Lect. 2. Fourier integral and Discrete Fourier Transforms

2.1 Fourier integral and its discrete representation

$$\alpha(f)=\int_{-\infty}^{\infty} a(x)\exp(i2\pi fx)dx \Rightarrow \alpha_r=\sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x\Delta f]\times$$

$$\int_{-\infty}^{\infty}\Phi^{(r)}[f+(r+v)\Delta f]\varphi^{(s)}(f)\exp[i2\pi f(k+u)\Delta x]df \propto \sum_{k=0}^{N-1} a_k \exp[i2\pi(k+u)(r+v)\Delta x\Delta f]$$

Cardinal sampling $\Delta x = 1/N\Delta f$; no sample grids shifts $\Rightarrow$

Canonical DFT and IDFT: $\alpha_r=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left(i2\pi\frac{kr}{N}\right)$; $a_k=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\alpha_r \exp\left(-i2\pi\frac{kr}{N}\right)$

Cardinal sampling $\Delta x = 1/N\Delta f$; sample grids shifts (u,v) $\Rightarrow$

Shifted DFTs $\alpha_r^{u,v}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+u)(r+v)}{N}\right]$ $a_k^{u,v}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\alpha_r^{u,v} \exp\left[-i2\pi\frac{(k+u)(r+v)}{N}\right]$

A reduced version of SDFT: $\alpha_r^{u,v}=\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k \exp\left(i2\pi\frac{kv}{N}\right)\right]\exp\left(i2\pi\frac{kr}{N}\right)\right\}\exp\left(i2\pi\frac{ur}{N}\right)$

Special cases of SDFT(1/2,0) for even-odd signals $\{a_k=\pm a_{2N-1-k}\}$:

DCT: $\alpha_r^{DCT}=\sum_{k=0}^{N-1} a_k \cos\left(\pi\frac{k+1/2}{N}r\right)$; DcST: $\alpha_r^{DcST}=\sum_{k=0}^{N-1} a_k \sin\left(\pi\frac{k+1/2}{N}r\right)$;

Other special cases of SDFTs: DCT(I-IV); DST(I-IV);

Sampling in σ -scaled coordinates (over/under sampling: $\Delta x = 1/\sigma N\Delta f$) ,no sampling grid shifts:

Scaled DFT: $\alpha_r^{\sigma}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+u)(r+v)}{\sigma N}\right]$;

Inverse ScDFT exists only if $\sigma N \in \mathbf{Z}$: $a_k^{\sigma}=\frac{1}{\sqrt{N}}\sum_{r=0}^{\sigma N-1}\alpha_r^{\sigma}\exp\left[-i2\pi\frac{(k+u)(r+v)}{\sigma N}\right]$

Computing ScDFT through the canonical DFT

$$\alpha_r^{\sigma}=IDFT\left\{DFT\left\{a_k\exp\left(i\pi\frac{k^2}{\sigma N}\right)\right\}\bullet DFT\left\{\exp\left(-i\pi\frac{k^2}{\sigma N}\right)\right\}\right\}\bullet\exp\left(i\pi\frac{r^2}{\sigma N}\right)$$

2-D DFTs:

Cardinal sampling, no sampling grid shifts:

Canonic separable 2-D DFT: $\alpha_{r,s}=\frac{1}{\sqrt{N_1N_2}}\sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1} a_{k,l}\exp\left[i2\pi\left(\frac{kr}{N_1}+\frac{ls}{N_1}\right)\right]$

Sampling in affine transformed coordinate system: $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}A & B\\C & D\end{bmatrix}\begin{bmatrix}\tilde{x}\\\tilde{y}\end{bmatrix}$:

Affine DFT (AffDFT): $\alpha_{r,s}=\sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1} a_{k,l}\exp\left[i2\pi\left(\frac{rk}{\sigma_A N_1}+\frac{sk}{\sigma_C N_1}+\frac{rl}{\sigma_B N_2}+\frac{sl}{\sigma_D N_2}\right)\right]$;

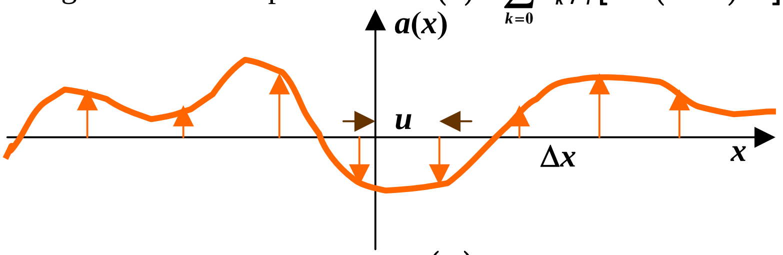
$\sigma_A=1/N_1A\Delta\tilde{x}\Delta f_x$; $\sigma_B=1/N_2B\Delta\tilde{y}\Delta f_x$; $\sigma_C=1/N_1C\Delta\tilde{x}\Delta f_y$ $\sigma_D=1/N_2D\Delta\tilde{y}\Delta f_y$

Rotated DFT (RotDFT,):

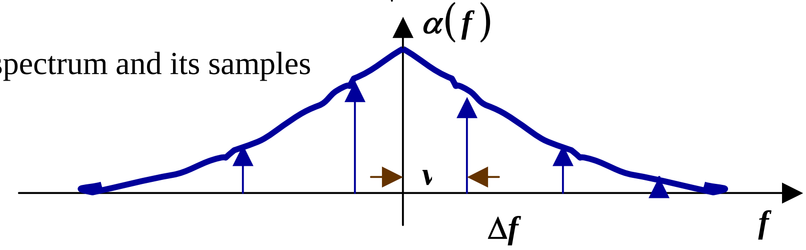
$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\tilde{x}\\\tilde{y}\end{bmatrix}\Rightarrow \alpha_{r,s}=\frac{1}{\sigma N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}\exp\left[i2\pi\left(\frac{r\cos\theta-s\sin\theta}{\sigma N}k+\frac{r\sin\theta+s\cos\theta}{\sigma N}l\right)\right]$$

