Lect. 3. Perfect resampling filter

Resampling task: applications and accuracy requirements

- Shift-variant image filtering
- Recovery of missing data and image super-resolution from multiple differently sampled frames
- Alignment video frames or images of different modalities
- Synthesis of stereo sequences in 3D visualization
- Signal differentiation and integration
- Image synthesis in 3D computer graphics.

Forward and backward resampling.

Convolutional resampling digital filter:

$$\tilde{a}_k^{\tilde{x}} = \sum_{n=0}^{N-1} h_n^{(intp)}(\tilde{x}) a_{k-n}$$

Most widely used resampling methods:

- nearest neighbor interpolation,
- linear (bilinear) interpolation,
- cubic (bicubic) spline,
- ………………………
- higher order splines.

Signal fractional shift as a basic resampling procedure.

Frequency response of the continuous signal ideal δx -shift filter: $H^{(intp)}(f) = \exp(i2\pi\delta x f)$

According to Theorem 2 (Lect. 1), DFT of the perfect δx -shifting resampling filter discrete PSF is:

For odd N:

$$\eta_{r,opt}^{(intp)}(\delta x) = \begin{cases} \dfrac{1}{\sqrt{N}} \exp(i2\pi r\delta x/N\Delta x), & r = 0,1,...,(N-1)/2 \\ \eta_{N-r,opt}^{*(intp)}(\delta x), & r = (N+1)/2,..., N-1 \end{cases}$$

For even N :

$$\eta_{r,opt}^{(intp)}(\delta x) = \begin{cases} \exp(i2\pi r\delta x/N\Delta x), & r = 0,1,..., N/2-1 \\ C\cos(\pi\delta x/\Delta x), & r = N/2 \\ \eta_{r,opt}^{*(intp)}(\delta x), & r = N/2+1,..., N-1 \end{cases}$$

Case 0: C=0; Case 1: C=1; Case 2, C=2

Impulse responses of the perfect interpolation filter (interpolation kernels):

For odd N: $h_n^{(intp)}(\delta x) = \mathrm{sincd}[N, \pi(n - \delta x/\Delta x)]$;

For even N:

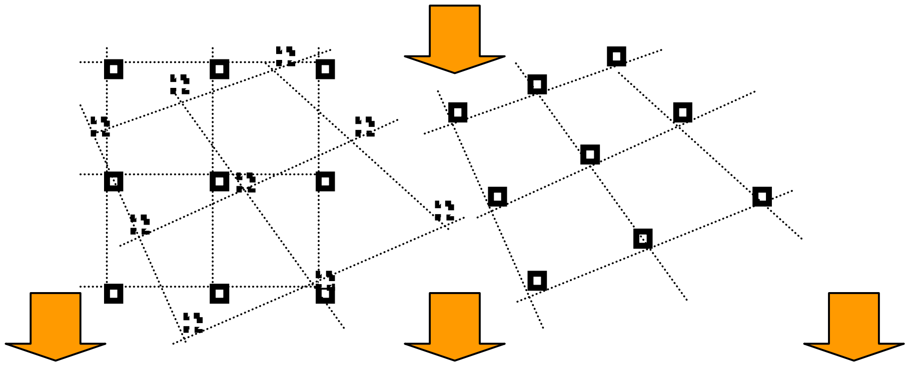
$$h_n^{(intp0)}(\delta x) = \{\overline{\mathrm{sincd}}[N-1; N; \pi(n-\delta x/\Delta x)] + \overline{\mathrm{sincd}}[N+1; N; \pi(n-\delta x/\Delta x)]\}/2 =$$

$$\cos\left(\pi\frac{n-\delta x/\Delta x}{N}\right)\mathrm{sincd}[N; \pi(n-\delta x/\Delta x)]$$

where

$$\mathrm{sincd}\{N; x\} = \frac{\sin(x)}{N\sin(x/N)} \ ; \qquad \overline{\mathrm{sincd}}\{M; N; x\} = \frac{\sin(Mx/N)}{N\sin(x/N)} \ .$$

<u>Theorem 3.</u> *Discrete sinc-interpolation is the only cyclic convolution based discrete interpolation method that, for odd **N**, does not introduce any signal distortion; for even **N**, only signal **N**/2-th spectral component is distorted*



y=x+3sin(2pix/32); mapping(lenna,y,y)

y=x+3cos(2pix/32); mapping(lenna,y,y)

y=x+20exp(((x-128)^2/1024); mapping(lenna,y,y)

The principle of image geometrical transformations through resampling

Point spread functions of optimal re-sampling filters

Find point spread function of the optimal re-sampling filter $h_n^{(intp)}(\delta x)$ as DFT of samples $\left\{\eta_{r,opt}^{(intp)}(\delta x)\right\}$ of its continuous frequency response.

