Sampling in affine transformed coordinate system: } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix};$$

$$\text{Affine DFT (AffDFT): } \alpha_{r,s} = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk}{\sigma_A N_1} + \frac{sk}{\sigma_C N_1} + \frac{rl}{\sigma_B N_2} + \frac{sl}{\sigma_D N_2}\right)\right];$$

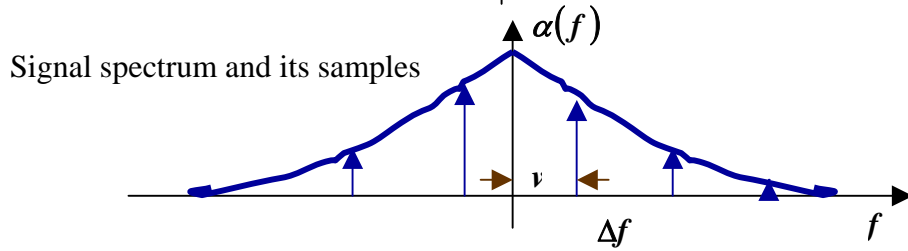
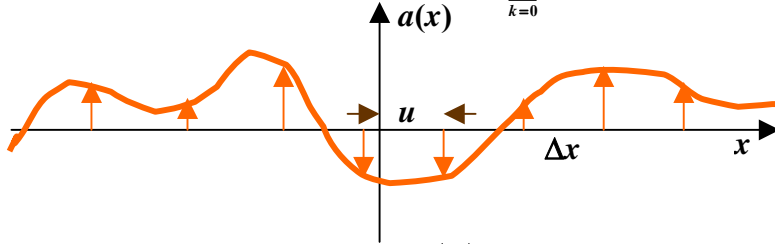
$$\sigma_A = 1/N_1 A \Delta \tilde{x} \Delta f_x; \sigma_B = 1/N_2 B \Delta \tilde{y} \Delta f_x; \sigma_C = 1/N_1 C \Delta \tilde{x} \Delta f_y; \sigma_D = 1/N_2 D \Delta \tilde{y} \Delta f_y$$

Rotated DFT (RotDFT):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \Rightarrow \alpha_{r,s} = \frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r \cos \theta - s \sin \theta}{\sigma N} k + \frac{r \sin \theta + s \cos \theta}{\sigma N} l\right)\right]$$

Integral Fourier Transform $\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi fx) dx$

Continuous signal and its samples $a(x) = \sum_{k=0}^{N-1} a_k \varphi_r [x - (k+u)\Delta x]$



$$\alpha(f) = \int_{-\infty}^{\infty} \left\{ \sum_{k=0}^{N-1} a_k \varphi_r [x - (k+u)\Delta x] \right\} \exp(i2\pi fx) dx = \sum_{k=0}^{N-1} a_k \exp[i2\pi f(k+u)\Delta x] \Phi_r(f)$$

$$\Phi_r(f) = \int_{-\infty}^{\infty} \varphi_r(x) \exp(i2\pi fx) dx \text{ - Frequency response of the signal reconstruction device}$$

$$\alpha_r = \int_{-\infty}^{\infty} \alpha(f) \phi_s [f - (r+v)\Delta f] df = \int_{-\infty}^{\infty} \left\{ \sum_{k=0}^{N-1} a_k \exp[i2\pi f(k+u)\Delta x] \Phi_r(f) \right\} \phi_s [f - (r+v)\Delta f] df =$$

$$\int_{-\infty}^{\infty} \Phi_r [f + (r+v)\Delta f] \phi_s(f) \exp[i2\pi f(k+u)\Delta x] df \times$$

$$\sum_{k=0}^{N-1} a_k \exp[i2\pi f(k+u)(r+v)\Delta x \Delta f]$$

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$$\alpha_r \propto \sum_{k=0}^{N-1} a_k \exp[i2\pi f(k+u)(r+v)\Delta x \Delta f]$$

Cardinal sampling $\Delta x = 1/N\Delta f$;

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Shifted Discrete Fourier Transforms:

$$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left[i2\pi \frac{(k+u)(r+v)}{N} \right]$$

$$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[a_k \exp \left(i2\pi \frac{kv}{N} \right) \right] \exp \left(i2\pi \frac{kr}{N} \right) \right\} \exp \left(i2\pi \frac{ur}{N} \right)$$

No sample shifts in signal and spectrum domains \Rightarrow

Canonical Discrete Fourier Transform

$$\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp \left(i2\pi \frac{kr}{N} \right)$$

Discrete Cosine Transform as SDFT(1/2,0) of evenly extended signals

$$\tilde{a}_k = \begin{cases} a_k, & k = 0, \dots, N-1 \\ a_{2N-1-k}, & k = N, \dots, 2N-1 \end{cases}$$

$$\begin{aligned} \tilde{\alpha}_r &= \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \tilde{a}_k \exp\left(i2\pi \frac{k+1/2}{2N} r\right) = \\ &= \frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r\right) + \sum_{k=N}^{2N-1} \tilde{a}_k \exp\left(i\pi \frac{k+1/2}{N} r\right) \right\} = \\ &= \frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r\right) + \sum_{k=0}^{N-1} \tilde{a}_{2N-1-k} \exp\left(i\pi \frac{2N-1-k+1/2}{N} r\right) \right\} = \\ &= \frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r\right) + \sum_{k=0}^{N-1} a_k \exp\left(-i\pi \frac{k+1/2}{N} r\right) \right\} = \\ &= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r\right) \end{aligned}$$

