

Lect. 3. Perfect resampling filter

Resampling task: applications and accuracy requirements

- Shift-variant image filtering
- Recovery of missing data and image super-resolution from multiple differently sampled frames
- Alignment video frames or images of different modalities
- Synthesis of stereo sequences in 3D visualization
- Signal differentiation and integration
- Image synthesis in 3D computer graphics.

Forward and backward resampling.

Convolutional resampling digital filter:

$$\tilde{a}_k^{\tilde{x}} = \sum_{n=0}^{N-1} h_n^{(intp)}(\tilde{x}) a_{k-n}$$

Most widely used resampling methods:

- nearest neighbor interpolation,
- linear (bilinear) interpolation,
- cubic (bicubic) spline,
-
- higher order splines.

Signal fractional shift as a basic resampling procedure.

Frequency response of the continuous signal ideal δx -shift filter: $H^{(intp)}(f) = \exp(i2\pi\delta x f)$

According to Theorem 2 (Lect. 1), DFT of the perfect δx -shifting resampling filter discrete PSF is:

$$\begin{aligned} \text{For odd } N: \quad \eta_{r,opt}^{(intp)}(\delta x) &= \begin{cases} \frac{1}{\sqrt{N}} \exp(i2\pi r \delta x / N \Delta x), & r = 0, 1, \dots, (N-1)/2 \\ \eta_{N-r,opt}^{*(intp)}(\delta x) & , r = (N+1)/2, \dots, N-1 \end{cases} \\ \text{For even } N: \quad \eta_{r,opt}^{(intp)}(\delta x) &= \begin{cases} \exp(i2\pi r \delta x / N \Delta x), & r = 0, 1, \dots, N/2-1 \\ C \cos(\pi \delta x / \Delta x), & r = N/2 \\ \eta_{r,opt}^{*(intp)}(\delta x), & r = N/2+1, \dots, N-1 \end{cases} \end{aligned}$$

Case 0: C=0; Case 1: C=1; Case 2, C=2

Impulse responses of the perfect interpolation filter (interpolation kernels):

For odd N: $h_n^{(intp)}(\delta x) = \text{sincd}[N, \pi(n - \delta x / \Delta x)]$;

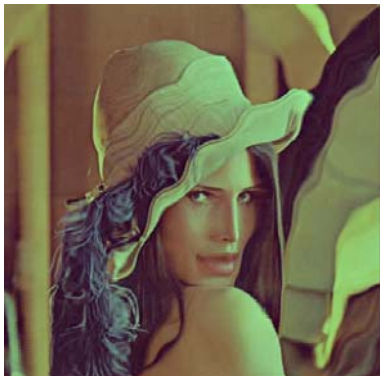
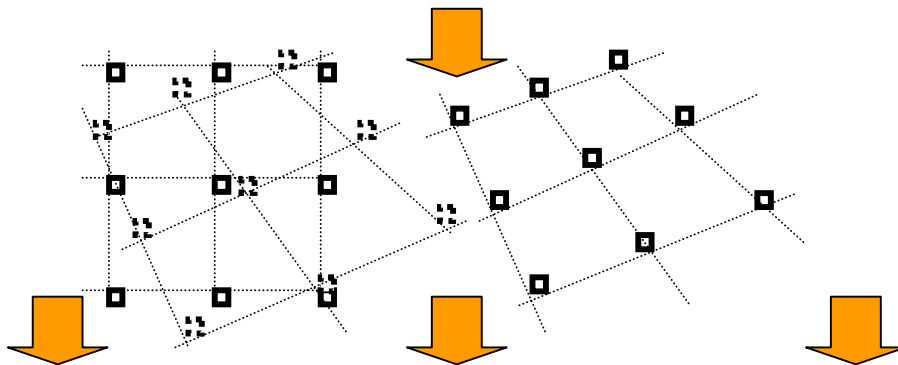
For even N:

$$h_n^{(intp^0)}(\delta x) = \left\{ \text{sincd}[N-1; N; \pi(n - \delta x / \Delta x)] + \text{sincd}[N+1; N; \pi(n - \delta x / \Delta x)] \right\} / 2 = \cos\left(\pi \frac{n - \delta x / \Delta x}{N}\right) \text{sincd}[N; \pi(n - \delta x / \Delta x)]$$

where

$$\text{sincd}\{N; x\} = \frac{\sin(x)}{N \sin(x/N)} ; \quad \overline{\text{sincd}}\{M; N; x\} = \frac{\sin(Mx/N)}{N \sin(x/N)} .$$