Integral Fourier Transform $\alpha(f)=\int_{-\infty}^{\infty} a(x)\exp(i2\pi fx)dx$

Continuous signal and its samples $a(x)=\sum_{k=0}^{N-1} a_k\varphi_r[x-(k+u)\Delta x]$



Signal spectrum and its samples



$$\alpha(f)=\int_{-\infty}^{\infty}\left\{\sum_{k=0}^{N-1} a_k\varphi_r[x-(k+u)\Delta x]\right\}\exp(i2\pi fx)dx=\sum_{k=0}^{N-1} a_k\exp[i2\pi f(k+u)\Delta x]\Phi_r(f)$$

$\Phi(f)=\int_{-\infty}^{\infty}\varphi_r(x)\exp(i2\pi fx)dx$ - Frequency response of the signal reconstruction device

$$\alpha_r=\int_{-\infty}^{\infty}\alpha(f)\phi_s[f-(r+v)\Delta f]df=\int_{-\infty}^{\infty}\left\{\sum_{k=0}^{N-1} a_k\exp[i2\pi f(k+u)\Delta x]\Phi_r(f)\right\}\phi_s[f-(r+v)\Delta f]df=$$

$$\int_{-\infty}^{\infty}\Phi_r[f+(r+v)\Delta f]\phi_s(f)\exp[i2\pi f(k+u)\Delta x]df\times$$

$$\sum_{k=0}^{N-1} a_k\exp[i2\pi f(k+u)(r+v)\Delta x\Delta f]$$

$$\Downarrow$$

$$\alpha_r\propto\sum_{k=0}^{N-1} a_k\exp[i2\pi f(k+u)(r+v)\Delta x\Delta f]$$

Cardinal sampling $\Delta x=1/N\Delta f$;

$$\Downarrow$$

Shifted Discrete Fourier Transforms:

$$\alpha_r^{u,v}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k\exp\left[i2\pi\frac{(k+u)(r+v)}{N}\right]$$

$$\alpha_r^{u,v}=\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k\exp\left(i2\pi\frac{kv}{N}\right)\right]\exp\left(i2\pi\frac{kr}{N}\right)\right\}\exp\left(i2\pi\frac{ur}{N}\right)$$

No sample shifts in signal and spectrum domains $\Rightarrow$

Canonical Discrete Fourier Transform

$$\alpha_r=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k\exp\left(i2\pi\frac{kr}{N}\right)$$

Discrete Cosine Transform as SDFT(1/2,0) of evenly extended signals

$$\tilde{a}_k = \begin{cases} a_k, & k = 0,...,N-1 \\ a_{2N-1-k}, & k = N,...,2N-1 \end{cases}$$

$$\tilde{\alpha}_r = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \tilde{a}_k \exp\left(i2\pi \frac{k+1/2}{2N} r \right) =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r \right) + \sum_{k=N}^{2N-1} \tilde{a}_k \exp\left(i\pi \frac{k+1/2}{N} r \right) \right\} =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r \right) + \sum_{k=0}^{N-1} \tilde{a}_{2N-1-k} \exp\left(i\pi \frac{2N-1-k+1/2}{N} r \right) \right\} =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left(i\pi \frac{k+1/2}{N} r \right) + \sum_{k=0}^{N-1} a_k \exp\left(-i\pi \frac{k+1/2}{N} r \right) \right\} =$$
$$\frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r \right)$$

DcST through DCT

$$\alpha_r^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1/2)r}{N} \right]$$
$$\alpha_{N-r}^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1/2)(N-r)}{N} \right] =$$
$$\frac{2}{\sqrt{2N}} \left(\sum_{k=0}^{N-1} a_k \left\{ \sin[\pi(k+1/2)] \cos\left[\pi \frac{(k+1/2)r}{N} \right] - \cos[\pi(k+1/2)] \sin\left[\pi \frac{(k+1/2)r}{N} \right] \right\} \right)$$
$$\frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} (-1)^k a_k \cos\left[\pi \frac{(k+1/2)r}{N} \right];$$

Inverse DcST through DCT

$$a_k = \frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_r^{DcST} \sin\left[\pi \frac{(k+1/2)r}{N} \right] \right) =$$
$$\frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \sin\left[\pi \frac{(k+1/2)(N-r)}{N} \right] \right) =$$
$$\frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2 \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \left\{ \sin[\pi(k+1/2)] \cos\left[\pi \frac{(k+1/2)r}{N} \right] - \right. \right.$$
$$\left. \left. \{\cos[\pi(k+1/2)]\} \sin\left[\pi \frac{(k+1/2)r}{N} \right] \right\} \right) =$$
$$\frac{1}{\sqrt{2N}} \left(\alpha_N^{DcST} + 2(-1)^k \sum_{r=1}^{N-1} \alpha_{N-r}^{DcST} \cos\left[\pi \frac{(k+1/2)r}{N} \right] \right);$$

Inverse Scaled DFT:

$$a_k^{\sigma} = \frac{1}{\sqrt{\sigma N}} \sum_{r=0}^{\sigma N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{\sigma N}\right) = \begin{cases} a_k, & k = 0,1,\ldots N-1 \\ 0, & k = N, N+1,\ldots,\sigma N - 1 \end{cases} \quad \text{for } \sigma > 1$$

