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FROM PHOTOGRAPHY TO *.GRAPHIES: UNCONVENTIONAL IMAGING TECHNIQUES

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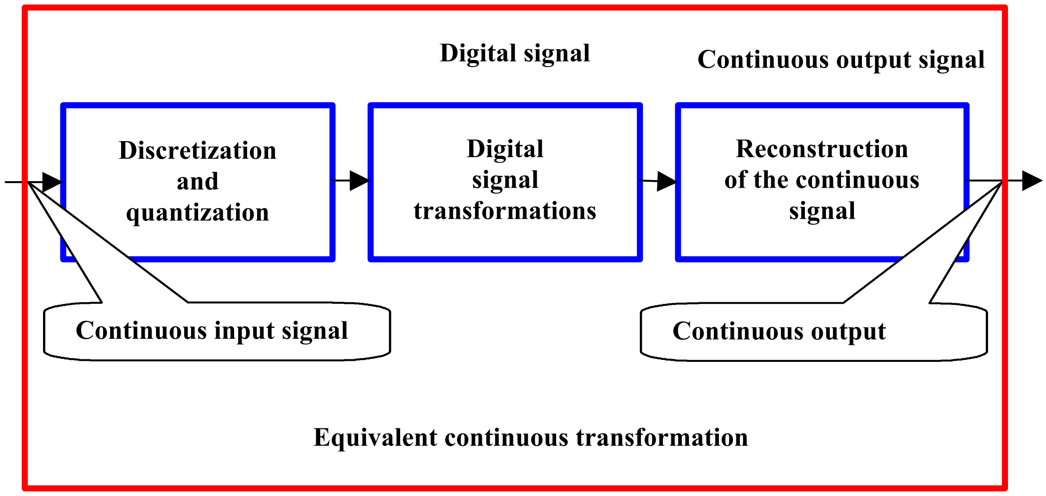
Lecture 5.

DISCRETE REPRESENTATION OF IMAGING TRANSFORMS

Lecture 5.
DISCRETE REPRESENTATION OF SIGNAL TRANSFORMATIONS

The Conformity principle between analogue and digital signal transformations



Discrete representation of signal transform parallels that of signals.

Discrete representation of the convolution integral

Convolution integral:

$$b(x) = \int_{-\infty}^{\infty} h(\xi) a(x-\xi) d\xi = \int_{-\infty}^{\infty} a(\xi) h(x-\xi) d\xi . \qquad (1)$$

For the signal restored from its discrete representation over shift basis functions:

$$a(\xi) = \sum_k a_k \varphi_{rest}(\xi - k\Delta x), \qquad (2)$$

$$b(x) = \int_{-\infty}^{\infty} a(\xi) h(x-\xi) d\xi = \int_{-\infty}^{\infty} \sum_k a_k \varphi_{rest}(\xi - k\Delta x) h(x-\xi) d\xi =$$

$$\sum_k a_k \int_{-\infty}^{\infty} \varphi_{rest}(\xi - k\Delta x) h(x-\xi) d\xi . \qquad (3)$$

Let $b(x)$ is also going to be restored

$$b(x) = \sum_n b_n \varphi_{rest}(x - k\Delta x);$$

from its discrete representation over shift basis functions:

$$\left\{ b_n = \int_{-\infty}^{\infty} b(x) \varphi_{discr}(x - n\Delta x) dx \right\} \qquad (4)$$

Then

$$b_n = \sum_k a_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\xi) \varphi_{rest}(\xi - k\Delta x) \varphi_{discr}(x - n\Delta x) d\xi dx =$$

$$\sum_k a_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - \xi - (n-k)\Delta x) \varphi_{rest}(\xi) \varphi_{discr}(x) d\xi dx = \sum_k a_k h_{n-k} . \qquad (5)$$

or

$$b_n = \sum_k a_k h_{n-k}\ , \tag{6),}$$

where

$$h_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - \xi - n\Delta x)\varphi_{rest}(x)\varphi_{discr}(\xi)dx d\xi$$

Shift-invariant Digital Filter is defined as:

$$b_k = \sum_n a_k h_{k-n} = \sum_n h_n a_{k-n}\ ;$$

Continuous and discrete impulse response and frequency response of digital filters:

For a digital filter defined by its coefficients $\{h_n\}$:

$$b(x) = \sum_{n=0}^{N_b-1} b_n \varphi_{rest}(x - n\Delta x) = \sum_{n=0}^{N_b-1}\left(\sum_{k=0}^{N_h-1} h_k a_{n-k}\right)\varphi_{rest}(x - n\Delta x) =$$

$$\sum_{n=0}^{N_b-1}\left(\sum_{k=0}^{N_h-1} h_k \int_{-\infty}^{\infty} a(\xi)\varphi_{discr}(\xi - (n-k)\Delta x)d\xi\right)\varphi_{rest}(x - n\Delta x) =$$

$$\int_{-\infty}^{\infty} a(\xi)\left[\sum_{n=0}^{N_b-1}\sum_{k=0}^{N_h-1} h_k \varphi_{rest}(x - n\Delta x)\varphi_{discr}(\xi - (n-k)\Delta x)\right]d\xi \tag{7}$$

Continuous impulse response of the digital filter is then:

$$h_{cnt}(x,\xi) = \sum_{n=0}^{N_b-1}\sum_{k=0}^{N_h-1} h_k \varphi_{discr}[\xi - (n-k)\Delta x]\varphi_{rest}(x - n\Delta x). \tag{8}$$

Discrete impulse response of the filter is defined as:

$$\mathbf{h}_{dscr} = \{h_n\}. \tag{9}$$

Continuous frequency response of the digital filter can be found as a Fourier Transform of its impulse response:

$$H_{cnt}(f,p) = \left[\sum_{k=0}^{N_h-1} h_k \exp(i2\pi pk\Delta x)\right]\cdot\frac{\sin[\pi(f-p)N_b\Delta x]}{\sin[\pi(f-p)\Delta x]}\cdot\Phi_{rest}(-p)\cdot\Phi_{discr}(f), \tag{10}$$

where $\Phi_{rest}(.)$ and $\Phi_{discr}(.)$ are frequency response functions of the signal restoration and discretization devices, respectively.