DcST through DCT

$$\begin{aligned} \alpha_r^{DcST} &= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1/2)r}{N}\right] \\ \alpha_{N-r}^{DcST} &= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1/2)(N-r)}{N}\right] = \\ &= \frac{2}{\sqrt{2N}} \left(\sum_{k=0}^{N-1} a_k \left\{ \sin[\pi(k+1/2)] \cos\left[\pi \frac{(k+1/2)r}{N}\right] - \cos[\pi(k+1/2)] \sin\left[\pi \frac{(k+1/2)r}{N}\right] \right\} \right) \\ &= \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} (-1)^k a_k \cos\left[\pi \frac{(k+1/2)r}{N}\right]; \end{aligned}$$

Inverse DcST through DCT

$$\begin{aligned} a_k &= \frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_r^{DcST} \sin\left[\pi \frac{(k+1/2)r}{N}\right] \right) = \\ &= \frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \sin\left[\pi \frac{(k+1/2)(N-r)}{N}\right] \right) = \\ &= \frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \left\{ \sin[\pi(k+1/2)] \cos\left[\pi \frac{(k+1/2)r}{N}\right] - \right. \right. \\ &\quad \left. \left. \cos[\pi(k+1/2)] \sin\left[\pi \frac{(k+1/2)r}{N}\right] \right\} \right) = \\ &= \frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2(-1)^k \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \cos\left[\pi \frac{(k+1/2)r}{N}\right] \right); \end{aligned}$$

Inverse Scaled DFT:

$$a_k^\sigma = \frac{1}{\sqrt{\sigma N}} \sum_{r=0}^{\sigma N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{\sigma N}\right) = \begin{cases} a_k, & k = 0, 1, \dots, N-1 \\ 0, & k = N, N+1, \dots, \sigma N-1 \end{cases} \quad \text{for } \sigma > 1$$

Proof:

$$\begin{aligned} a_k^\sigma &= \frac{1}{\sqrt{\sigma N}} \sum_{r=0}^{\sigma N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{\sigma N}\right) = \frac{1}{\sigma N} \sum_{r=0}^{\sigma N-1} \left[\sum_{n=0}^{N-1} a_n \exp\left(i2\pi \frac{nr}{\sigma N}\right) \right] \exp\left(-i2\pi \frac{kr}{\sigma N}\right) = \\ &= \frac{1}{\sigma N} \sum_{n=0}^{N-1} a_n \left[\sum_{r=0}^{\sigma N-1} \exp\left(i2\pi \frac{(n-k)r}{\sigma N}\right) \right] = \frac{1}{\sigma N} \sum_{n=0}^{N-1} a_n \frac{\exp[i2\pi(n-k)] - 1}{\exp\left(i2\pi \frac{n-k}{\sigma N}\right) - 1} = \\ &= \frac{1}{\sigma N} \sum_{n=0}^{N-1} a_n \frac{\sin[\pi(n-k)]}{\sin\left(\pi \frac{n-k}{\sigma N}\right)} \exp\left[i\pi \frac{\sigma N-1}{\sigma N}(n-k)\right] = \begin{cases} a_n, & k = 0, 1, \dots, N-1 \\ 0, & k = N, N+1, \dots, \sigma N-1 \end{cases} \end{aligned}$$

as for $k = N, N+1, \dots, \sigma N-1$, $\sin[\pi(n-k)] = 0$ and $\sin\left(\pi \frac{n-k}{\sigma N}\right) \neq 0$

Scaled DFT as cyclic convolution:

$$\alpha_r^\sigma = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{\sigma N}\right) = \frac{\exp\left(i\pi \frac{\tilde{r}^2}{\sigma N}\right)}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} \left[a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N}\right) \right] \exp\left[-i\pi \frac{(\tilde{k}-\tilde{r})^2}{\sigma N}\right],$$

FFT based algorithm for computing Scaled DFT:

$$\alpha_r^{(\sigma)} = \text{IFFT}_{[\sigma N]} \left\{ \text{ZP}_{[\sigma N]} \left[\text{FFT}_N \left\{ a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N}\right) \right\} \right] \bullet \text{FFT}_{[\sigma N]} \left\{ \exp\left(-i\pi \frac{\tilde{n}^2}{\sigma N}\right) \right\} \right\},$$

where $\text{FFT}_M\{\}$ and $\text{IFFT}_M\{\}$ denote M -point direct and inverse FFTs, $[\sigma N]$ is a integer number defined by the inequality $\sigma N \leq [\sigma N] < \sigma N + 1$, and $\text{ZP}_M[\cdot]$ is a zero-padding operator. If $\sigma > 1$, it pads the array of N samples with zeros to the array of $[\sigma N]$ samples. If $\sigma < 1$, it cuts array of N samples to size of $[\sigma N]$ samples.

