Lecture 3. Signal transformations and mathematical models of imaging systems.

1. Signal transformations

Generally, signal transformations are mappings in the signal space. In practice, a hierarchical description is adopted that regard arbitrary signal transformations as a combination of point-wise non-linear and linear transformations.

Point-wise transformations: $b(x=\xi) = \mathrm{PWT}[a(x=\xi)]$

Linear transformations satisfy the superposition principle: $\mathrm{L}\left(\sum_{k=0}^{N-1}\alpha_k a_k(x)\right) = \sum_{k=0}^{N-1}\alpha_k \mathrm{L}(a_k(x))$

Linear signal transformations are defined only for linear signal space. Linearly transformed signals also form a linear signal space. Therefore for linear transforms, one may introduce the notion of the sum of transforms, i.e. $\sum_n \mathrm{L}_n$, as well as the product of transforms, $\prod_n \mathrm{L}_n$. The physical equivalent of the product is a series (cascade) connection of units realizing the transform-factors. The physical equivalent of the sum of linear transforms is a parallel connection of units realizing the transform-items. Thanks to the linearity of the transforms, their multiplication is distributive with respect to addition:

$$\mathrm{L}_1(\mathrm{L}_2+\mathrm{L}_3) = \mathrm{L}_1\mathrm{L}_2+\mathrm{L}_1\mathrm{L}_3;\quad (\mathrm{L}_1+\mathrm{L}_2)\mathrm{L}_3 = \mathrm{L}_1\mathrm{L}_3+\mathrm{L}_2\mathrm{L}_3.$$

If a linear transformation L performs a one-to-one mapping of a signal, then there exists an inverse transformation L^{-1} such that

$$\mathbf{LL}^{-1}\mathbf{a} = \mathbf{L}^{-1}\mathbf{La} = \mathbf{a}$$

A linear transformation of discrete signals is fully characterized by a matrix $\mathbf{L} = \{\lambda_{k,n}\}$ linking the input $\mathbf{a} = \{\alpha_k\}$ and output $\mathbf{b} = \{\beta_k\}$ pairs of the linear transformation $\mathbf{b} = \mathbf{La}$, respectively:

$$\mathbf{b} = \left\{\beta_n = \sum_k \lambda_{k,n}\alpha_k\right\}.$$

Thus, in order to describe a linear transformation of a sequence of N numbers, it suffices to specify a matrix of N^2 numbers. In the general, linear transforms are not point-wise, and that they become so only if transformation matrix is a diagonal one ($\{\lambda_{k,n} = \lambda_k\delta_{k-n}\}$.

For continuous signals, integral transforms are the instruments by which one produces a meaningful description of linear transformations. Linear transformation

$$b(\xi) = \mathrm{L}a(x) = \mathrm{L}\int_F \alpha(f)\varphi(x,f)df = \int_F \alpha(f)[\mathrm{L}\varphi(x,f)]df = \int_F \alpha(f)h(\xi,f)df$$

is fully specified by its response $h(\xi, f) = \mathrm{L}(\varphi(x,f))$ to the transform kernel. Spectra $\beta(f)$ and $\alpha(f)$ of a signal and its linear transformation are also related with an integral transform

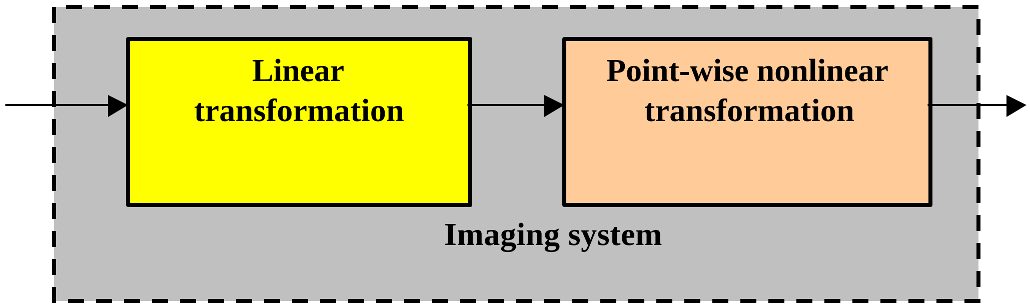
$$\beta(f) = \int_F \alpha(p)H(f,p)dp, \text{ where } H(f,p) = \int h(\xi,p)\phi(f,\xi)d\xi$$