Discrete frequency response of the digital filter is defined as

$$H_{dscr}(f) = \left[\sum_{k=0}^{N_h-1} h_n\, exp(i2\pi fk\Delta x)\right] \tag{11}$$

Boundary effects and methods for signal extension: extrapolation and mirror reflection

Discrete Fourier Transform as a discrete representation of the Fourier integral:

For signal restoration basis functions $\varphi_{rest}(k,x)=\varphi_{rest}[x-(k+u)\Delta x]$, where ***k*** is an integer index of signal samples and ***u*** is a shift of signal samples with respect to the signal co-ordinate system,

$$a(x)=\sum_k a_k \varphi_{rest}[x-(k+u)\Delta x] \qquad (12)$$

where a_k are signal samples, integral Fourier Transform:

$$\alpha(f)=\int_{-\infty}^{\infty} a(x)\exp(i2\pi fx)dx \qquad (13)$$

can be found as

$$\alpha(f)=\int_{-\infty}^{\infty}\left\{\sum_k a_k \varphi_{rest}[x-(k+u)\Delta x]\right\}\exp(i2\pi fx)dx=$$

$$\sum_k a_k \int_{-\infty}^{\infty}\varphi_{rest}[x-(k+u)\Delta x]\exp(i2\pi fx)dx=\sum_k a_k \exp[i2\pi(k+u)\Delta xf]\cdot\Phi_{rest}(f). \qquad (14)$$

Let signal spectrum $\alpha(f)$ is going to be also approximated from its samples $\{\alpha_r\}$:

$$\alpha_r=\int_{-\infty}^{\infty}\alpha(f)\varphi_{dicr}[f-(r+v)\Delta f] \qquad (15)$$

for spectrum discretization basis functions $\{\varphi_{discr}[f-(r+v)\Delta f]\}$, where r is an integer index of spectrum samples and v is an arbitrary shift parameter that indicates shifts of spectrum samples with respect to its co-ordinate system. Then:

$$\alpha_r=\int_{-\infty}^{\infty}\left\{\sum_k a_k \exp[i2\pi(k+u)\Delta xf]\cdot\Phi_{rest}(f)\right\}\varphi_{discr}[f-(r+v)\Delta f]df=$$

$$\sum_k a_k \int_{-\infty}^{\infty}\Phi_{rest}(f)\varphi_{discr}[f-(r+v)\Delta f]\exp[i2\pi(k+u)\Delta xf]df=$$

$$\sum_k a_k \int_{-\infty}^{\infty}\Phi_{rest}[f+(r+v)\Delta f]\varphi_{discr}(f)\exp[i2\pi(k+u)\Delta x(f+(r+v)\Delta f)]df=$$

$$\sum_k a_k \exp[i2\pi(k+u)(r+v)\Delta f\Delta x]\int_{-\infty}^{\infty}\Phi_{rest}[f+(r+v)\Delta f]\varphi_{discr}(f)\exp[i2\pi(k+u)f\Delta x]df. \qquad (15)$$

The summation term $\sum_k a_k \exp[i2\pi(k+u)(r+v)\Delta f\Delta x]$ of this equation is used to define discrete representation of integral Fourier Transform:

$$\alpha_r = \frac{1}{\sqrt{N}} \sum_k a_k \exp\left[i2\pi \frac{(k+u)(r+v)}{N}\right], \qquad N = 1/\Delta x \Delta f \tag{16}$$

Introduce ***(Shifted) Discrete Fourier Transforms*** by removing from this equation an exponential factor $\exp(i2\pi uv/N)$ that depends neither on k nor on r:

$$\alpha_r^{(u,v)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{k(r+v)}{N}\right] \exp\left(i2\pi \frac{ru}{N}\right). \tag{17}$$

These transforms are invertible:

$$a_k^{(u,v)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_r \exp\left[-i2\pi \frac{r(k+u)}{N}\right] \exp\left(i2\pi \frac{kv}{N}\right). \tag{18}$$

For signal and its spectrum sample shifts $u=v=0$, SDFTs are known as direct and inverse Discrete Fourier Transforms:

$$\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{kr}{N}\right];$$

$$a_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_r \exp\left[-i2\pi \frac{kr}{N}\right]. \tag{19}$$

The presence of shift parameters makes SDFT to be more flexible in simulating integral Fourier Transform: it allows to

- perform continuous spectrum analysis;
- perform convolution and correlation with sub-pixel resolution;
- flexibly imitate sinc-interpolation of discrete data required by the sampling theorem.

In order to illustrate this property, we will, for a discrete signal $\{a_k\}$, $k = 0,1,\ldots,N-1$, compute its direct SDFT with shift parameters ***(u,v)*** and inverse SDTF from subset of K spectral samples with shift parameters ***(p,q)***:

$$\tilde{a}_n^{u/p,v/q} = \frac{1}{\sqrt{N}} \sum_{r=0}^{K-1} \left\{ \alpha_r^{u,v} \exp\left(-i2\pi \frac{rp}{N}\right) \right\} \exp\left(-i2\pi \frac{n(r+q)}{N}\right) =$$

$$\frac{1}{\sqrt{N}} \sum_{r=0}^{K-1} \left\{ \left[\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kv}{N}\right) \exp\left(i2\pi \frac{(k+u)r}{N}\right) \right] \exp\left(-i2\pi \frac{rp}{N}\right) \right\},$$