Rotated Scaled DFT as a 2-D cyclic convolution

$$\begin{aligned} \alpha_{r,s}^\theta &= \frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{a}_{k,l} \exp\left[i2\pi \left(\frac{\tilde{k} \cos \theta + \tilde{l} \sin \theta}{\sigma N} \tilde{r} - \frac{\tilde{k} \sin \theta - \tilde{l} \cos \theta}{\sigma N} \tilde{s} \right)\right] = \\ &= \frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{a}_{k,l} \exp\left[i2\pi \left(\frac{\tilde{k}\tilde{r} + \tilde{l}\tilde{s}}{\sigma N} \cos \theta + \frac{\tilde{l}\tilde{r} - \tilde{k}\tilde{s}}{\sigma N} \sin \theta \right)\right] \end{aligned}$$

Using identities

$$2(\tilde{k}\tilde{r} + \tilde{l}\tilde{s}) = \tilde{r}^2 + \tilde{k}^2 - \tilde{s}^2 - \tilde{l}^2 - (\tilde{r} - \tilde{k})^2 + (\tilde{s} + \tilde{l})^2;$$

$$2(\tilde{l}\tilde{r} - \tilde{k}\tilde{s}) = 2\tilde{k}\tilde{l} - 2\tilde{r}\tilde{s} + 2(\tilde{r} - \tilde{k})(\tilde{s} + \tilde{l}).$$

obtain:

$$\begin{aligned} \alpha_{r,s}^\theta &= \frac{1}{\sigma N} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \tilde{a}_{k,l} \exp \left[i 2\pi \left(\frac{\tilde{k}\tilde{r} + \tilde{l}\tilde{s}}{\sigma N} \cos \theta + \frac{\tilde{l}\tilde{r} - \tilde{k}\tilde{s}}{\sigma N} \sin \theta \right) \right] = \\ &= \frac{1}{\sigma N_a} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{a}_{k,l} \exp \left[i\pi \frac{\tilde{r}^2 + \tilde{k}^2 - \tilde{s}^2 - \tilde{l}^2 - (\tilde{r} - \tilde{k})^2 + (\tilde{s} + \tilde{l})^2}{\sigma N} \cos \theta \right] \times \\ &\exp \left[-i 2\pi \frac{2\tilde{k}\tilde{l} - 2\tilde{r}\tilde{s} + 2(\tilde{r} - \tilde{k})(\tilde{s} + \tilde{l})}{\sigma N} \sin \theta \right] = \\ &\left\{ \frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} (\alpha_{r,s} A_{r,s}) \text{ChF}(\tilde{s} + \tilde{l}, \tilde{r} - \tilde{k}) \right\} \exp \left[-i\pi \frac{(\tilde{k}^2 - \tilde{l}^2) \cos \theta - 2\tilde{k}\tilde{l} \sin \theta}{\sigma N} \right], \end{aligned}$$

$$\text{where } \text{ChF}(\tilde{s} + \tilde{l}, \tilde{r} - \tilde{k}) = \exp \left[i\pi \frac{(\tilde{s} + \tilde{l})^2 \cos \theta - (\tilde{r} - \tilde{k})^2 \cos \theta - 2(\tilde{r} - \tilde{k})(\tilde{s} + \tilde{l}) \sin \theta}{\sigma N} \right];$$

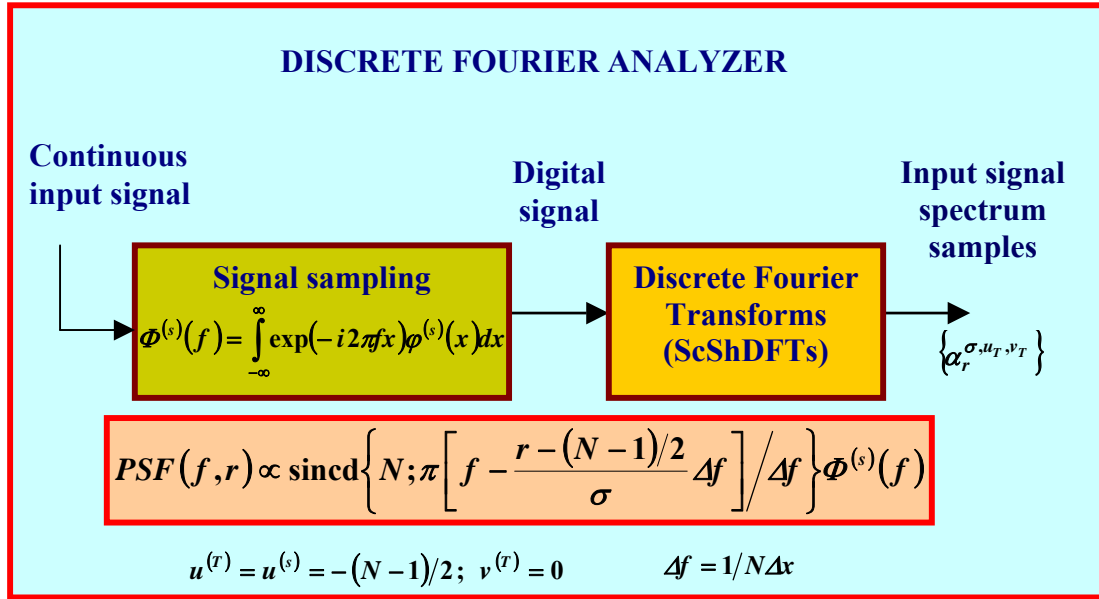
$$\text{and } A_{r,s} = \left\{ \exp \left[-i\pi \frac{(\tilde{r}^2 - \tilde{s}^2) \cos \theta + 2\tilde{r}\tilde{s} \sin \theta}{\sigma N} \right] \right\}.$$