Proof:

$$a_k^{\sigma} = \frac{1}{\sqrt{\sigma N}} \sum_{r=0}^{\sigma N-1} \alpha_r \exp\left(-i2\pi \frac{kr}{\sigma N}\right) = \frac{1}{\sigma N} \sum_{r=0}^{\sigma N-1} \left[\sum_{n=0}^{N-1} a_k \exp\left(i2\pi \frac{nr}{\sigma N}\right)\right] \exp\left(-i2\pi \frac{kr}{\sigma N}\right) =$$

$$\frac{1}{\sigma N} \sum_{n=0}^{N-1} a_k \left[\sum_{r=0}^{\sigma N-1} \exp\left(i2\pi \frac{n-k}{\sigma N} r\right)\right] = \frac{1}{\sigma N} \sum_{n=0}^{N-1} a_k \frac{\exp[i2\pi(n-k)] - 1}{\exp\left(i2\pi \frac{n-k}{\sigma N}\right) - 1} =$$

$$\frac{1}{\sigma N} \sum_{n=0}^{N-1} a_n \frac{\sin[\pi(n-k)]}{\sin\left(\pi \frac{n-k}{\sigma N}\right)} \exp\left[i\pi \frac{\sigma N - 1}{\sigma N}(n-k)\right] = \begin{cases} a_n, & k = 0,1,\ldots, N-1 \\ 0, & k = N, N = 1,\ldots,\sigma N - 1 \end{cases}$$

as for $k = N, N = 1,\ldots,\sigma N - 1$, $\sin[\pi(n-k)] = 0$ and $\sin\left(\pi \frac{n-k}{\sigma N}\right) \neq 0$

Scaled DFT as cyclic convolution:

$$\alpha_r^{\sigma} = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{\sigma N}\right) = \frac{\exp\left(i\pi \frac{\tilde{r}^2}{\sigma N}\right)}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} \left[a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N}\right)\right] \exp\left[-i\pi \frac{(\tilde{k} - \tilde{r})^2}{\sigma N}\right],$$

FFT based algorithm for computing Scaled DFT:

$$\alpha_r^{(\sigma)} = \mathbf{IFFT}_{\lfloor \sigma N \rfloor}\left\{\mathbf{ZP}_{\lfloor \sigma N \rfloor}\left[\mathbf{FFT}_N\left\{a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N}\right)\right\}\right] \bullet \mathbf{FFT}_{\lfloor \sigma N \rfloor}\left\{\exp\left(-i\pi \frac{\tilde{n}^2}{\sigma N}\right)\right\}\right\},$$

where $\mathbf{FFT}_M\{\cdot\}$ and $\mathbf{IFFT}_M\{\cdot\}$ denote M-point direct and inverse FFTs, $\lfloor \sigma N \rfloor$ is a integer number defined by the inequality $\sigma N \le \lfloor \sigma N \rfloor < \sigma N + 1$, and $\mathbf{ZP}_M[\cdot]$ is a zero-padding operator. If $\sigma > 1$, it pads the array of N samples with zeros to the array of $\lfloor \sigma N \rfloor$ samples. If $\sigma < 1$, it cuts array of N samples to size of $\lfloor \sigma N \rfloor$ samples.

Rotated Scaled DFT as a 2-D cyclic convolution

$$\alpha_{r,s}^{\theta} = \frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{a}_{k,l} \exp\left[i2\pi\left(\frac{\tilde{k}\cos\theta + \tilde{l}\sin\theta}{\sigma N}\tilde{r} - \frac{\tilde{k}\sin\theta - \tilde{l}\cos\theta}{\sigma N}\tilde{s}\right)\right] =$$

$$\frac{1}{\sigma N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{a}_{k,l} \exp\left[i2\pi\left(\frac{\tilde{k}\tilde{r} + \tilde{l}\tilde{s}}{\sigma N}\cos\theta + \frac{\tilde{l}\tilde{r} - \tilde{k}\tilde{s}}{\sigma N}\sin\theta\right)\right]$$

Using identities

$$2(\tilde{k}\tilde{r} + \tilde{l}\tilde{s}) = \tilde{r}^2 + \tilde{k}^2 - \tilde{s}^2 - \tilde{l}^2 - (\tilde{r} - \tilde{k})^2 + (\tilde{s} + \tilde{l})^2;$$

$$2(\tilde{l}\tilde{r} - \tilde{k}\tilde{s}) = 2\tilde{k}\tilde{l} - 2\tilde{r}\tilde{s} + 2(\tilde{r} - \tilde{k})(\tilde{s} + \tilde{l}).$$

obtain:

$$\alpha_{r,s}^{\theta} = \frac{1}{\sigma N}\sum_{r=0}^{N-1}\sum_{s=0}^{N-1}\tilde{a}_{k,l}\exp\left[i2\pi\left(\frac{\tilde{k}\tilde{r}+\tilde{l}\tilde{s}}{\sigma N}\cos\theta + \frac{\tilde{l}\tilde{r}-\tilde{k}\tilde{s}}{\sigma N}\sin\theta\right)\right] =$$

$$\frac{1}{\sigma N_a}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\tilde{a}_{k,l}\exp\left[i\pi\frac{\tilde{r}^2+\tilde{k}^2-\tilde{s}^2-\tilde{l}^2-\left(\tilde{r}-\tilde{k}\right)^2+\left(\tilde{s}+\tilde{l}\right)^2}{\sigma N}\cos\theta\right]\times$$

$$\exp\left[-i2\pi\frac{2\tilde{k}\tilde{l}-2\tilde{r}\tilde{s}+2\left(\tilde{r}-\tilde{k}\right)\left(\tilde{s}+\tilde{l}\right)}{\sigma N}\sin\theta\right] =$$

$$\left\{\frac{1}{\sigma N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left(\alpha_{r,s}\mathrm{A}_{r,s}\right)\mathbf{ChF}\left(\tilde{s}+\tilde{l},\tilde{r}-\tilde{k}\right)\right\}\exp\left[-i\pi\frac{\left(\tilde{k}^2-\tilde{l}^2\right)\cos\theta-2\tilde{k}\tilde{l}\sin\theta}{\sigma N}\right],$$

where $\mathbf{ChF}\left(\tilde{s}+\tilde{l},\tilde{r}-\tilde{k}\right) = \exp\left[i\pi\frac{\left(\tilde{s}+\tilde{l}\right)^2\cos\theta-\left(\tilde{r}-\tilde{k}\right)^2\cos\theta-2\left(\tilde{r}-\tilde{k}\right)\left(\tilde{s}+\tilde{l}\right)\sin\theta}{\sigma N}\right]$;