$$\exp\left(-i2\pi \frac{n(r+q)}{N}\right) =$$

$$\frac{1}{N} \sum_{k=0}^{N-1} a_k \sum_{r=0}^{K-1} \exp\left(i2\pi \frac{kv}{N}\right) \exp\left(i2\pi \frac{(k+u)r}{N}\right) \exp\left(-i2\pi \frac{rp}{N}\right) \exp\left(-i2\pi \frac{n(r+q)}{N}\right) =$$

$$\frac{1}{N}\sum_{k=0}^{N-1} a_k\, exp\left(i2\pi\,\frac{kv-nq}{N}\right)\sum_{r=0}^{K-1} exp\left(i2\pi\,\frac{(k-n+u-p)r}{N}\right)=$$

$$\frac{1}{N}\sum_{k=0}^{N-1} a_k\, exp\left(i2\pi\,\frac{kv-nq}{N}\right)\frac{exp\left(i2\pi\,(k-n+u-p)K/N\right)-1}{exp\left(i2\pi\,(k-n+u-p)/N\right)-1}=$$

$$\sum_{k=0}^{N-1} a_k sincd\left(K;N;(k-n+u-p)\right) exp\left(i\pi\,\frac{K-1}{N}\left((k-n+u-p)+2(kv-nq)\right)\right)=$$

$$\left\{\sum_{k=0}^{N-1} a_k exp\left(i\pi k\left(\frac{K-1}{N}+2v\right)\right) sincd\left(K;N;(k-n+u-p)\right)\right\}\times$$

$$exp\left(-i\pi\left(\frac{K-1}{N}+2q\right)n\right) exp\left(i\pi\,\frac{K-1}{N}(u-p)\right). \qquad (20)$$

In lecture 6, we will demonstrate how this property can be implemented in efficient algorithms for signal interpolation in signal spectrum analysis and convolution.
An important role in these applications plays inverse DFT of a function

$$\varphi_r^{(0)}=\begin{cases} exp\left(i2\pi ur/N\right), & r=0,1,\ldots,N/2-1\\ 0, & r=N/2\\ \varphi_{N-r}^{*}, & r=N/2+1,\ldots,N-1\end{cases} \qquad (21)$$

which is discrete version of the sinc-function $\mathrm{sinc}(x)=\sin x/x$:

$$\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\varphi_r \exp\left(-i2\pi\,\frac{kr}{N}\right)=$$

$$\frac{1}{\sqrt{N}}\left\{\varphi_o+\sum_{r=1}^{N/2-1}\varphi_r\, exp\left(-i2\pi\,\frac{kr}{N}\right)+\sum_{r=N/2+1}^{N-1}\varphi_r\, exp\left(-i2\pi\,\frac{kr}{N}\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{\varphi_o+\sum_{r=1}^{N/2-1}\varphi_r\, exp\left(-i2\pi\,\frac{kr}{N}\right)+\sum_{r=1}^{N/2-1}\varphi_{N-r}\, exp\left(-i2\pi\,\frac{k(N-r)}{N}\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{\varphi_o+\sum_{r=1}^{N/2-1}\varphi_r\, exp\left(-i2\pi\,\frac{kr}{N}\right)+\sum_{r=1}^{N/2-1}\varphi_r^{*}\, exp\left(i2\pi\,\frac{kr}{N}\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{1+\sum_{r=1}^{N/2-1} exp\left(i2\pi\,\frac{ur}{N}\right) exp\left(-i2\pi\,\frac{kr}{N}\right)+\sum_{r=1}^{N/2-1} exp\left(-i2\pi\,\frac{ur}{N}\right) exp\left(i2\pi\,\frac{kr}{N}\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{1+\sum_{r=1}^{N/2-1} exp\left(-i2\pi\,\frac{(k-u)r}{N}\right)+\sum_{r=1}^{N/2-1} exp\left(i2\pi\,\frac{(k-u)r}{N}\right)\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{1+\frac{\exp(-i\pi(k-u))-\exp\left(-i2\pi\frac{(k-u)}{N}\right)}{\exp\left(-i2\pi\frac{(k-u)}{N}\right)-1}+\right.$$

$$\left.+\frac{\exp(i\pi(k-u))-\exp\left(i2\pi\frac{(k-u)}{N}\right)}{\exp\left(i2\pi\frac{(k-u)}{N}\right)-1}\right\}=$$

$$\frac{1}{\sqrt{N}}\left\{1-\frac{\exp\left(-i\pi\frac{N-1}{N}(k-u)\right)-\exp\left(-i\pi\frac{(k-u)}{N}\right)}{\exp\left(i\pi\frac{(k-u)}{N}\right)-\exp\left(-i\pi\frac{(k-u)}{N}\right)}+\right.$$

$$\left.+\frac{\exp\left(i\pi\frac{N-1}{N}(k-u)\right)-\exp\left(i\pi\frac{(k-u)}{N}\right)}{\exp\left(i\pi\frac{(k-u)}{N}\right)-\exp\left(-i\pi\frac{(k-u)}{N}\right)}\right\}=$$

$$\frac{1}{\sqrt{N}}\frac{\exp\left(i\pi\frac{N-1}{N}(k-u)\right)-\exp\left(i\pi\frac{N-1}{N}\frac{(k-u)}{N}\right)}{\exp\left(i\pi\frac{(k-u)}{N}\right)-\exp\left(-i\pi\frac{(k-u)}{N}\right)}=$$

$$\sqrt{N}\frac{\sin\left[\pi\frac{N-1}{N}(k-u)\right]}{N\sin[\pi(k-u)/N]}=\sqrt{N}\,\mathrm{sincd}(\mathrm{N-1;N;}(k-u)). \qquad (22)$$