FFT based algorithm for computing Rotated Scaled DFT:

$$\{\tilde{a}_{k,l}\} = \text{IFFT}_{2[\sigma N]} \left\{ \text{FFT}_{2[\sigma N]} \left[\text{ZP}_{[N\sigma]} \left[\text{FFT}_{2N} (a_{k,l}) \right] \bullet A_{r,s} \right] \bullet \text{FFT}_{2[\sigma N]} \left[\text{ChF}(r,s) \right] \right\},$$

where $\text{FFT}_{2[\sigma N]}[\cdot]$ and $\text{IFFT}_{2[\sigma N]}[\cdot]$ are operators of direct and inverse $[N\sigma]$ -point 2D FFT, $[N\sigma]$ is the smallest integer larger than $N\sigma$, $\text{ZP}_{[N\sigma]}[\cdot]$ is a 2-D zero-padding operator. For $\sigma > 1$, it pads array of $N \times N$ samples with zeros to the array of $[\sigma N] \times [\sigma N]$ samples with $[\sigma N]$ defined, as above, by the inequality $\sigma N \leq [\sigma N] < \sigma N + 1$. For $\sigma < 1$, the padding operator cuts array of $N \times N$ samples to the size of $[\sigma N] \times [\sigma N]$ samples.

Point spread function and resolving power of discrete Fourier analysis.



$$\alpha_r^{\sigma, u, v} = \int_{-\infty}^{\infty} \alpha(f) PSF_{DFA}(r, f) df$$

$$\alpha_r^{\sigma, v^r, u^r} = \frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[a_k \exp\left(i2\pi \frac{kv^{(r)}}{\sigma N}\right) \right] \exp\left(i2\pi \frac{kr}{\sigma N}\right) \right\} \exp\left(i2\pi \frac{ru^{(r)}}{\sigma N}\right) =$$

$$\frac{1}{\sqrt{N}} \left\{ \sum_{k=0}^{N-1} \left[\int_{-\infty}^{\infty} a(x) \phi^{(s)}[x - (k + u^{(s)})\Delta x] dx \right] \exp\left(i2\pi \frac{kv^{(r)}}{\sigma N}\right) \exp\left(i2\pi \frac{kr}{\sigma N}\right) \right\} \exp\left(i2\pi \frac{u^{(r)}r}{\sigma N}\right) =$$

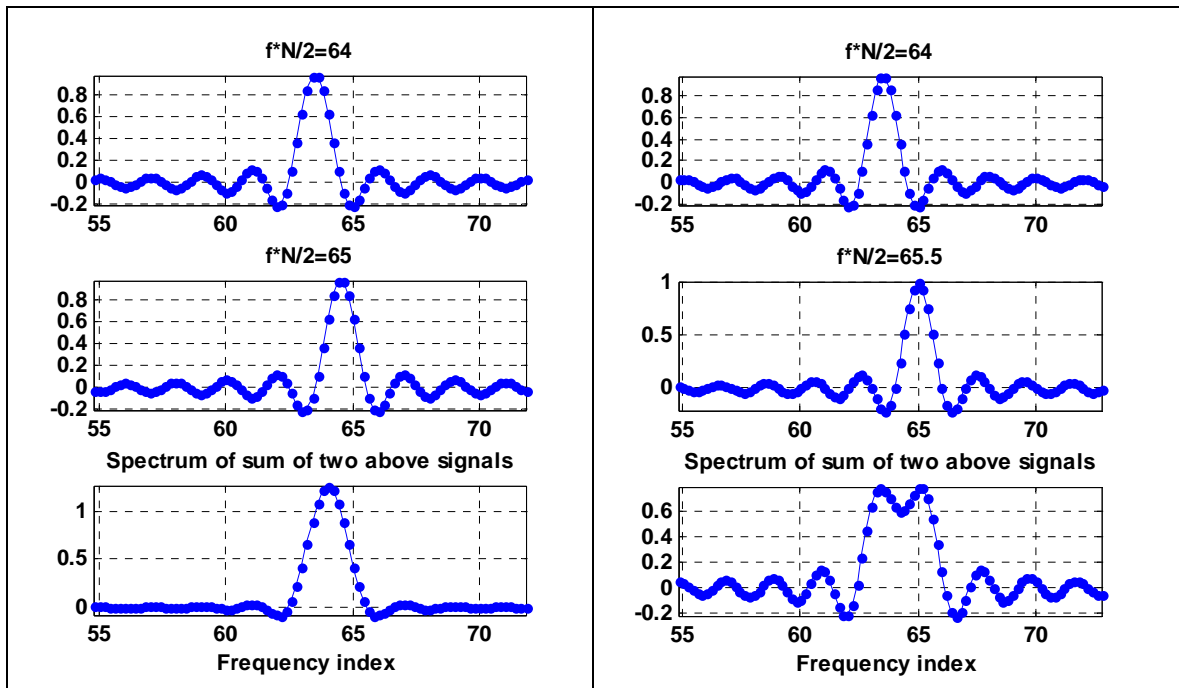
$$\frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} \alpha(f) \left\{ \sum_{k=0}^{N-1} \left[\int_{-\infty}^{\infty} \exp(-i2\pi fx) \phi^{(s)}[x - (k + u^{(s)})\Delta x] dx \right] \exp\left[i2\pi \frac{k(r + v^{(r)})}{\sigma N}\right] \exp\left(i2\pi \frac{u^{(r)}r}{\sigma N}\right) \right\} df$$

$$PSF_{DFA} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\int_{-\infty}^{\infty} \exp(-i2\pi fx) \phi^{(s)}[x - (k + u^{(s)})\Delta x] dx \right] \exp\left[i2\pi \frac{k(r + v^{(r)})}{\sigma N}\right] \exp\left(i2\pi \frac{u^{(r)}r}{\sigma N}\right) \Rightarrow$$

$$PSF_{DFA} = \sqrt{N} \text{sincd}\left[N; \pi \left(\frac{r + v^{(r)}}{\sigma} - fN\Delta x \right)\right] \times$$