Signal integral representation via delta-function is of especial importance for the characterization of linear transformations. For such a representation, the relationship between output signal $b(x)$ and input signal $a(x)$ of a linear transformation is reduced to

$$b(x) = \int_S a(\xi)h(x,\xi)d\xi, \text{ where } h(x,\xi) = \mathrm{L}(\delta(x,\xi))$$

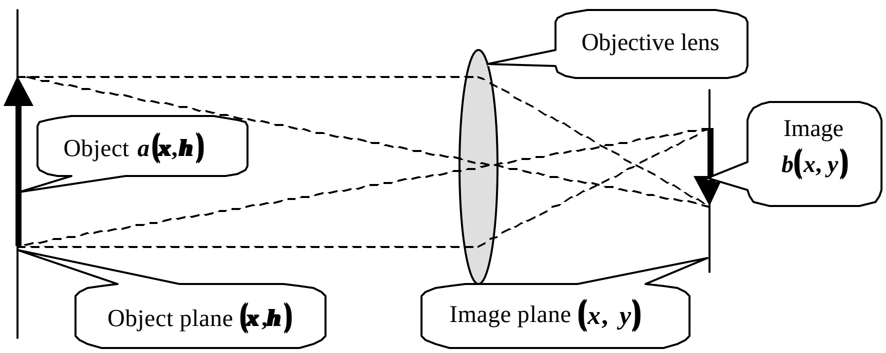
is linear transformation response to a delta-function called ***impulse response*** or ***point spread function*** of a linear system that implements this transformation.

Cascade combination of units performing linear and point-wise signal transformations is the basic model used to describe mathematically imaging and holographic systems.



Basic model of imaging and holographic systems

2. Direct image plane imaging

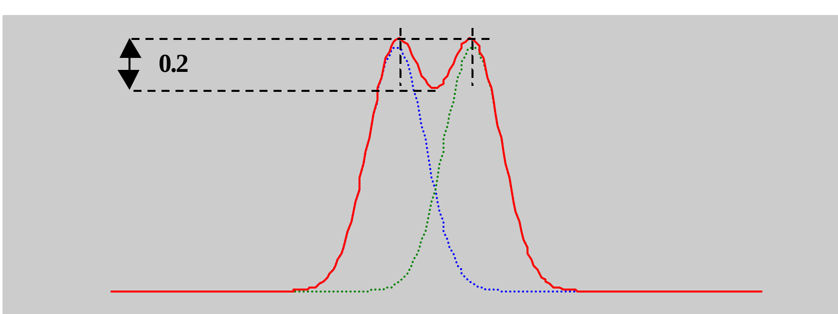


$$b(x,y)=\int_X\int_Y a(\xi,\eta)h(x,y;\xi,\eta)d\xi d\eta$$

In space-invariant imaging: $h(x,y;\xi,\eta)=h(x-\xi,y-\eta)$. It is modeled by the convolution integral

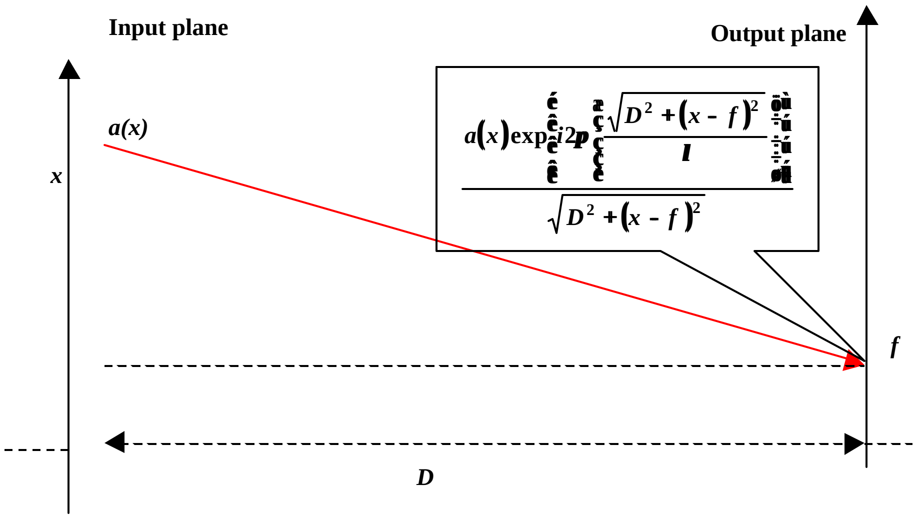
$$b(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$

Point spread function is the basic specification characteristic of imaging systems. It defines imaging system ***resolving power***. On a qualitative level it is commonly accepted that resolving power of imaging systems is defined by the ***Rayleigh's criterion***: two point sources are considered resolved if minimum between two corresponding PSF peaks does not exceed 80% or so of the peak maxima.