In a similar way, one can show that inverse DFT of a function

$$\varphi_r^{(2)}=\begin{cases}\exp(i2\pi ur/N), & r=0,1,...,N/2-1\\ 2\cos(\pi u), & r=N/2\\ \varphi_{N-r}^{*}, & r=N/2+1,...,N-1\end{cases}$$

is function

$$\sqrt{N}\,\mathrm{sincd}(\mathrm{N+1;N;}(k-u))=\sqrt{N}\frac{\sin\left[\pi\frac{N+1}{N}(k-u)\right]}{N\sin[\pi(k-u)/N]}. \qquad (23)$$

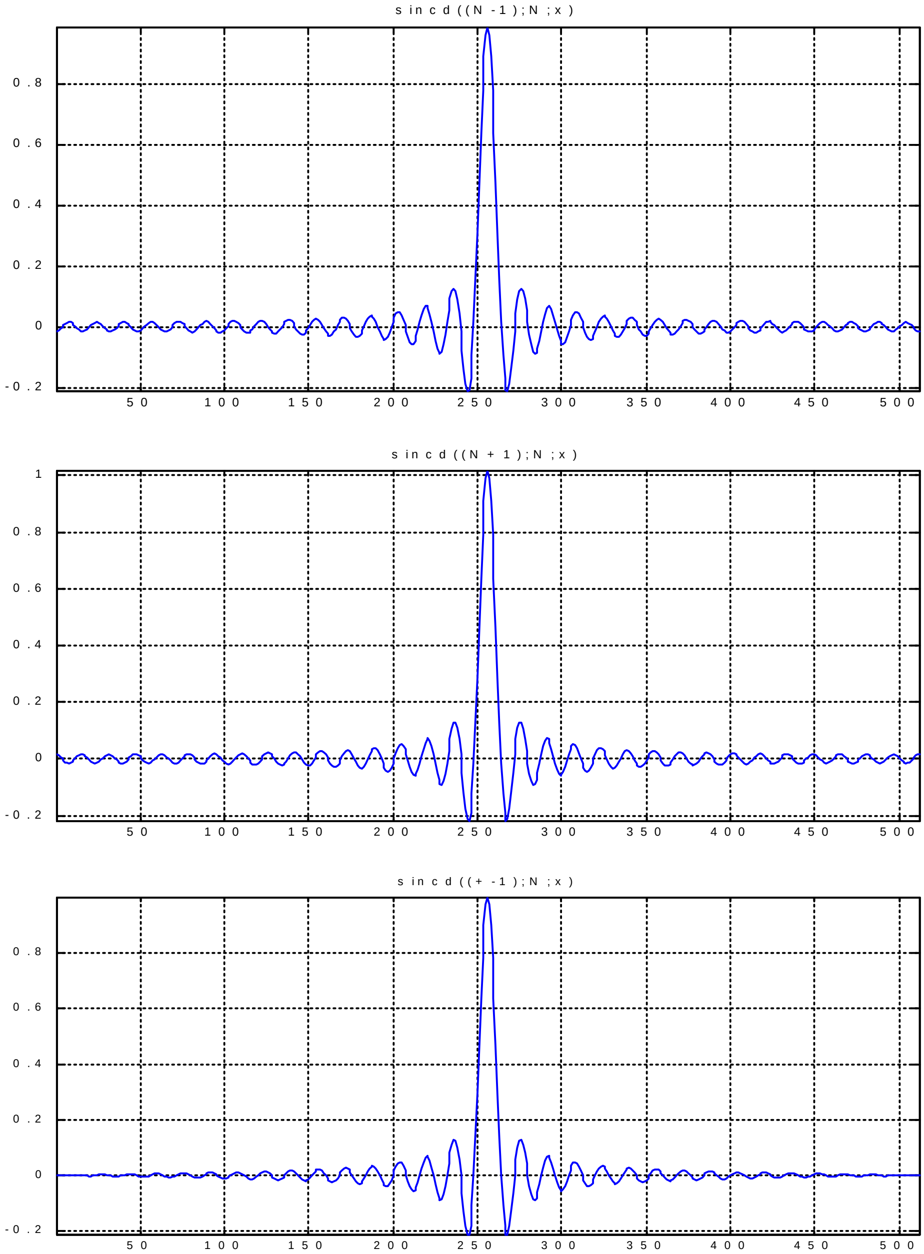
Combination of these two functions:

$$\varphi_r^{(1)} = \frac{\varphi_r^{(0)} + \varphi_r^{(2)}}{2} = \begin{cases} exp(i2\pi ur / N), & r = 0,1,..., N/2 - 1 \\ cos(\pi u), & r = N/2 \\ \varphi_{N-r}^{*}, & r = N/2+1,..., N-1 \end{cases} \tag{24}$$