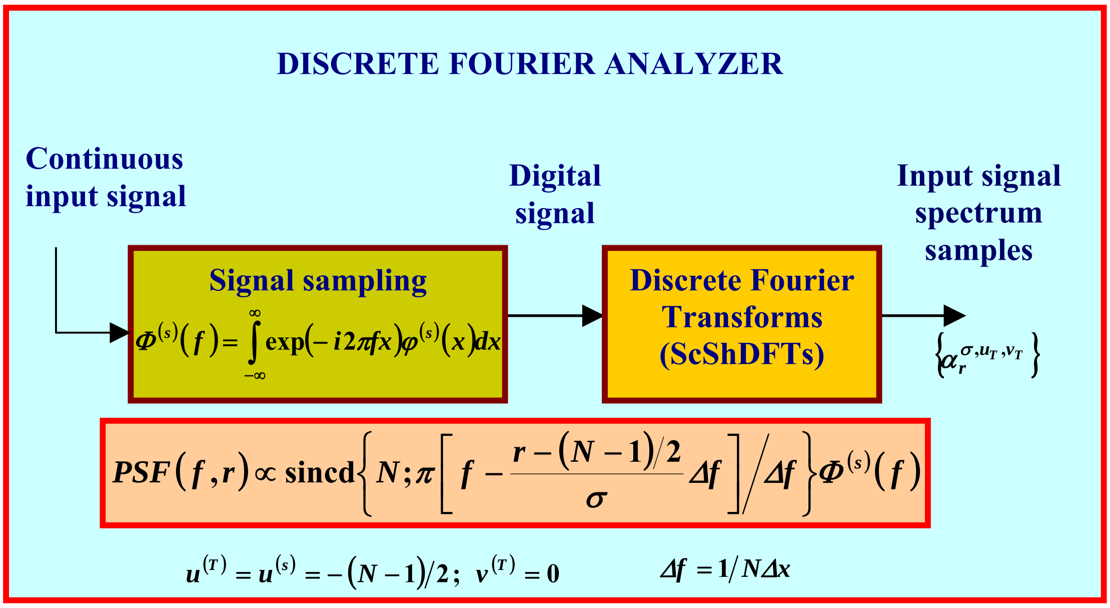
and $A_{r,s} = \left\{\exp\left[-i\pi\frac{\left(\tilde{r}^2-\tilde{s}\right)^2\cos\theta+2\tilde{r}\tilde{s}\sin\theta}{\sigma N}\right]\right\}$.

FFT based algorithm for computing Rotated Scaled DFT:

$$\left\{\tilde{a}_{k,l}\right\} = \mathbf{IFFT\,2}_{\lfloor\sigma N\rfloor}\left\{\mathbf{FFT\,2}_{\lfloor\sigma N\rfloor}\left[\mathbf{ZP}_{\lfloor N\sigma\rfloor}\left[\mathbf{FFT\,2}_{N}\left(a_{k,l}\right)\right]\bullet A_{r,s}\right]\bullet \mathbf{FFT\,2}_{\lfloor\sigma N\rfloor}\left[\mathbf{ChF}(r,s)\right]\right\},$$

where $\mathbf{FFT\,2}_{\lfloor N\sigma\rfloor}[\cdot]$ and $\mathbf{IFFT\,2}_{\lfloor N\sigma\rfloor}[\cdot]$ are operators of direct and inverse $\lfloor N\sigma\rfloor$-point 2D FFT, $\lfloor N\sigma\rfloor$ is the smallest integer larger then $N\sigma$, $\mathbf{ZP}_{\lfloor N\sigma\rfloor}[\cdot]$ is a 2-D zero-padding operator. For $\sigma > 1$, it pads array of $N\times N$ samples with zeros to the array of $\lfloor\sigma N\rfloor\times\lfloor\sigma N\rfloor$ samples with $\lfloor\sigma N\rfloor$ defined, as above, by the inequality $\sigma N \le \lfloor\sigma N\rfloor < \sigma N + 1$. For $\sigma < 1$, the padding operator cuts array of $N\times N$ samples to the size of $\lfloor\sigma N\rfloor\times\lfloor\sigma N\rfloor$ samples.

Point spread function and resolving power of discrete Fourier analysis.



$$\alpha_r^{\sigma,u,v}=\int_{-\infty}^{\infty}\alpha(f)PSF_{DFA}(r,f)df$$

$$\alpha_r^{\sigma,v_T,u_T}=\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[a_k\exp\left(i2\pi\frac{kv^{(T)}}{\sigma N}\right)\right]\exp\left(i2\pi\frac{kr}{\sigma N}\right)\right\}\exp\left(i2\pi\frac{ru^{(T)}}{\sigma N}\right)=$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty}a(x)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left(i2\pi\frac{kv^{(T)}}{\sigma N}\right)\exp\left(i2\pi\frac{kr}{\sigma N}\right)\right\}\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right)=$$