For odd number of signal samples N :

$$
\begin{aligned}
&h_n^{(intp)}(\delta x) = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{nr}{N}\right) = \\
&\frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=(N+1)/2}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) \right\} = \\
&\frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp\left[-i2\pi \frac{n}{N}(N-r)\right] \right\} = \\
&\frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp[-i2\pi n] \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
&\frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
&\frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left(i2\pi \frac{\delta x}{N\Delta x}\right) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \exp\left(-i2\pi \frac{\delta x}{N\Delta x}\right) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
&\frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left[-i2\pi \frac{n - \delta x/\Delta x}{N} r\right] + \sum_{r=1}^{(N-1)/2} \exp\left[i2\pi \frac{n - \delta x/\Delta x}{N} r\right] \right\}. \qquad (1)
\end{aligned}
$$

Denote $\tilde{n} = n - \delta x/\Delta x$

Then

$$
\begin{aligned}
&h_n^{(intp)}(\delta x) = \frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left(-i2\pi \frac{\tilde{n} r}{N}\right) + \sum_{r=1}^{(N-1)/2} \exp\left(i2\pi \frac{\tilde{n} r}{N}\right) \right\} = \\
&\frac{1}{N} \left\{ \frac{\exp\left[\frac{-i\pi(N+1)\tilde{n}}{N}\right] - 1}{\exp\left(-i2\pi \frac{\tilde{n}}{N}\right) - 1} + \frac{\exp\left[\frac{i\pi(N+1)\tilde{n}}{N}\right] - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i2\pi \frac{\tilde{n}}{N}\right) - 1} \right\} = \\
&\frac{1}{N} \left\{ \frac{\left\{1 - \exp\left[\frac{-i\pi(N+1)\tilde{n}}{N}\right]\right\} \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} + \right. \\
&\left. + \frac{\left\{\exp\left[\frac{i\pi(N+1)\tilde{n}}{N}\right] - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)\right\} \exp\left(-i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} = \\
&\frac{1}{N} \left\{ \frac{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp(-i\pi\tilde{n}) + \exp(i\pi\tilde{n}) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right. =
\end{aligned}
$$

$$\frac{1}{N}\frac{\sin(\pi\tilde{n})}{\sin\left(\pi\frac{\tilde{n}}{N}\right)}=\frac{\sin[\pi(n-\delta x/\Delta x)]}{N\sin\left(\pi\frac{n-\delta x/\Delta x}{N}\right)}=\mathrm{sincd}(N;[\pi(n-\delta x/\Delta x)]) \qquad (2)$$

For even number of signal samples N, we have:

$$h_n^{(intp)}(\delta x)=\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)=$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{r=0}^{N/2-1}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)+\sum_{r=N/2}^{N-1}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{r=0}^{N/2-1}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)+\sum_{r=1}^{N/2}\eta_{N-r,opt}^{(intp)}(\delta x)\exp\left[-i2\pi\frac{n}{N}(N-r)\right]\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{r=0}^{N/2-1}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)+\sum_{r=1}^{N/2}\eta_{N-r,opt}^{(intp)}(\delta x)\exp(-i2\pi n)\exp\left(i2\pi\frac{n}{N}r\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{\sum_{r=0}^{(N-1)/2}\eta_{r,opt}^{(intp)}(\delta x)\exp\left(-i2\pi\frac{n}{N}r\right)+\sum_{r=1}^{N/2}\eta_{r,opt}^{*(intp)}(\delta x)\exp\left(i2\pi\frac{n}{N}r\right)\right\}=$$

$$\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(i2\pi\frac{\delta x}{N\Delta x}\right)\exp\left(-i2\pi\frac{n}{N}r\right)+\right.$$

$$\left.\sum_{r=1}^{N/2-1}\exp\left(-i2\pi\frac{\delta x}{N\Delta x}\right)\exp\left(i2\pi\frac{n}{N}r\right)+\eta_{N/2,opt}^{(intp)}(\delta x)\exp\left(i\pi N\frac{n}{N}\right)\right\}=$$

$$\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{n-\delta x/\Delta x}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{n-\delta x/\Delta x}{N}r\right)+\eta_{N/2,opt}^{(intp)}(\delta x)\exp(i\pi n)\right\}$$