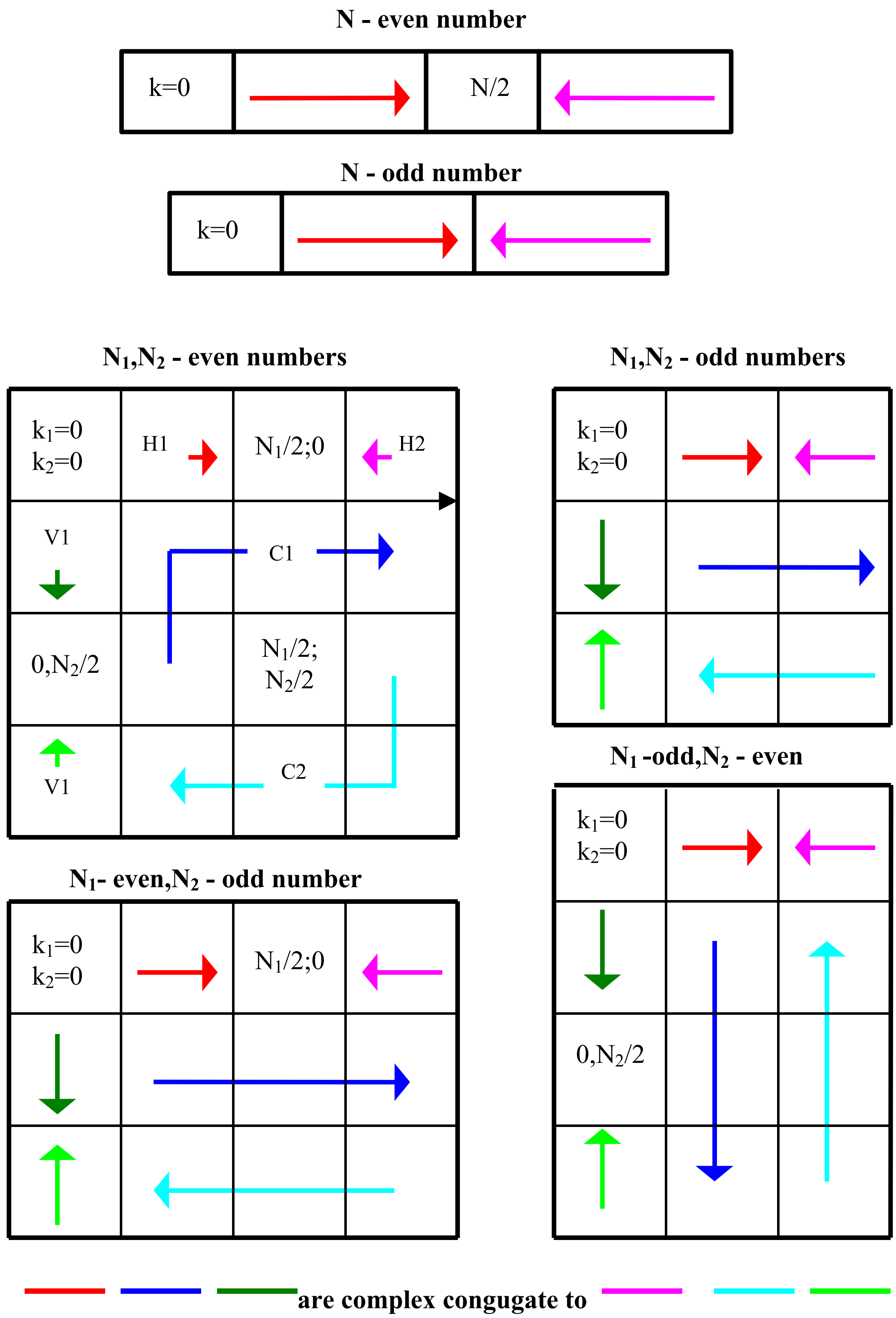
corresponds to the function

$$sincd(\pm 1; N; x) = [sincd(N-1; N; x) + sincd(N+1; N; x)]/2 . \tag{25}$$

We call these functions discrete sinc-functions.