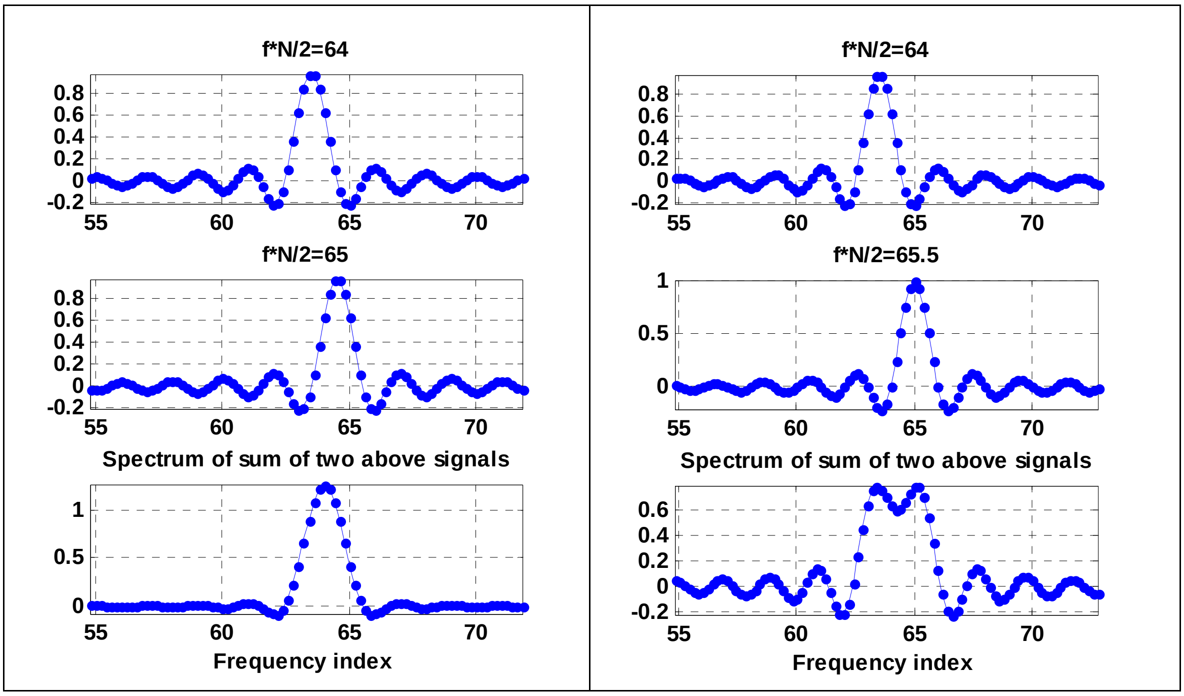
$$\frac{1}{\sqrt{N}}\int_{-\infty}^{\infty}\alpha(f)\left\{\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty}\exp(-i2\pi fx)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left[i2\pi\frac{k\left(r+v^{(T)}\right)}{\sigma N}\right]\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right)\right\}df$$

$$PSF_{DFA}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left[\int_{-\infty}^{\infty}\exp(-i2\pi fx)\varphi^{(s)}\left[x-\left(k+u^{(s)}\right)\Delta x\right]dx\right]\exp\left[i2\pi\frac{k\left(r+v^{(T)}\right)}{\sigma N}\right]\exp\left(i2\pi\frac{u^{(T)}r}{\sigma N}\right)\Rightarrow$$

$$PSF_{DFA}=\sqrt{N}\,\mathrm{sincd}\left[N;\pi\left(\frac{r+v^{(T)}}{\sigma}-fN\Delta x\right)\right]\times$$

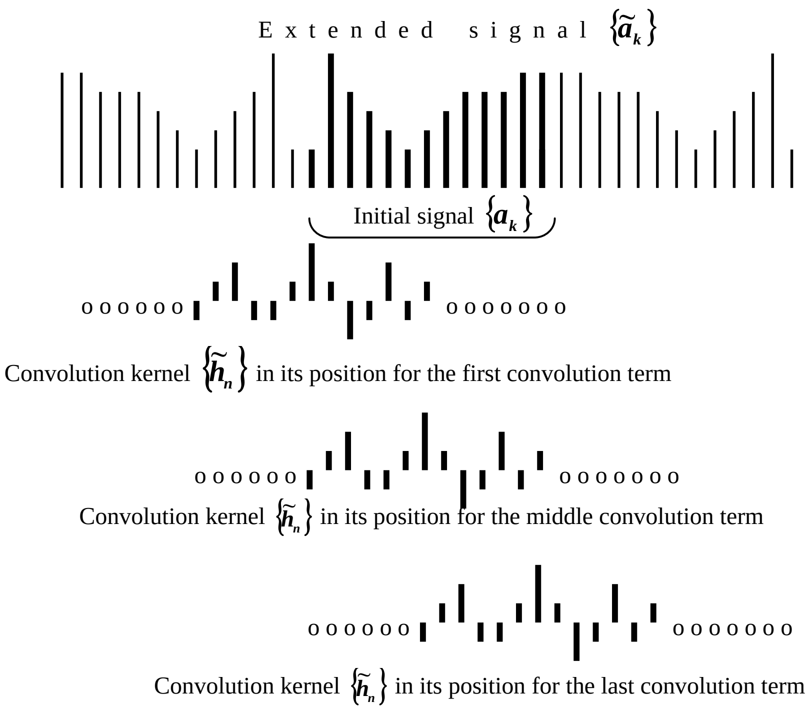
$$\Phi^{(s)}(f)\exp\left\{i2\pi\left[\left(u^{(T)}+\frac{N-1}{2}\right)\frac{r}{\sigma N}-\left(u^{(s)}+\frac{N-1}{2}\right)f\Delta x+\frac{(N-1)v^{(T)}}{2\sigma N}\right]\right\}$$

Resolving power of discrete Fourier analysis



DFT spectra (plotted with sub-sample resolution) of sinusoidal signals of 2 different frequencies and of sum of these signals