Denote $\tilde{n}=n-\delta x/\Delta x$. Then

$$h_n^{(intp)}(\delta x)=\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)+\eta_{N/2,opt}^{(intp)}(\delta x)\exp(i\pi n)\right\}; \qquad (3)$$

Case 0: $\eta_{N/2,opt}^{(intp)}(\delta x)=0;$

$$h_n^{(intp)}(\delta x)=\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp(-i\pi\tilde{n})-1}{\exp\left(-i2\pi\frac{\tilde{n}}{N}\right)-1}+\frac{\exp(-i\pi\tilde{n})-\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)}{\exp\left(i2\pi\frac{\tilde{n}}{N}\right)-1}\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp\left(-i\pi\frac{N-1}{N}\tilde{n}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}\right)}{\exp\left(-i\pi\frac{\tilde{n}}{N}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}\right)}+\frac{\exp\left(i\pi\frac{N-1}{N}\tilde{n}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}r\right)}{\exp\left(i\pi\frac{\tilde{n}}{N}\right)-\exp\left(-i\pi\frac{\tilde{n}}{N}\right)}\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp\left(i\pi\frac{N-1}{N}\tilde{n}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}r\right)-\exp\left(-i\pi\frac{N-1}{N}\tilde{n}\right)+\exp\left(i\pi\frac{\tilde{n}}{N}\right)}{\exp\left(i\pi\frac{\tilde{n}}{N}\right)-\exp\left(-i\pi\frac{\tilde{n}}{N}\right)}\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp\left(i\pi\frac{N-1}{N}\tilde{n}\right)-\exp\left(-i\pi\frac{N-1}{N}\tilde{n}\right)}{\exp\left(i\pi\frac{\tilde{n}}{N}\right)-\exp\left(-i\pi\frac{\tilde{n}}{N}\right)}\right\}=\frac{1}{N}\frac{\sin\left(\pi\frac{N-1}{N}\tilde{n}\right)}{N\sin\left(\pi\frac{\tilde{n}}{N}\right)} \qquad (4)$$

$$h_n^{(intp0)}(\delta x)=\overline{\mathrm{sincd}}\{N;N-1;\pi[n-\delta x/\Delta x]\}, \qquad (5)$$

where

$$\overline{\mathrm{sincd}}(M;N;x)=\frac{\sin(Mx/N)}{N\sin(x/N)}$$

Case_2: $\eta_{N/2,opt}^{(intp2)}(\delta x)=2\cos\left(\pi\frac{\delta x}{\Delta x}\right)=\exp\left(i\pi\frac{\delta x}{\Delta x}\right)+\exp\left(-i\pi\frac{\delta x}{\Delta x}\right);$

Proceed from Eq. (3):

$$h_n^{(intp)}(\delta x)=\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)+\eta_{N/2,opt}^{(intp)}(\delta x)\exp(i\pi n)\right\}=$$

$$\frac{1}{N}\left\{\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)+\left[\exp\left(i\pi\frac{\delta x}{\Delta x}\right)+\exp\left(-i\pi\frac{\delta x}{\Delta x}\right)\right]\exp(i\pi n)\right\}$$

$=$

$$\frac{1}{N}\left\{\begin{matrix}\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)+\\ \exp(i\pi n)\exp\left(i\pi\frac{\delta x}{\Delta x}\right)+\exp(i\pi n)\exp\left(-i\pi\frac{\delta x}{\Delta x}\right)\end{matrix}\right\}=$$

$$\frac{1}{N}\left\{\begin{matrix}\sum_{r=0}^{N/2-1}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2-1}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)+\\ \exp(-i\pi n)\exp\left(i\pi\frac{\delta x}{\Delta x}\right)+\exp(i\pi n)\exp\left(-i\pi\frac{\delta x}{\Delta x}\right)\end{matrix}\right\}=$$