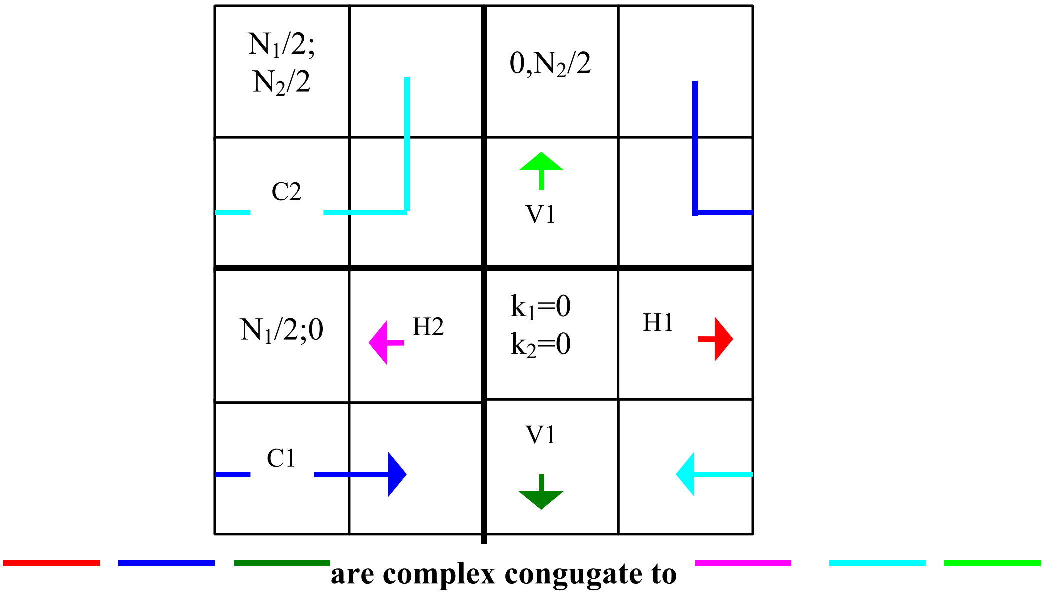


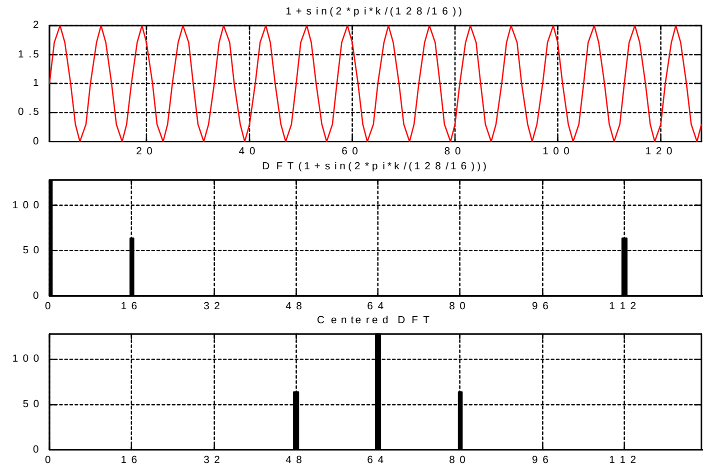
Discrete sinc-functions

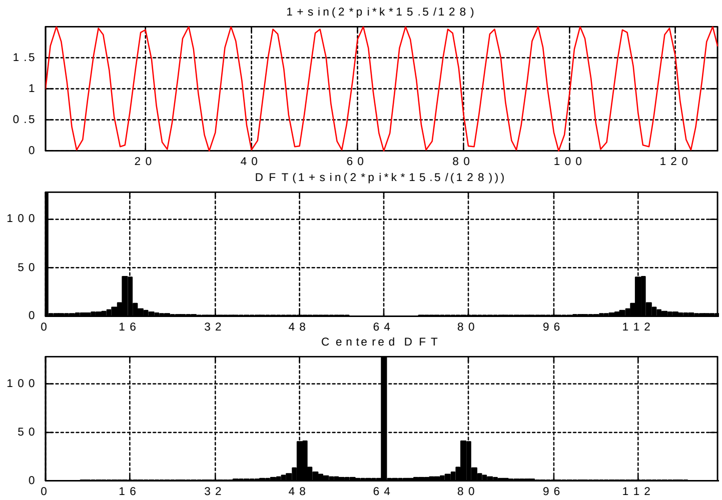


Types of the DFT spectra symmetry for one and two-dimensional real-valued signals

Centering DFT spectrum representation





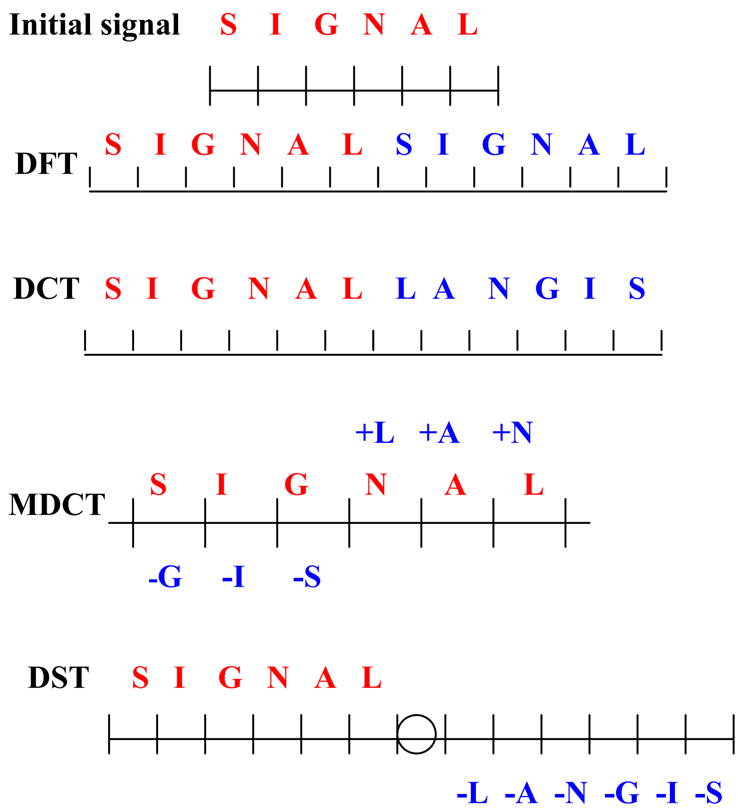


DFTs of sinusoidal signals with integer and non integer number of periods

Shift parameters in SDFTs provide them also with a flexibility in dealing with signal symmetries. There exist quite a number of special cases of SDFTs for different integer and semi integer shift parameters and different types of signal symmetries. Here are some examples