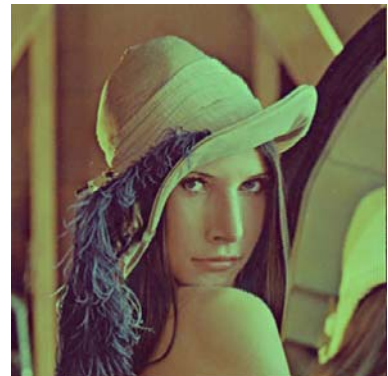
Theorem 3. Discrete sinc-interpolation is the only cyclic convolution based discrete interpolation method that, for odd N, does not introduce any signal distortion; for even N, only signal N/2-th spectral component is distorted



$y=x+3\sin(2\pi x/32)$; mapping(lenna,y,y)



$y=x+3\cos(2\pi x/32)$; mapping(lenna,y,y)



$y=x+20\exp(-((x-128)^2/1024))$; mapping(lenna,y,y)



The principle of image geometrical transformations through resampling

Point spread functions of optimal re-sampling filters

Find point spread function of the optimal re-sampling filter $h_n^{(intp)}(\delta x)$ as DFT of samples $\{\eta_{r,opt}^{(intp)}(\delta x)\}$ of its continuous frequency response.

For odd number of signal samples N :

$$\begin{aligned}
 h_n^{(intp)}(\delta x) &= \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{nr}{N}\right) = \\
 &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=(N+1)/2}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) \right\} = \\
 &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp\left[-i2\pi \frac{n}{N} (N-r)\right] \right\} = \\
 &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp[-i2\pi n] \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
 &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
 &= \frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left(i2\pi \frac{\delta x}{N\Delta x}\right) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{(N-1)/2} \exp\left(-i2\pi \frac{\delta x}{N\Delta x}\right) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\
 &= \frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left[-i2\pi \frac{n - \delta x/\Delta x}{N} r\right] + \sum_{r=1}^{(N-1)/2} \exp\left[i2\pi \frac{n - \delta x/\Delta x}{N} r\right] \right\}. \quad (1)
 \end{aligned}$$

Denote $\tilde{n} = n - \delta x/\Delta x$

Then

$$\begin{aligned}
 h_n^{(intp)}(\delta x) &= \frac{1}{N} \left\{ \sum_{r=0}^{(N-1)/2} \exp\left(-i2\pi \frac{\tilde{n}r}{N}\right) + \sum_{r=1}^{(N-1)/2} \exp\left(i2\pi \frac{\tilde{n}r}{N}\right) \right\} = \\
 &= \frac{1}{N} \left\{ \frac{\exp\left[\frac{-i\pi(N+1)\tilde{n}}{N}\right] - 1}{\exp\left(-i2\pi \frac{\tilde{n}}{N}\right) - 1} + \frac{\exp\left[\frac{i\pi(N+1)\tilde{n}}{N}\right] - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i2\pi \frac{\tilde{n}}{N}\right) - 1} \right\} = \\
 &= \frac{1}{N} \left\{ \frac{\left\{1 - \exp\left[\frac{-i\pi(N+1)\tilde{n}}{N}\right]\right\} \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} + \right. \\
 &+ \left. \frac{\left\{\exp\left[\frac{i\pi(N+1)\tilde{n}}{N}\right] - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)\right\} \exp\left(-i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} = \\
 &= \frac{1}{N} \left\{ \frac{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp(-i\pi\tilde{n}) + \exp(i\pi\tilde{n}) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} =
 \end{aligned}$$

$$\frac{1}{N} \frac{\sin(\pi \tilde{n})}{\sin\left(\pi \frac{\tilde{n}}{N}\right)} = \frac{\sin[\pi(n - \delta x / \Delta x)]}{N \sin\left(\pi \frac{n - \delta x / \Delta x}{N}\right)} = \text{sincd}(N; [\pi(n - \delta x / \Delta x)]) \quad (2)$$