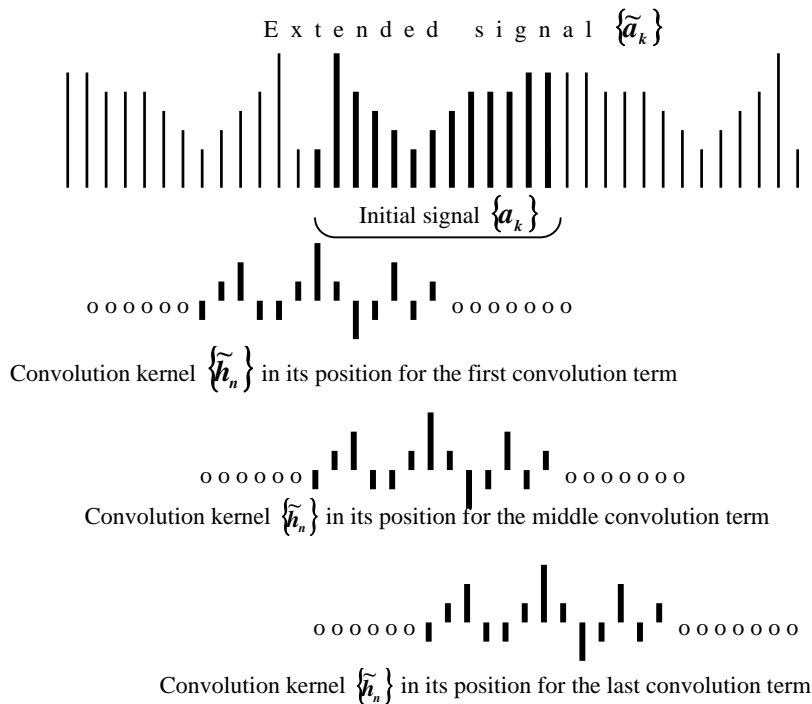
$$\Phi^{(s)}(f) \exp\left\{i2\pi \left[\left(u^{(r)} + \frac{N-1}{2} \right) \frac{r}{\sigma N} - \left(u^{(s)} + \frac{N-1}{2} \right) f\Delta x + \frac{(N-1)v^{(r)}}{2\sigma N} \right]\right\}$$

Resolving power of discrete Fourier analysis



DFT spectra (plotted with sub-sample resolution) of sinusoidal signals of 2 different frequencies and of sum of these signals

2.2 Boundary effect free convolution in DCT domain.



Cyclic convolution of a signal extended by mirror reflection from its borders

$$\tilde{a}_{(k) \bmod 2N} = \begin{cases} a_k, & k = 0, 1, \dots, N-1; \\ a_{2N-k-1}, & k = N, N+1, \dots, 2N-1 \end{cases}$$

$$\tilde{\alpha}_r = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \tilde{a}_k \exp\left(i2\pi \frac{kr}{2N}\right) = \alpha_r^{(DCT)} \exp\left(-i\pi \frac{r}{2N}\right);$$

$$\tilde{h}_{(n) \bmod 2N} = \begin{cases} 0, & n = 0, \dots, [N/2] - 1 \\ h_{n-[N/2]}, & n = [N/2], \dots, [N/2] + N - 1; \\ 0, & n = [N/2] + N, \dots, 2N - 1 \end{cases}; \quad \left[\frac{N}{2}\right] = \begin{cases} N/2, & \text{for even } N \\ (N-1)/2, & \text{for odd } N \end{cases}$$

$$\tilde{\eta}_r = \frac{1}{\sqrt{2N}} \sum_{n=0}^{2N-1} \tilde{h}_n \exp\left(i2\pi \frac{nr}{2N}\right)$$

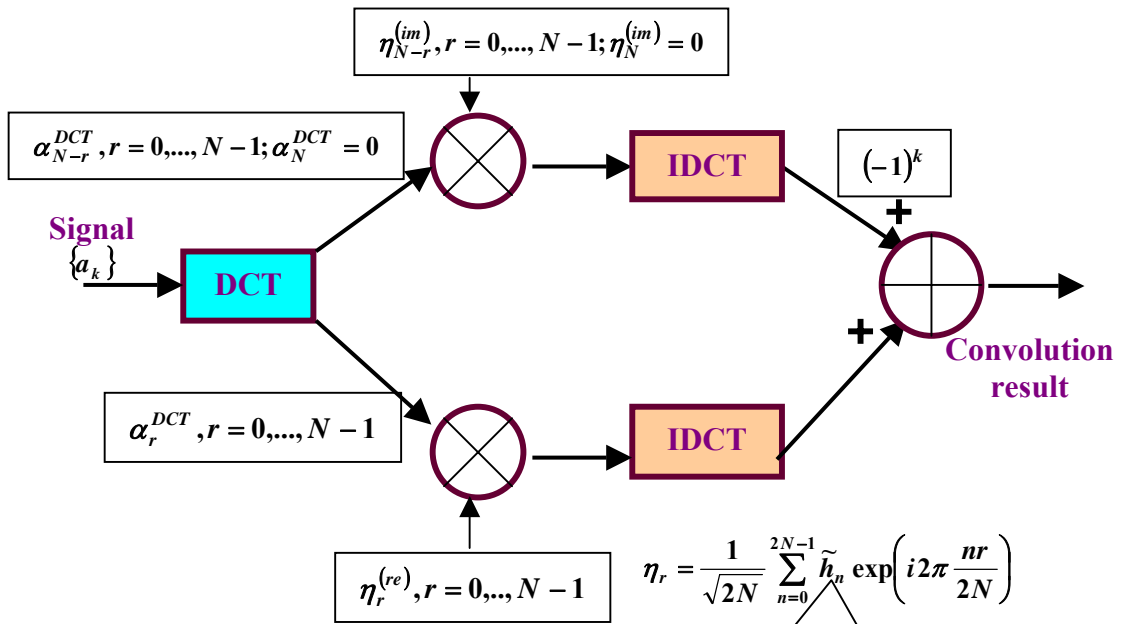
$$b_k = \frac{1}{\sqrt{2N}} \sum_{r=0}^{2N-1} \alpha_r^{(DCT)} \exp\left(-i\pi \frac{r}{2N}\right) \tilde{\eta}_r \exp\left(-i2\pi \frac{kr}{2N}\right) =$$