Discrete cosine transform DCT= $SDFT_{1/2,0}\left(a_k = a_{2N-1-k}\right)$; IDC= $ISDFT_{1/2,0}\left(\alpha_r = -\alpha_{2N-r}\right)$	$\alpha_r = \frac{2}{\sqrt{2N}}\sum_{k=0}^{N-1} a_k \cos\left(\pi\frac{(k+1/2)r}{N}\right)$ $a_k = \frac{1}{\sqrt{2N}}\left\{\alpha_0 + 2\sum_{r=1}^{N-1}\alpha_r \cos\left(\pi\frac{(k+1/2)r}{N}\right)\right\}$
DCST= $SDFT_{1/2,0}\left(a_k = -a_{2N-1-k}\right)$ IDCST= $SDFT_{1/2,0}\left(\alpha_r = -\alpha_{2N-r}\right)$	$\alpha_r = \frac{2}{\sqrt{2N}}\sum_{k=0}^{N-1} a_k \sin\left(\pi\frac{(k+1/2)r}{N}\right) =$ $\frac{1}{i\sqrt{2N}}\sum_{k=0}^{N-1} a_k \exp\left(i2\pi\frac{(k+1/2)r}{2N}\right)$; $a_k = \frac{1}{\sqrt{2N}}\left(\alpha_N + 2\sum_{r=1}^{N}\alpha_r \sin\left(\pi\frac{(k+1/2)r}{N}\right)\right)$
DCT-IV= $SDFT_{1/2,1/2}\left(a_k = -a_{2N-1-k}\right)$ IDCT-IV= $ISDFT_{1/2,1/2}\left(\alpha_r = -\alpha_{2N-1-r}\right)$	$\alpha_r = \frac{\sqrt{2}}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \cos\left(\pi\frac{(k+1/2)(r+1/2)}{N}\right)$ $a_k = \frac{\sqrt{2}}{\sqrt{N}}\sum_{r=0}^{N-1} \alpha_r \cos\left(\pi\frac{(k+1/2)(r+1/2)}{N}\right)$
Discrete Sine Transform DST= $SDFT_{1,1}\left(a_k = -a_{2N-k}; a_n = 0\right)$ IDST= $ISDFT_{1,1}\left(\alpha_r = -\alpha_{2N-r}; a_N = a_{2N+1} = 0\right)$	$\alpha_r = \frac{1}{\sqrt{2N}}\sum_{k=0}^{N-1} a_k \sin\left(2\pi\frac{(k+1)(r+1)}{2N+1}\right)$ $a_k = \frac{1}{\sqrt{2N}}\sum_{r=0}^{N-1} \alpha_r \sin\left(2\pi\frac{(k+1)(r+1)}{2N+1}\right)$
DST-IV= $SDFT_{1/2,1/2}\left(a_k = -a_{2N-k}; a_n = 0\right)$	$\alpha_r = \frac{1}{\sqrt{2N}}\sum_{k=0}^{N-1} a_k \sin\left(2\pi\frac{(k+1/2)(r+1/2)}{2N}\right)$ $a_k = \frac{1}{\sqrt{2N}}\sum_{r=0}^{N-1} \alpha_r \sin\left(2\pi\frac{(k+1/2)(r+1/2)}{2N}\right)$
"Modified DCT", or "lapped transform MDCT= $SDFT_{\frac{N+2}{4},\frac{1}{2}}\left(\tilde{a}_k\right)$ $\tilde{a}_k = \begin{cases} a_k - a_{N-1-k}, & k = 0,...,N-1 \\ a_k + a_{3N-1-k}, & k = N,...,2N-1 \end{cases}$	$\alpha_r = \sum_{k=0}^{N-1} a_k \cos\left[2\pi\frac{\left(k+\frac{N+2}{4}\right)\left(r+\frac{1}{2}\right)}{N}\right] =$ $\frac{1}{2}\sum_{k=0}^{N-1} \tilde{a}_k \exp\left[i2\pi\frac{\left(k+\frac{N+2}{4}\right)\left(r+\frac{1}{2}\right)}{N}\right]$

Types of signal symmetry for DFT, DCT, DST and MDCT



Discrete Fresnel Transform

Because of the existing correlation between the Fresnel transform (2.37) and the Fourier transform (2.27) we may use the Fourier transform's discrete representation to construct that of the Fresnel transform. In doing so, it is particularly important to take into account a possible shift of the sample raster of a signal $\boldsymbol{a(x)exp(i2\pi f^2/D^2)}$ relative to the origins of the signal and its spectrum coordinate systems, which serve as the centers of symmetry of the exponential terms $\boldsymbol{a(x)exp(i2\pi f^2/D^2)}$ and $\boldsymbol{a(x)exp(-i2\pi x^2/D^2)}$. Thus, for the bases of the sample functions

$$
\begin{aligned}
&\{\varphi_k(x) = sinc[\pi(x - (k+u)\Delta x)/\Delta x]\},\\
&\{\chi_r(f) = sinc[\pi(f - (r+v)\Delta f)/\Delta f]\}
\end{aligned}
\qquad (25)
$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$, as above, are the samples raster shifts relative to the origin of coordinates over $\boldsymbol{x}$ and $\boldsymbol{f}$ respectively, we obtain the following discrete representation of the direct and inverse Fresnel transforms

$$
\alpha_r^{\kappa} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \, exp\left\{- i\pi \left[(k+u)\kappa - (r+v)/\kappa\right]^2 / N\right\}, \qquad (26a)
$$

$$
a_k^{\kappa} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_r^{\kappa} \, exp\left\{i\pi \left[(k+u)\kappa - (r+v)/\kappa\right]^2 / N\right\}, \qquad (26b)
$$

where $N = 1/\Delta x \Delta f D^2$; $\{a_k\}$ is the sequence of the signal samples such that

$$
a_k = \frac{1}{\Delta x} exp\left[i\pi (k+u)^2 \kappa^2 / N\right] \int_{-\infty}^{\infty} a(x) exp\left(- i\pi \frac{x^2}{D^2}\right) sinc\left[\pi\left(x - (k+u)\Delta x\right)/\Delta x\right] dx, \qquad (27)
$$

$\{\alpha_r^{\kappa}\}$ is the sequence of the Fresnel spectrum such that

$$
\alpha_r^{\kappa} = \frac{1}{\Delta f} exp\left[- i\pi (r+v)^2 \kappa^2\right] \int_{-\infty}^{\infty} \alpha_D(f) exp\left(i\pi \frac{f^2}{D^2}\right) sinc\left[\pi\left(f - (r+v)\Delta f\right)/\Delta f\right] df \qquad (28)
$$

Δx and Δf are the discretization intervals of the signal and its spectrum, and κ is a dimensionless parameter

$$
\kappa = (\Delta f / \Delta x)^{1/2}. \qquad (29)
$$

This kind of representation assumes the following approximation of a continuous signal and its Fresnel spectrum

$$
a(x) \cong exp\left(i\pi x^2 / D^2\right) \sum_{k=0}^{N-1} a_k \, exp\left[i\pi \frac{(k+u)^2 \kappa^2}{N}\right] sinc\left[\pi\left(x - (k+u)\Delta x\right)/\Delta x\right] \qquad (30a)
$$

$$a_D(x) \cong \exp\left(-i\varphi^2/D^2\right)\sum_{k=0}^{N-1}\alpha_r^k \exp\left[i\pi\frac{(r+v)^2/\kappa^2}{N}\right]\mathrm{sinc}\left[\pi\left(f-(r+v)\Delta f\right)/\Delta f\right]$$

(30b)

Transforms (26) involve three parameters: ***u***,***v*** and κ. However, these shift parameters are additive and consequently may be replaced with just a single parameter representing (26) by

$$\alpha_r^{\kappa,w} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left\{-i\pi\left(\kappa k - r/\kappa + w\right)^2/N\right\}, \qquad (31a)$$

$$a_k^{\kappa,w} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \alpha_r^k \exp\left\{i\pi\left(\kappa k - r/\kappa + w\right)^2/N\right\}, \qquad (31b)$$

where ***w*** is the overall shift $w=u\kappa-v/\kappa$. We call the pair of transforms (31) the direct and inverse discrete Fresnel Transforms (DFrT), respectively.