For even number of signal samples N , we have:

$$\begin{aligned} h_n^{(intp)}(\delta x) &= \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) = \\ &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{N/2-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=N/2}^{N-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) \right\} = \\ &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{N/2-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{N/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp\left[-i2\pi \frac{n}{N} (N-r)\right] \right\} = \\ &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{N/2-1} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{N/2} \eta_{N-r,opt}^{(intp)}(\delta x) \exp(-i2\pi n) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\ &= \frac{1}{\sqrt{N}} \left\{ \sum_{r=0}^{(N-1)/2} \eta_{r,opt}^{(intp)}(\delta x) \exp\left(-i2\pi \frac{n}{N} r\right) + \sum_{r=1}^{N/2} \eta_{r,opt}^{*(intp)}(\delta x) \exp\left(i2\pi \frac{n}{N} r\right) \right\} = \\ &= \frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(i2\pi \frac{\delta x}{N \Delta x}\right) \exp\left(-i2\pi \frac{n}{N} r\right) + \right. \\ &\quad \left. \sum_{r=1}^{N/2-1} \exp\left(-i2\pi \frac{\delta x}{N \Delta x}\right) \exp\left(i2\pi \frac{n}{N} r\right) + \eta_{N/2,opt}^{(intp)}(\delta x) \exp\left(i\pi n \frac{n}{N}\right) \right\} = \\ &= \frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{n - \delta x / \Delta x}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{n - \delta x / \Delta x}{N} r\right) + \eta_{N/2,opt}^{(intp)}(\delta x) \exp(i\pi n) \right\} \end{aligned}$$

Denote $\tilde{n} = n - \delta x / \Delta x$. Then

$$h_n^{(intp)}(\delta x) = \frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) + \eta_{N/2,opt}^{(intp)}(\delta x) \exp(i\pi n) \right\}; \quad (3)$$

Case 0: $\eta_{N/2,opt}^{(intp)}(\delta x) = 0$;

$$\begin{aligned} h_n^{(intp)}(\delta x) &= \frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) \right\} = \\ &= \frac{1}{N} \left\{ \frac{\exp(-i\pi \tilde{n}) - 1}{\exp\left(-i2\pi \frac{\tilde{n}}{N}\right) - 1} + \frac{\exp(-i\pi \tilde{n}) - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i2\pi \frac{\tilde{n}}{N}\right) - 1} \right\} = \\ &= \frac{1}{N} \left\{ \frac{\exp\left(-i\pi \frac{N-1}{N} \tilde{n}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(-i\pi \frac{\tilde{n}}{N}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)} + \frac{\exp\left(i\pi \frac{N-1}{N} \tilde{n}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} = \end{aligned}$$

$$\frac{1}{N} \left\{ \frac{\exp\left(i\pi \frac{N-1}{N} \tilde{n}\right) - \exp\left(i\pi \frac{\tilde{n}}{N} r\right) - \exp\left(-i\pi \frac{N-1}{N} \tilde{n}\right) + \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} =$$

$$\frac{1}{N} \left\{ \frac{\exp\left(i\pi \frac{N-1}{N} \tilde{n}\right) - \exp\left(-i\pi \frac{N-1}{N} \tilde{n}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} = \frac{1}{N} \frac{\sin\left(\pi \frac{N-1}{N} \tilde{n}\right)}{N \sin\left(\pi \frac{\tilde{n}}{N}\right)} \quad (4)$$

$$h_n^{(inp0)}(\delta x) = \overline{\text{sincd}}\{N; N-1; \pi[n - \delta x/\Delta x]\}, \quad (5)$$

where

$$\overline{\text{sincd}}(M; N; x) = \frac{\sin(Mx/N)}{N \sin(x/N)}$$

$$\text{Case 2: } \eta_{N/2, opt}^{(inp2)}(\delta x) = 2 \cos\left(\pi \frac{\delta x}{\Delta x}\right) = \exp\left(i\pi \frac{\delta x}{\Delta x}\right) + \exp\left(-i\pi \frac{\delta x}{\Delta x}\right);$$

Proceed from Eq. (3):

$$h_n^{(inp)}(\delta x) = \frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) + \eta_{N/2, opt}^{(inp)}(\delta x) \exp(i\pi r) \right\} =$$

$$\frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) + \left[\exp\left(i\pi \frac{\delta x}{\Delta x}\right) + \exp\left(-i\pi \frac{\delta x}{\Delta x}\right) \right] \exp(i\pi r) \right\}$$

$$=$$

$$\frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) + \right. \\ \left. \exp(i\pi r) \exp\left(i\pi \frac{\delta x}{\Delta x}\right) + \exp(i\pi r) \exp\left(-i\pi \frac{\delta x}{\Delta x}\right) \right\} =$$