2.2 Boundary effect free convolution in DCT domain.



Cyclic convolution of a signal extended by mirror reflection from its borders

$$\tilde{a}_{(k)\bmod 2N} = \begin{cases} a_k, & k = 0,1,\dots, N-1; \\ a_{2N-k-1}, & k = N, N+1,\dots,2N-1 \end{cases}$$

$$\tilde{\alpha}_r = \frac{1}{\sqrt{2N}}\sum_{k=0}^{2N-1}\tilde{a}_k \exp\left(i2\pi\frac{kr}{2N}\right) = \alpha_r^{(DCT)}\exp\left(-i\pi\frac{r}{2N}\right);$$

$$\tilde{h}_{(n)\bmod 2N} = \begin{cases} 0, & n = 0,\dots,[N/2]-1 \\ h_{n-[N/2]}, & n = [N/2],\dots,[N/2]+N-1 \\ 0, & n = [N/2]+N,\dots,2N-1 \end{cases}; \quad \left[\frac{N}{2}\right] = \begin{cases} N/2, & \text{for even } N \\ (N-1)/2, & \text{for odd } N \end{cases}$$

$$\tilde{\eta}_r = \frac{1}{\sqrt{2N}}\sum_{n=0}^{2N-1}\tilde{h}_n \exp\left(i2\pi\frac{nr}{2N}\right)$$

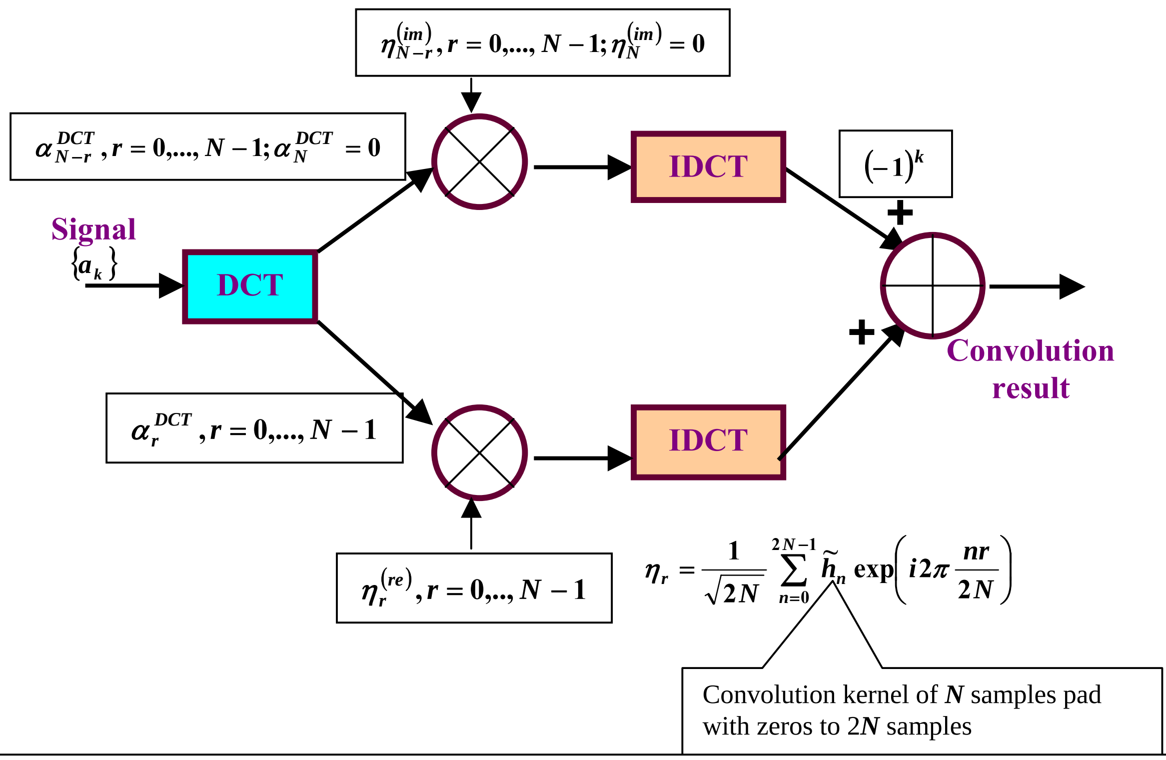
$$b_k = \frac{1}{\sqrt{2N}}\sum_{r=0}^{2N-1}\alpha_r^{(DCT)}\exp\left(-i\pi\frac{r}{2N}\right)\tilde{\eta}_r \exp\left(-i2\pi\frac{kr}{2N}\right) =$$

$$\frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\tilde{\eta}_0 + \sum_{r=1}^{N-1}\alpha_r^{(DCT)}\left[\tilde{\eta}_r \exp\left(-i2\pi\frac{k+1/2}{2N}r\right) + \tilde{\eta}_r^* \exp\left(i2\pi\frac{k+1/2}{2N}r\right)\right]\right\} \Rightarrow$$

$$b_k = \frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\tilde{\eta}_0 + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\tilde{\eta}_r^{re}\cos\left(\pi\frac{k+1/2}{N}r\right) + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\tilde{\eta}_r^{im}\sin\left(\pi\frac{k+1/2}{N}r\right)\right\} = .$$