It is quite evident that the discrete Fresnel transform may be calculated by means of the SDFT(w/κ,0):

$$\alpha_r^{\kappa,w} = \frac{1}{\sqrt{N}}\left\{\sum_{k=0}^{N-1} a_k \exp\left\{-i\pi\left(\kappa k + w\right)^2/N\right\}\exp\left[i2\pi\frac{(k+w/\kappa)r}{N}\right]\right\}\exp\left(-i\pi\frac{r^2}{N\kappa^2}\right)$$

(32)

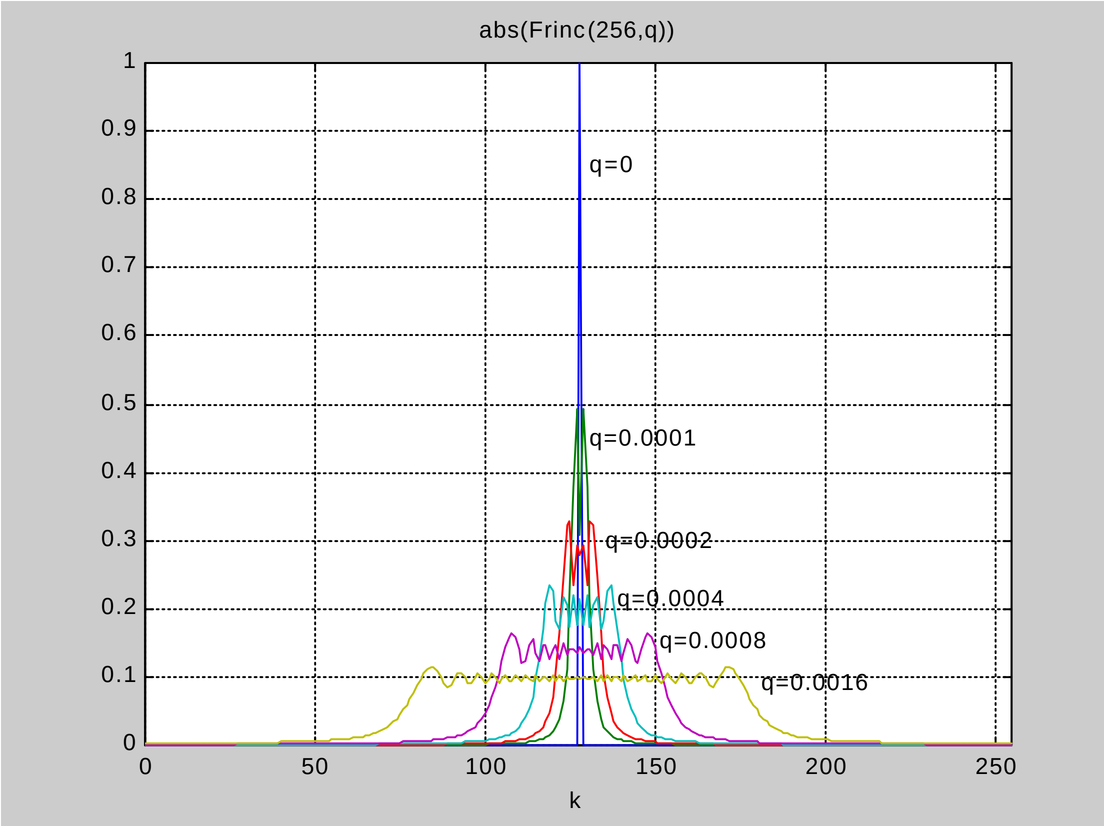
Introduce a periodic function ***frinc***(***N***;***q***;***r***) of period ***N,*** discrete Fourier transform of a “chirp-function” $\exp\left(i\pi q k^2\right)$:

$$\mathrm{frinc}(N;q;r) = \frac{1}{N}\sum_{k=0}^{N-1}\exp\left(i\pi q k^2\right)\exp\left(-i2\pi\frac{kr}{N}\right) \qquad (33)$$

Function ***frinc***(***N***;***q***;***r***) plays the same role in the DFrT as the function ***sincd***(***N***; ***x***) plays in DFT. Function ***frinc***(***N***;***q***;***r***) is a symmetric one with the symmetry center that changes with ***N*** and ***q***. This change is compensated in a modified frinc-function:

$$\overline{\mathrm{frinc}}(N;q;r) = \frac{1}{N}\sum_{k=0}^{N-1}\exp\left[i\pi(k-N)kq\right]\exp\left(-i2\pi\frac{kr}{N}\right) \qquad (34)$$

that is illustrated in figure.



Absolute values of function $\overline{\boldsymbol{frinc}}(\boldsymbol{N};\boldsymbol{q};\boldsymbol{r})$ for ***N*=256** and different "focusing" parameter ***q***

Test questions

1. Explain the conformity principle between analog and digital signal transformations.
2. Derive general formula of digital filtering in signal domain. What are continuous and discrete impulse and frequency responses of digital filters.
3. From the Fourier integral, derive basic formula of SDFT and demonstrate similarities and dissimilarities between SDFT and integral Fourier Transform.
4. Derive DCT, DST as special cases of SDFTs. Explain energy compaction capabilitiy of DCT.