$$\frac{1}{\sqrt{2N}} \left\{ \alpha_0^{(DCT)} \tilde{\eta}_0 + \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \left[\tilde{\eta}_r \exp\left(-i2\pi \frac{k+1/2}{2N} r\right) + \tilde{\eta}_r^* \exp\left(i2\pi \frac{k+1/2}{2N} r\right) \right] \right\} \Rightarrow$$

$$b_k = \frac{1}{\sqrt{2N}} \left\{ \alpha_0^{(DCT)} \tilde{\eta}_0 + 2 \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \tilde{\eta}_r^{re} \cos\left(\pi \frac{k+1/2}{N} r\right) + 2 \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \tilde{\eta}_r^{im} \sin\left(\pi \frac{k+1/2}{N} r\right) \right\} = .$$

$$\frac{1}{\sqrt{2N}} \left\{ \alpha_0^{(DCT)} \tilde{\eta}_0 + 2 \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \tilde{\eta}_r^{re} \cos\left(\pi \frac{k+1/2}{N} r\right) + 2 \sum_{r=1}^{N-1} \alpha_{N-r}^{(DCT)} \tilde{\eta}_{N-r}^{im} \sin\left(\pi \frac{k+1/2}{N} (N-r)\right) \right\} \Rightarrow$$

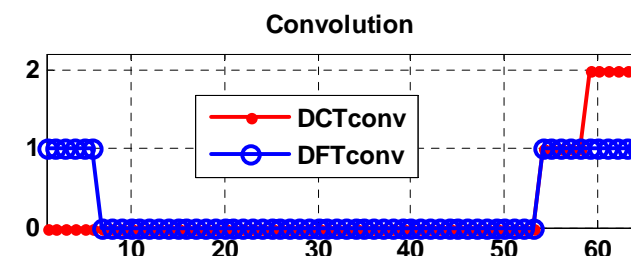
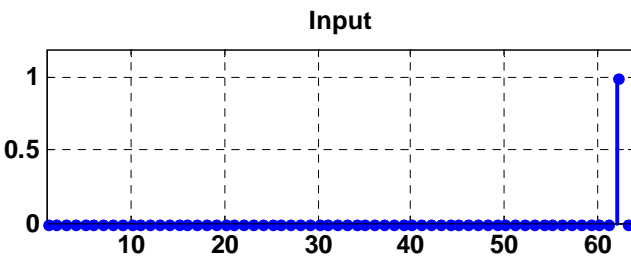
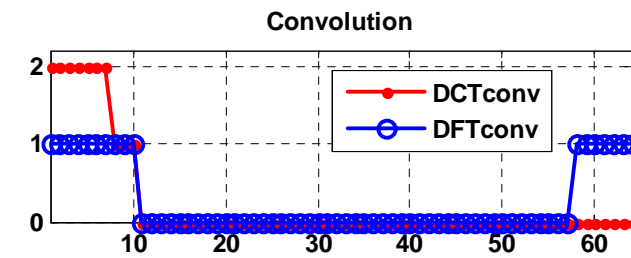
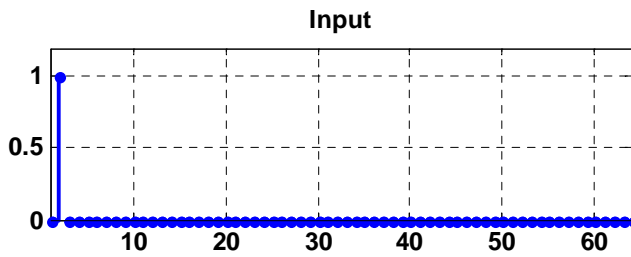
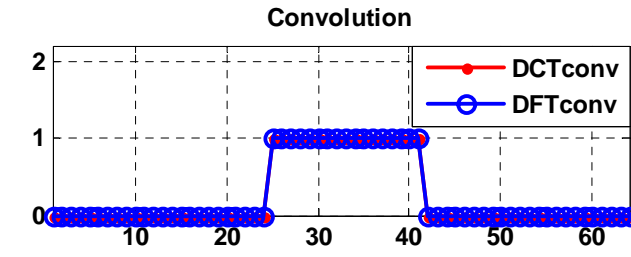
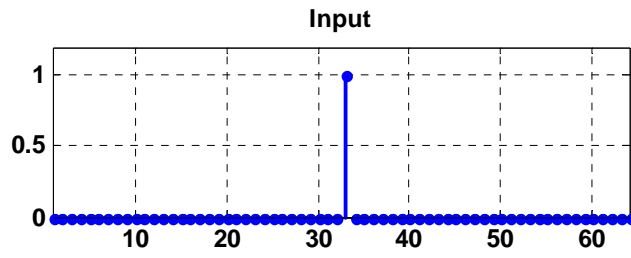
$$b_k = \frac{1}{\sqrt{2N}} \left\{ \alpha_0^{(DCT)} \eta_0 + 2 \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \tilde{\eta}_r^{re} \cos\left(\pi \frac{k+1/2}{N} r\right) + 2(-1)^k \sum_{r=1}^{N-1} \alpha_{N-r}^{(DCT)} \tilde{\eta}_{N-r}^{im} \cos\left(\pi \frac{k+1/2}{N} r\right) \right\}$$