Rayleigh's criterion of the resolving power

3. Transform domain imaging



Free space wave propagation point spread function

$$h(x, y; f_x, f_y) = \frac{\exp\left[i2\pi\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2}\Big/\lambda\right]}{\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2}},$$

Kirchhoff's integral : the relationship between complex amplitudes $a(x, y)$ and $\alpha(f_x, f_y)$, respectively of the wave front in the input and output planes

$$\alpha(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x, y)\frac{\exp\left[i2\pi\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2}\Big/\lambda\right]}{\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2}}dxdy.$$

In "near zone" approximation: $\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2} \cong D$;

$$\exp\left[i2\pi\sqrt{D^2 + (x - f_x)^2 + (y - f_y)^2}\Big/\lambda\right] \cong \exp\left[i\pi\frac{(x - f_x)^2 + (y - f_y)^2}{\lambda D}\right].$$

Kirchhoff's integral is reduced to its Fresnel approximation:

$$\alpha(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x, y)\exp\left[i\pi\frac{(x - f_x)^2 + (y - f_y)^2}{\lambda D}\right]dxdy.$$

Far zone approximation: $\pi\left(f_x^2 + f_y^2\right)/\lambda D \approx 0$ and $\pi\left(x^2 + y^2\right)/\lambda D \approx 0$, Fourier integral can be used:

$$\alpha(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x, y)\exp\left[-i2\pi\frac{xf_x + yf_y}{\lambda D}\right]dxdy.$$