$$\frac{1}{N} \left\{ \sum_{r=0}^{N/2-1} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2-1} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) + \right. \\ \left. \exp(-i\pi r) \exp\left(i\pi \frac{\delta x}{\Delta x}\right) + \exp(i\pi r) \exp\left(-i\pi \frac{\delta x}{\Delta x}\right) \right\} =$$

$$\frac{1}{N} \left\{ \sum_{r=0}^{N/2} \exp\left(-i2\pi \frac{\tilde{n}}{N} r\right) + \sum_{r=1}^{N/2} \exp\left(i2\pi \frac{\tilde{n}}{N} r\right) \right\} =$$

$$\frac{1}{N} \left\{ \frac{\exp\left(-i\pi \frac{(N+2)}{N} \tilde{n}\right) - 1}{\exp\left(-i2\pi \frac{\tilde{n}}{N}\right) - 1} + \frac{\exp\left(i\pi \frac{(N+2)}{N} \tilde{n}\right) - \exp\left(i2\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i2\pi \frac{\tilde{n}}{N}\right) - 1} \right\} =$$

$$\frac{1}{N} \left\{ \frac{\exp\left(-i\pi \frac{(N+1)}{N} \tilde{n}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(-i\pi \frac{\tilde{n}}{N}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)} + \frac{\exp\left(i\pi \frac{(N+1)}{N} \tilde{n}\right) - \exp\left(i\pi \frac{\tilde{n}}{N}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} =$$

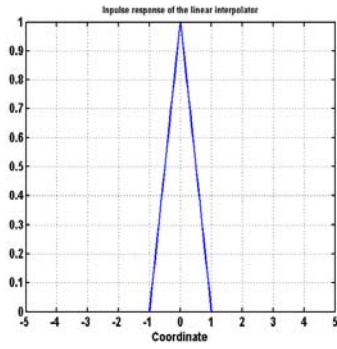
$$\frac{1}{N} \left\{ \frac{\exp\left(i\pi \frac{(N+1)}{N} \tilde{n}\right) - \exp\left(-i\pi \frac{(N+1)}{N} \tilde{n}\right)}{\exp\left(i\pi \frac{\tilde{n}}{N}\right) - \exp\left(-i\pi \frac{\tilde{n}}{N}\right)} \right\} = \frac{1}{N} \frac{\sin\left(\pi \frac{(N+1)}{N} \tilde{n}\right)}{\sin\left(\pi \frac{\tilde{n}}{N}\right)}; \quad (6)$$

$$h_n^{(intp)}(\delta x) = \text{sincd}[N+1; N; \pi(n - \delta x / \Delta x)] \quad (7)$$

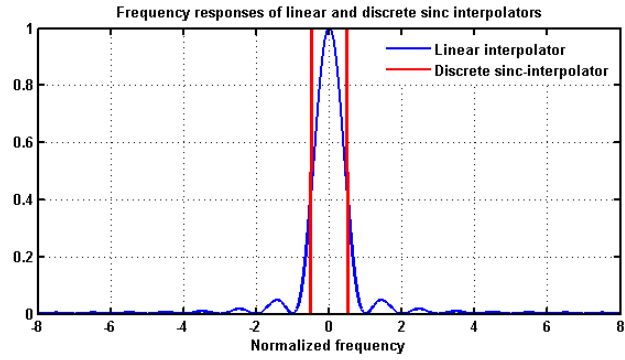
Case 1: $\eta_{N/2,opt}^{(intp1)}(\delta x) = \cos\left(\pi \frac{\delta x}{\Delta x}\right) = \frac{\eta_{N/2,opt}^{(intp0)}(\delta x) + \eta_{N/2,opt}^{(intp2)}(\delta x)}{2} =$

$$\frac{1}{N} \frac{\sin\left(\pi \frac{(N+1)}{N} \tilde{n}\right) + \sin\left(\pi \frac{(N-1)}{N} \tilde{n}\right)}{2 \sin\left(\pi \frac{\tilde{n}}{N}\right)} = \frac{1}{N} \frac{\sin(\pi \tilde{n})}{\sin\left(\pi \frac{\tilde{n}}{N}\right)} \cos\left(\pi \frac{\tilde{n}}{N}\right);$$

$$\eta_{N/2,opt}^{(intp1)}(\delta x) = \cos\left(\pi \frac{n - \delta x / \Delta x}{N}\right) \text{sincd}[N; \pi(n - \delta x / \Delta x)]$$