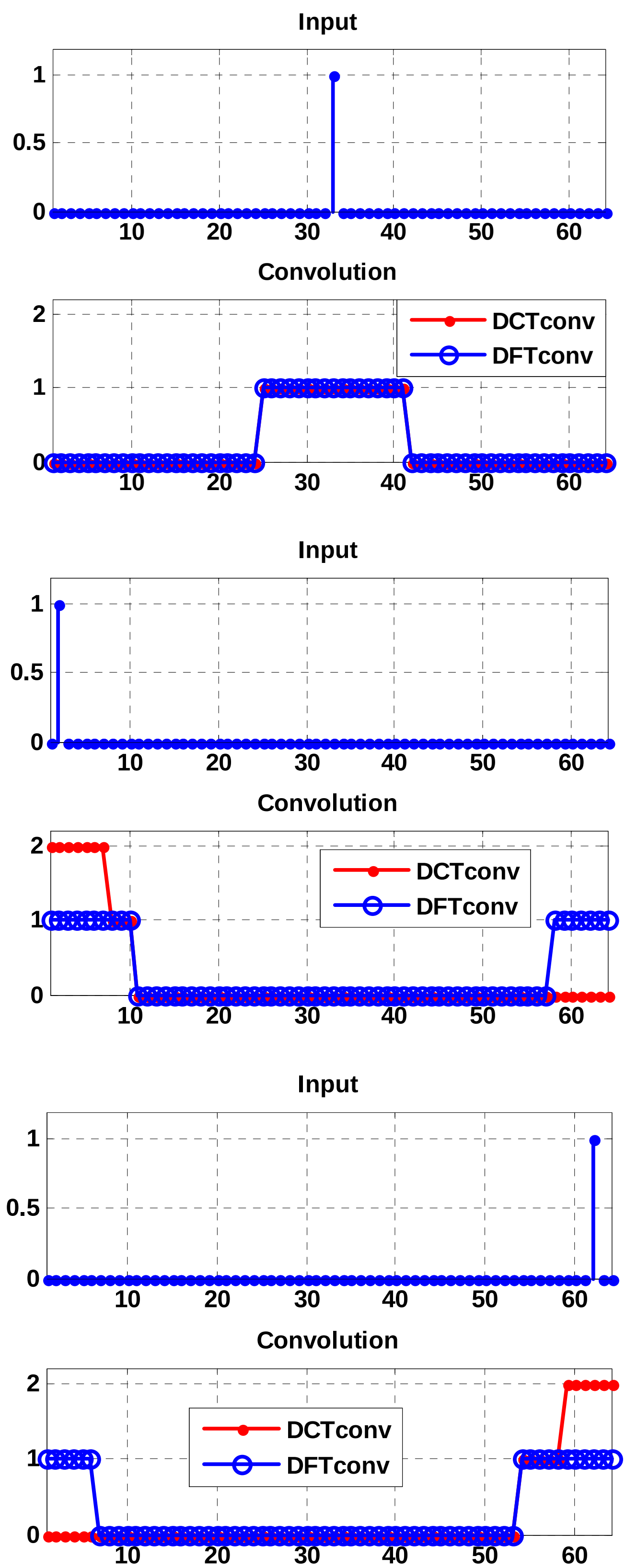
$$\frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\tilde{\eta}_0 + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\tilde{\eta}_r^{re}\cos\left(\pi\frac{k+1/2}{N}r\right) + 2\sum_{r=1}^{N-1}\alpha_{N-r}^{(DCT)}\tilde{\eta}_{N-r}^{im}\sin\left(\pi\frac{k+1/2}{N}(N-r)\right)\right\} \Rightarrow$$

$$b_k = \frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\eta_0 + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\tilde{\eta}_r^{re}\cos\left(\pi\frac{k+1/2}{N}r\right) + 2(-1)^k\sum_{r=1}^{N-1}\alpha_{N-r}^{(DCT)}\tilde{\eta}_{N-r}^{im}\cos\left(\pi\frac{k+1/2}{N}r\right)\right\}$$



$$b_k = \frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)}\eta_0 + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\tilde{\eta}_r^{re}\cos\left(\pi\frac{k+1/2}{N}r\right) + 2(-1)^k\sum_{r=1}^{N-1}\alpha_{N-r}^{(DCT)}\tilde{\eta}_{N-r}^{im}\cos\left(\pi\frac{k+1/2}{N}r\right)\right\}$$

Flow chart of an algorithm for signal convolution in DCT domain



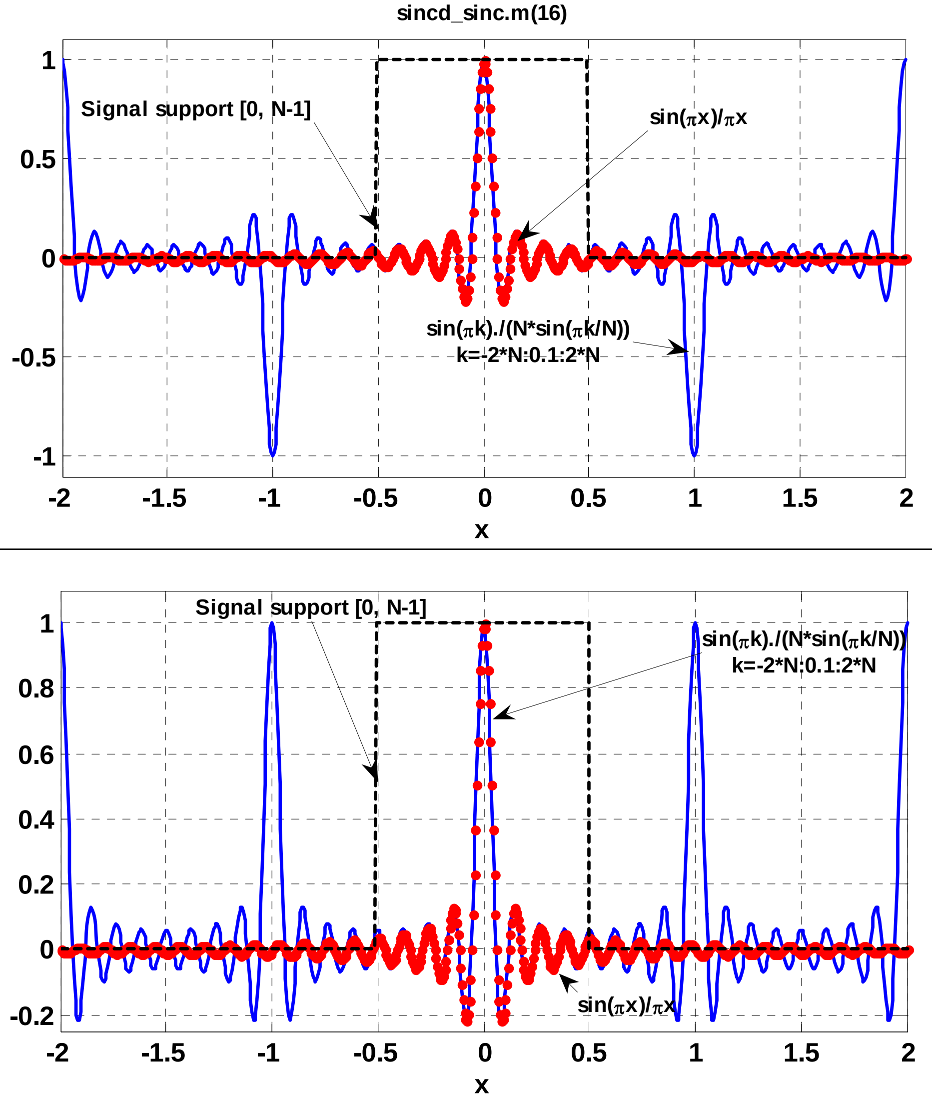
Cyclic convolution in DCT domain vs that in DFT domain