Convolution kernel of N samples pad with zeros to $2N$ samples

$$b_k = \frac{1}{\sqrt{2N}} \left\{ \alpha_0^{(DCT)} \eta_0 + 2 \sum_{r=1}^{N-1} \alpha_r^{(DCT)} \tilde{\eta}_r^{re} \cos\left(\pi \frac{k+1/2}{N} r\right) + 2(-1)^k \sum_{r=1}^{N-1} \alpha_{N-r}^{(DCT)} \tilde{\eta}_{N-r}^{im} \cos\left(\pi \frac{k+1/2}{N} r\right) \right\}$$

Flow chart of an algorithm for signal convolution in DCT domain

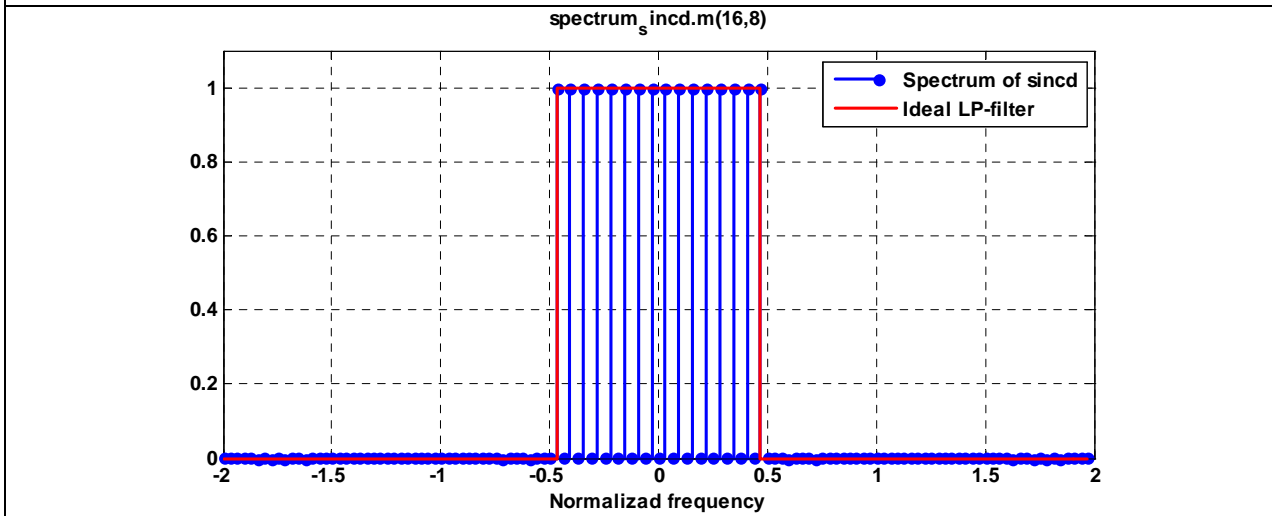
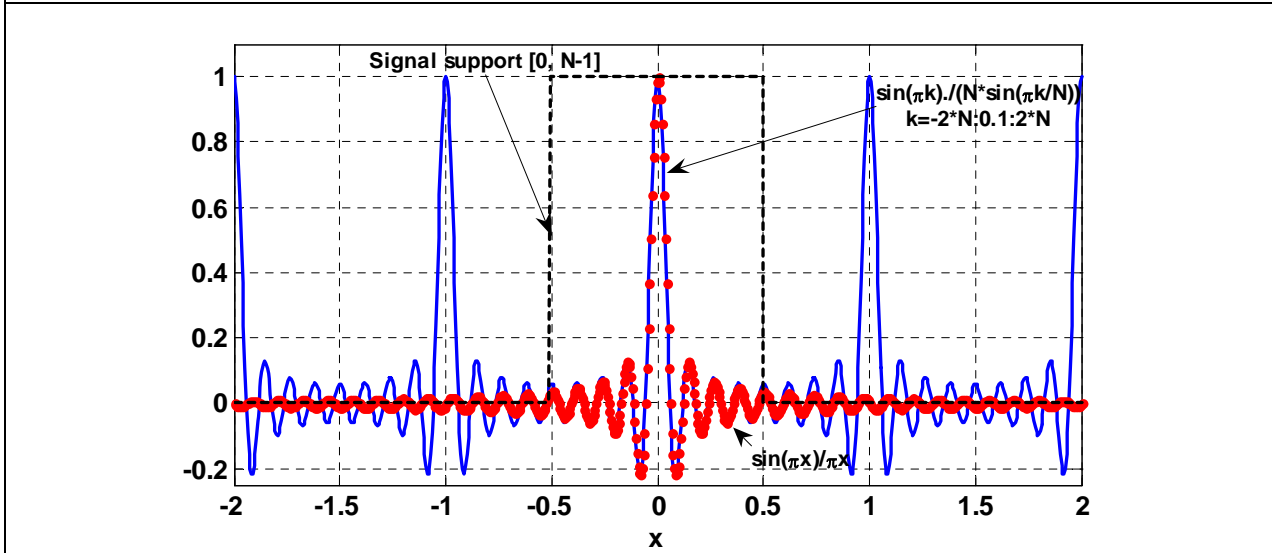
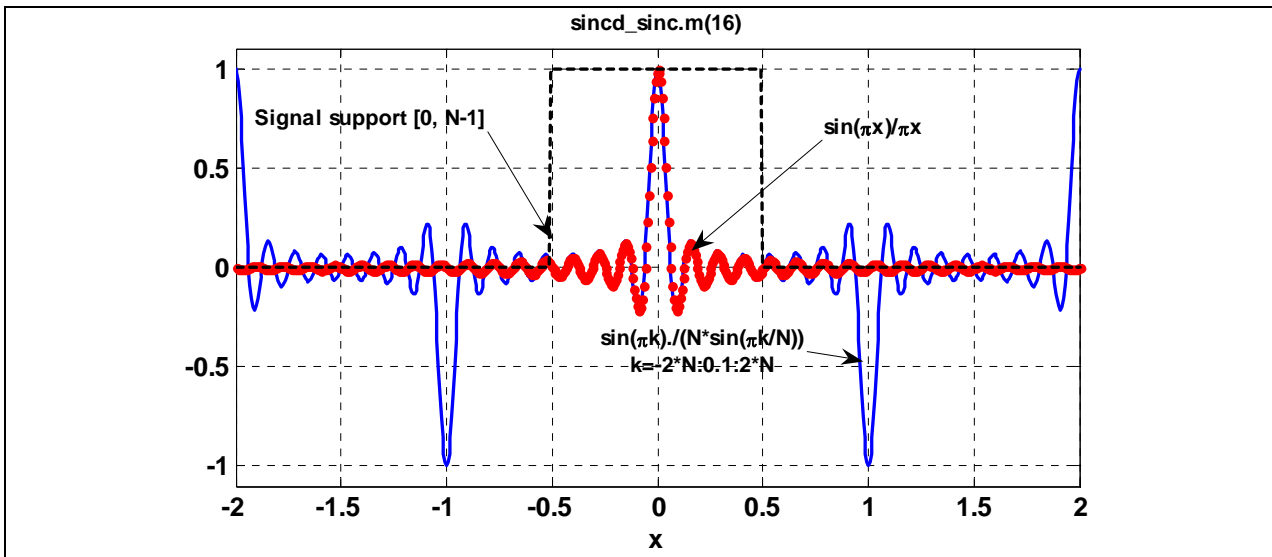