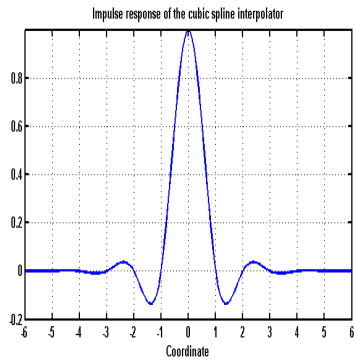


a)

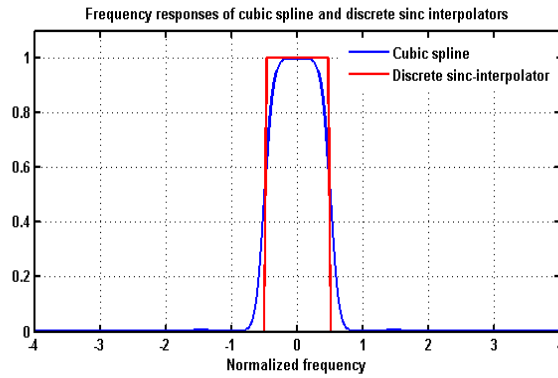


b)

Linear interpolation kernel (a) and frequency response (b)

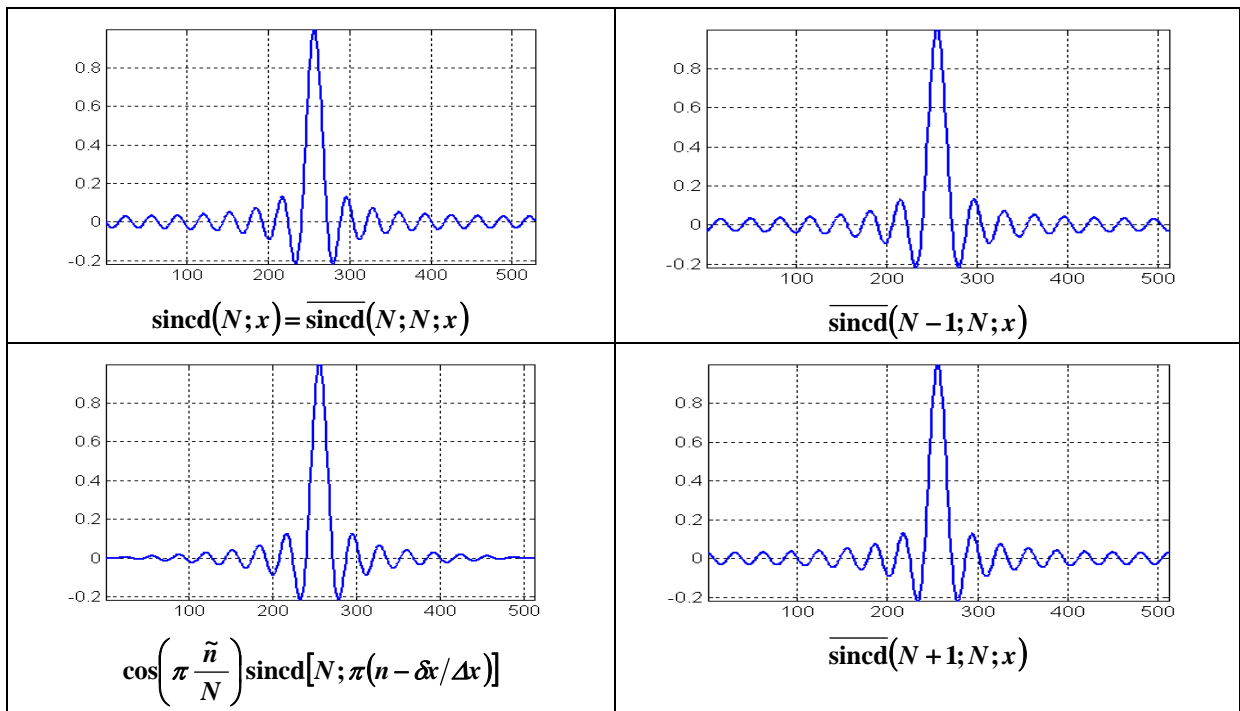


a)



b)

Cardinal cubic spline interpolation kernel (a) and frequency response (b)



Four versions of discrete sinc-function