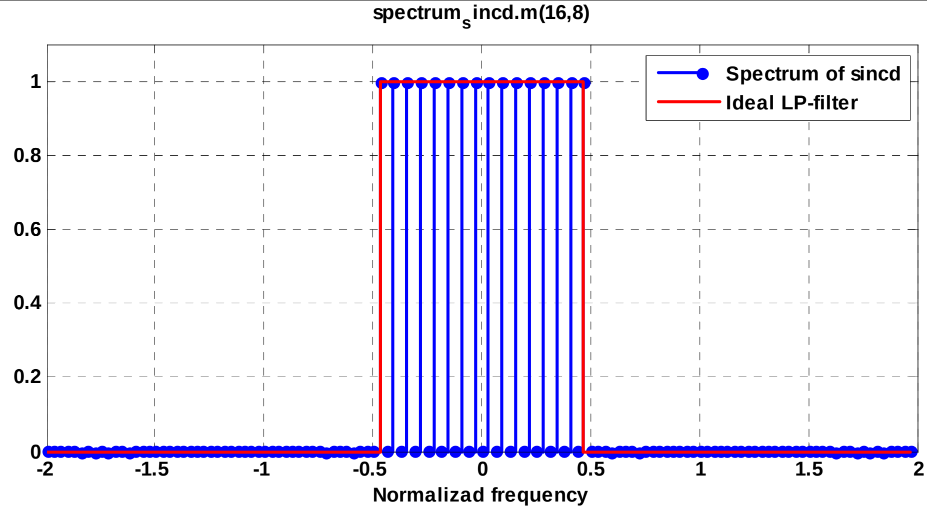
dct_vs_dft_conv.m

Discrete Fourier Transforms

Transform	
Canonical Discrete Fourier Transform (DFT)	$\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N} \right)$
Shifted DFT	$\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{N} \right]$
Discrete Cosine Transform (DCT)	$\alpha_r^{DCT} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{k+1/2}{N} r \right)$
Discrete Cosine-Sine Transform (DcST)	$\alpha_r^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left(\pi \frac{k+1/2}{N} r \right); \ \alpha_{N-r} = \sum_{k=0}^{N-1} (-1)^k a_k \cos\left(\pi \frac{k+1/2}{N} r \right)$
Scaled DFT	$\alpha_r^{\sigma} = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{\sigma N} \right] = \frac{1}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{\sigma N} \right)$
Scaled DFT as a cyclic convolution	$\alpha_r^{\sigma} = \frac{\exp\left(i\pi \frac{\tilde{r}^2}{\sigma N} \right)}{\sqrt{\sigma N}} \sum_{k=0}^{N-1} \left[a_k \exp\left(i\pi \frac{\tilde{k}^2}{\sigma N} \right) \right] \exp\left[-i\pi \frac{(\tilde{k}-\tilde{r})^2}{\sigma N} \right]$
Canonical 2-D DFT	$\alpha_{r,s} = \frac{1}{\sqrt{N_1 N_2}} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{kr}{N_1} + \frac{ls}{N_1} \right) \right]$
Affine DFT	$\alpha_{r,s} = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk}{\sigma_A N_1} + \frac{sk}{\sigma_C N_1} + \frac{rl}{\sigma_B N_2} + \frac{sl}{\sigma_D N_2} \right) \right]$
Rotated DFT (RotDFT)	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r\cos\theta - s\sin\theta}{N} k + \frac{r\sin\theta + s\cos\theta}{N} l \right) \right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk+sl}{N} \cos\theta - \frac{sk-rl}{N} \sin\theta \right) \right]$
Rotated Scaled DFT	$\alpha_{r,s} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{r\cos\theta - s\sin\theta}{\sigma N} k + \frac{r\sin\theta + s\cos\theta}{\sigma N} l \right) \right] =$ $\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_{k,l} \exp\left[i2\pi \left(\frac{rk+sl}{\sigma N} \cos\theta - \frac{sk-rl}{\sigma N} \sin\theta \right) \right]$
Discrete Sinc-function	$\mathrm{sincd}(N, x) = \frac{\sin x}{N \sin(x/N)}$

Discrete sinc-function vs continuous sinc-function





Frequency response of the ideal low pass filter (red) and Fourier transform of the discrete sinc-function (blue)