$$\frac{1}{N}\left\{\sum_{r=0}^{N/2}\exp\left(-i2\pi\frac{\tilde{n}}{N}r\right)+\sum_{r=1}^{N/2}\exp\left(i2\pi\frac{\tilde{n}}{N}r\right)\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp\left(-i\pi\frac{(N+2)}{N}\tilde{n}\right)-1}{\exp\left(-i2\pi\frac{\tilde{n}}{N}\right)-1}+\frac{\exp\left(i\pi\frac{(N+2)}{N}\tilde{n}\right)-\exp\left(i2\pi\frac{\tilde{n}}{N}\right)}{\exp\left(i2\pi\frac{\tilde{n}}{N}\right)-1}\right\}=$$

$$\frac{1}{N}\left\{\frac{\exp\left(-i\pi\frac{(N+1)}{N}\tilde{n}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}\right)}{\exp\left(-i\pi\frac{\tilde{n}}{N}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}\right)}+\frac{\exp\left(i\pi\frac{(N+1)}{N}\tilde{n}\right)-\exp\left(i\pi\frac{\tilde{n}}{N}\right)}{\exp\left(i\pi\frac{\tilde{n}}{N}\right)-\exp\left(-i\pi\frac{\tilde{n}}{N}\right)}\right\}=$$

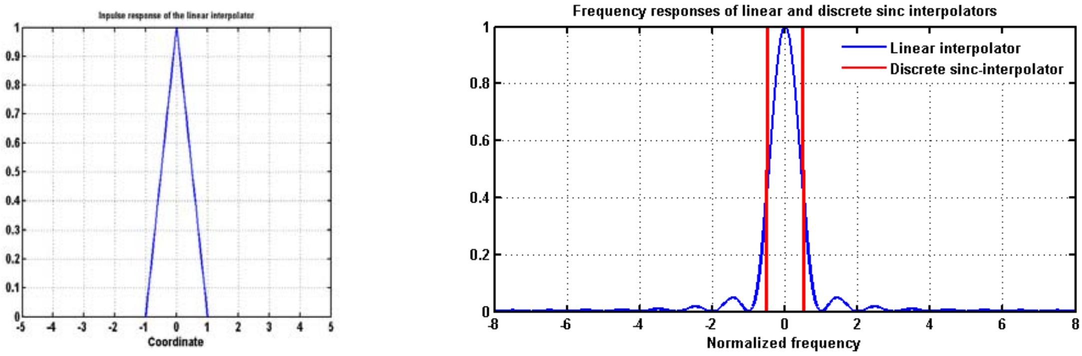
$$\frac{1}{N}\left\{\frac{\exp\left(i\pi\frac{(N+1)}{N}\tilde{n}\right)-\exp\left(-i\pi\frac{(N+1)}{N}\tilde{n}\right)}{\exp\left(i\pi\frac{\tilde{n}}{N}\right)-\exp\left(-i\pi\frac{\tilde{n}}{N}\right)}\right\}=\frac{1}{N}\frac{\sin\left(\pi\frac{(N+1)}{N}\tilde{n}\right)}{\sin\left(\pi\frac{\tilde{n}}{N}\right)}; \qquad (6)$$

$$h_n^{(intp)}(\delta x)=\mathrm{sincd}\left[N+1;N;\pi(n-\delta x/\Delta x)\right] \qquad (7)$$

Case 1: $$\eta_{N/2,opt}^{(intp1)}(\delta x)=\cos\left(\pi\frac{\delta x}{\Delta x}\right)=\frac{\eta_{N/2,opt}^{(intp0)}(\delta x)+\eta_{N/2,opt}^{(intp2)}(\delta x)}{2}=$$