Cyclic convolution in DCT domain vs that in DFT domain

Discrete Fourier Transforms

Transform	
Canonical Discrete Fourier Transform (DFT)	$\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N}\right)$
Shifted DFT	$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{N}\right]$
Discrete Cosine Transform (DCT)	$\alpha_r^{DCT} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r\right)$
Discrete Cosine-Sine Transform (DcST)	$\alpha_r^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left(\pi \frac{k+1/2}{N} r\right); \alpha_{N-r} = \sum_{k=0}^{N-1} (-1)^k a_k \cos\left(\pi \frac{k+1/2}{N} r\right)$
Scaled DFT	$\alpha_r^\sigma = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{\sigma N}\right] = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{\sigma N}\right)$
Scaled DFT as a cyclic convolution	$\alpha_r^\sigma = \frac{\exp\left(i\pi \frac{\tilde{r}^2}{\sigma N}\right)}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} \left[a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N}\right) \right] \exp\left[-i\pi \frac{(\tilde{k}-\tilde{r})^2}{\sigma N}\right]$
Canonical 2-D DFT	$\alpha_{r,s} = \frac{1}{\sqrt{N_1 N_2}} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{kr}{N_1} + \frac{ls}{N_1}\right)\right]$
Affine DFT	$\alpha_{r,s} = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk}{\sigma_A N_1} + \frac{sk}{\sigma_C N_1} + \frac{rl}{\sigma_B N_2} + \frac{sl}{\sigma_D N_2}\right)\right]$
Rotated DFT (RotDFT)	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r \cos \theta - s \sin \theta}{N} k + \frac{r \sin \theta + s \cos \theta}{N} l\right)\right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk + sl}{N} \cos \theta - \frac{sk - rl}{N} \sin \theta\right)\right]$
Rotated Scaled DFT	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r \cos \theta - s \sin \theta}{\sigma N} k + \frac{r \sin \theta + s \cos \theta}{\sigma N} l\right)\right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk + sl}{\sigma N} \cos \theta - \frac{sk - rl}{\sigma N} \sin \theta\right)\right]$
Discrete Sinc-function	$\text{sincd}(N, x) = \frac{\sin x}{N \sin(x/N)}$

Discrete sinc-function vs continuous sinc-function



Frequency response of the ideal low pass filter (red) and Fourier transform of the discrete sinc-function (blue)