Properties of Fourier transform

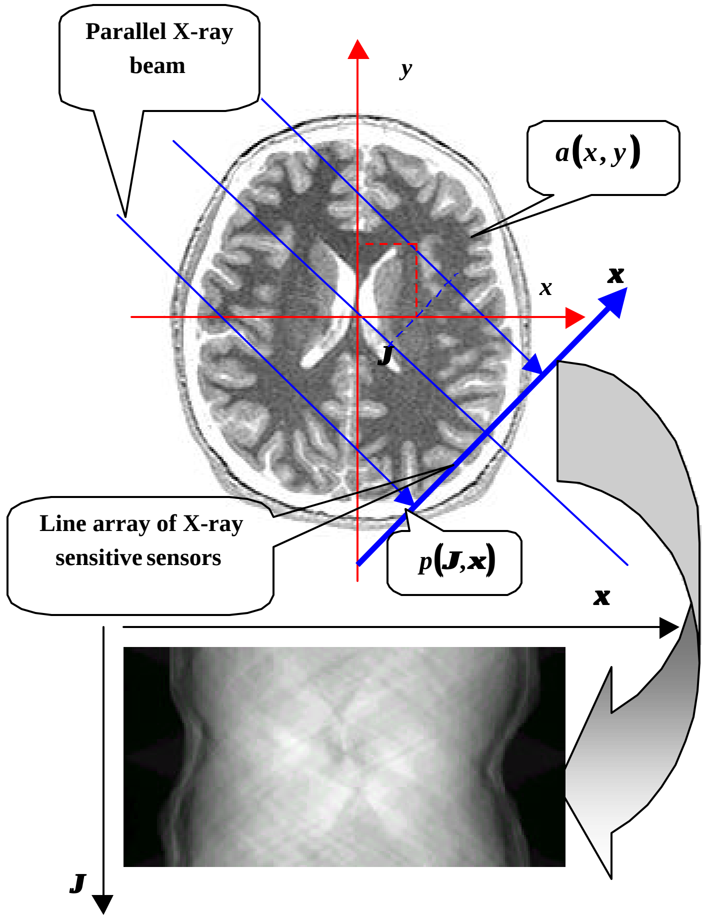
Signal	Fourier Transform
$a(x)=\lim_{F\to\infty}\int_{-F}^{F}\alpha(f)\exp(-i2\pi fx)df = \int_{-\infty}^{\infty}\alpha(f)\exp(-i2\pi fx)df$	$\alpha(f)=\int_{-\infty}^{\infty}a(x)\exp(i2\pi fx)dx$
$a(0)=\int_{-\infty}^{\infty}\alpha(f)df$	$\alpha(0)=\int_{-\infty}^{\infty}a(x)dx$
$a(-x)$	$\alpha(-f)$
$a(x)=\pm a(-x)$	$\alpha(f)=\pm\alpha(-f)$
$a^*(x)$	$\alpha^*(-f)$
$a(x)=\pm a^*(x)$	$\alpha(f)=\pm\alpha^*(-f)$
$a(x)=\pm a(-x)=a^*(x)$	$\alpha(f)=\pm\alpha(-f)=\alpha^*(-f)$
$\frac{d^n}{(dx)^n}a(x)$	$(i2\pi f)^n\alpha(f)$
$\frac{1}{\Delta_x}rect\left(\frac{x+\Delta_x/2}{\Delta_x}\right)$	$\frac{1}{\Delta_x}\int_{-\Delta_x/2}^{\Delta_x/2}\exp(i2\pi fx)dx=\frac{\sin(\pi f\Delta_x)}{\pi f\Delta_x}$
$\int_{-F}^{F}\exp(-i2\pi fx)df=2F\frac{\sin(2\pi Fx)}{2\pi Fx}=2F\,\mathrm{sinc}(2\pi Fx)$	$rect\left(\frac{x+F}{2F}\right)$
$\exp(-i2\pi f_0x)$	$\alpha(f)=\lim_{F\to\infty}\int_{-F}^{F}\exp[i2\pi(f-f_0)x]df = \lim_{X\to\infty}2X\,\mathrm{sinc}[2\pi X(f-f_0)]=\delta(f-f_0)$
$\int_{-\infty}^{X}a(x)dx$	$\frac{i}{2\pi f}\alpha(f)+\alpha(0)\delta(f)$
$\cos(2\pi f_0x)$	$[\delta(f+f_0)+\delta(f-f_0)]/2$
$\sin(2\pi f_0x)$	$[\delta(f+f_0)-\delta(f-f_0)]/2i$
Shift theorem: $a(x-x_0)$	$\alpha(f)\exp(-i2\pi fx_0)$
Convolution theorem $b(x)=\int_{-\infty}^{\infty}a(\xi)h(x-\xi)d\xi$	$\beta(f)=\alpha(f)\eta(f)$, where $\alpha(f)$ and $\eta(f)$ are Fourier Transforms of $a(x)$ and $h(x)$
Parceval's identities: $\int_{-\infty}^{\infty}a(\xi)h^*(\xi)d\xi=\int_{-\infty}^{\infty}\alpha(f)\eta^*(f)df$;	$\int_{-\infty}^{\infty}\lvert a(x)\rvert^2d\xi=\int_{-\infty}^{\infty}\lvert\alpha(f)\rvert^2df$
$a(x)\cos(2\pi f_0x)$	$[\alpha(f+f_0)+\alpha(f-f_0)]/2$
$\sum_{k=-\infty}^{\infty}a(k\Delta_x)\delta(x-k\Delta_x)$	$\frac{1}{\Delta_x}\sum_{r=-\infty}^{\infty}\alpha\left(f-\frac{r}{\Delta_x}\right)$

2-D Fourier Transform:

$a(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(f_x,f_y)\exp[-i2\pi(f_xx+f_yy)]df_xdf_y$	$\alpha(f_x,f_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(x,y)\exp[i2\pi(f_xx+f_yy)]dxdy$

For shift-invariant imaging systems, Fourier Transform of impulse response is called ***frequency response*** of ***modulation transfer function*** (**MTF**).

4. Imaging from projections



Radon transform: $$p(\theta,\xi)=\int_X\int_Y a(x,y)\,\delta(\xi - x\cos\theta - y\sin\theta)\,dxdy$$

Projection theorem:

$$\rho(\theta, f_\xi)=\int_{-\infty}^{\infty} p(\theta,\xi)\exp(i2\pi f_\xi \xi)\,d\xi = \int_X\int_Y a(x,y)\exp\left[i2\pi f_\xi(x\cos\theta + y\sin\theta)\right]dxdy =$$
$$\alpha(f_x = f_\xi\cos\theta,\ f_y = f_\xi\sin\theta)$$

5. Stochastic transforms and noise models

Additive signal independent noise model (ASIN-model) :

$$b(x)=a(x)+n(x),$$

Multiplicative noise (MN-) model assumes signal transformation defined as

$$b(x)=m(x)\cdot a(x),$$

Impulse noise model (ImpN-) model

$$b(x)=[1-e(x)]\cdot a(x)+e(x)n(x)$$

Home work HW-1: Derive, following the table of properties of 1-D Fourier Transform, as many properties of 2-D Fourier Transform as you can