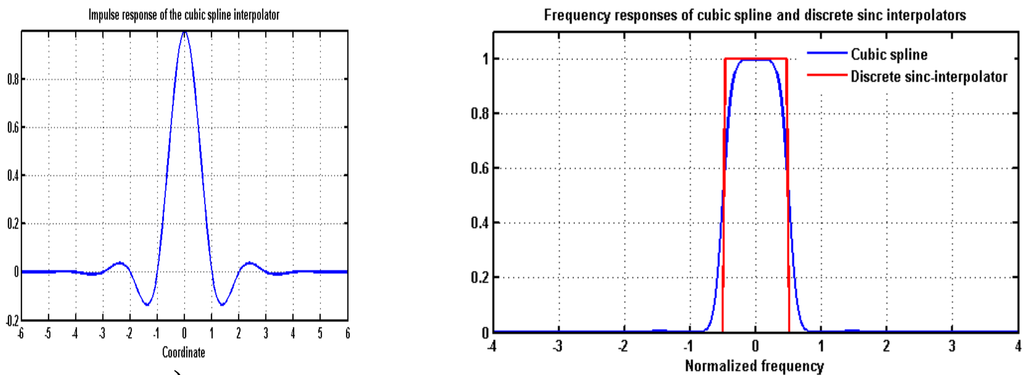
$$\frac{1}{N}\frac{\sin\left(\pi\frac{(N+1)}{N}\tilde{n}\right)+\sin\left(\pi\frac{(N-1)}{N}\tilde{n}\right)}{2\sin\left(\pi\frac{\tilde{n}}{N}\right)}=\frac{1}{N}\frac{\sin(\pi\tilde{n})}{\sin\left(\pi\frac{\tilde{n}}{N}\right)}\cos\left(\pi\frac{\tilde{n}}{N}\right);$$

$$\eta_{N/2,opt}^{(intp1)}(\delta x)=\cos\left(\pi\frac{n-\delta x/\Delta x}{N}\right)\mathrm{sincd}\left[N;\pi(n-\delta x/\Delta x)\right]$$



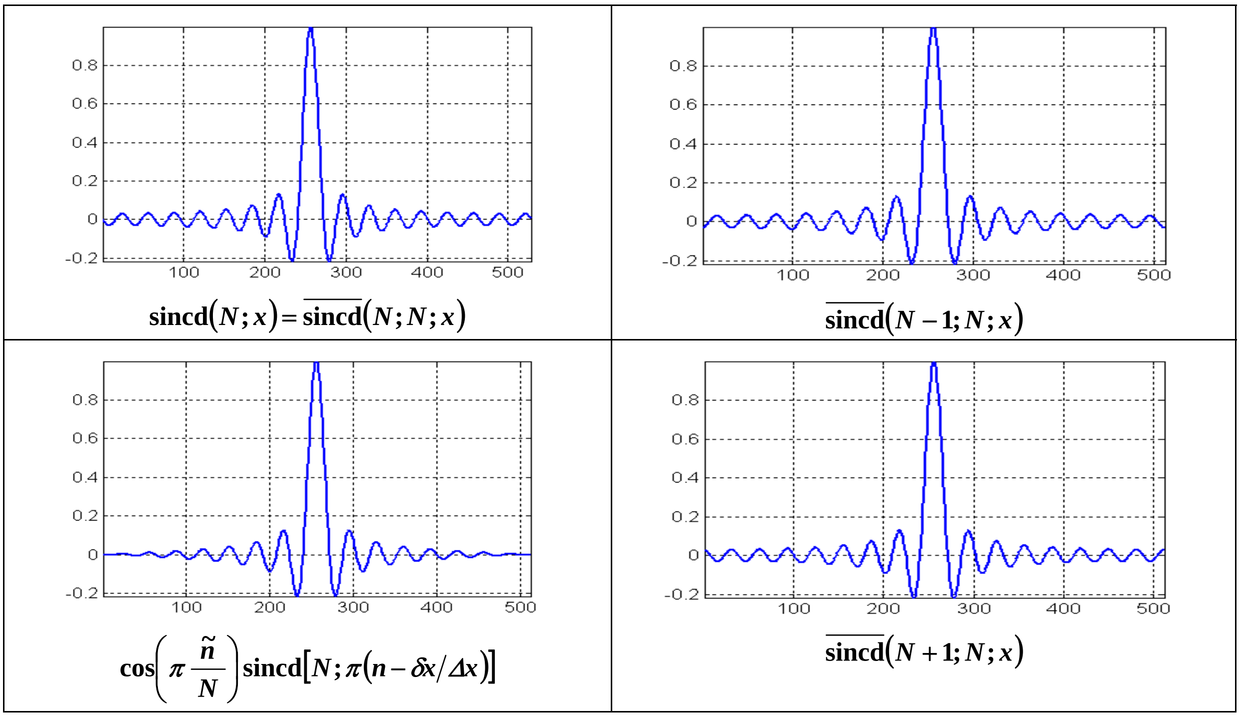
a) b)

Linear interpolation kernel (a) and frequency response (b)



a) b)

Cardinal cubic spline interpolation kernel (a) and frequency response (b)



Four versions of discrete sinc-function