L. Yaroslavsky. . Course ***0510.7211 “Digital Image Processing: Applications”***

Lecture 6. Imaging transforms.

Basic types of signal transforms: element-wise and linear transforms, stochastic transformations. Linear transforms and the superposition principle.

Representation of linear transforms. Point-spread function. $b(x)=\int_X a(\xi)h(x,\xi)d\xi$;

Shift invariant transforms: $h(x,\xi)=h(x-\xi)$.

Signal convolution $b(x)=\int_{-\infty}^{\infty} a(\xi)h(x-\xi)d\xi$ $\qquad \delta(x,\xi)=\lim_{\Delta x\to 0} rect[(x-\xi)/\Delta x]$

Fourier Transform and its properties. $\alpha(f)=\int_{-\infty}^{\infty} a(x)exp(i2\pi fx)dx;$

$$a(x)=\lim_{F\to\infty}\int_{-F}^{F}\alpha(f)\exp(-i2\pi fx)df; \qquad \delta(x)=\lim_{F\to\infty} 2F\operatorname{sinc}(2\pi F(x-\xi))$$

Frequency response of linear transforms:

$$\beta(f)=\int_F \alpha(p)H(f,p)dp\ ; \qquad H(f,p)=\int_{-\infty}^{\infty} h(x,\xi)\exp[i2\pi(fx-p\xi)]dxd\xi\ .$$

Fourier domain representation of shift invariant linear transforms:

$$H(f,p)=\eta(f)\delta(f-p); \qquad \eta(f)=\int_{-\infty}^{\infty} h(x)\exp(i2\pi fx)dx; \qquad \beta(f)=\eta(f)\alpha(f);$$

Windowed Fourier Transform: $\alpha_w(f,x_0)=\int_{-\infty}^{\infty} w(x-x_0)a(x)\exp(i2\pi fx)dx;$

Hartley Transform: $\alpha^H(f)=\int_{-\infty}^{\infty} a(x)[\cos(2\pi fx)+\sin(2\pi fx)]dx \propto \int_{-\infty}^{\infty} a(x)\cos(2\pi fx-\pi/4)dx$

Hankel Transform: $\alpha^{Hn}(f)=\int_0^{\infty} xa(x)J_0(fx)dx\ ; \quad J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(ix\cos\theta)d\theta$

Mellin transform- Fourier Transform in co-ordinate $z=\exp(x)$: $\mu(f)=\int_0^{\infty} a(z)z^{i2\pi f-1}dz$.

Scale invariance: $\mathbf{MellTrsf}(a(kz))=k^{-i2\pi}\,\mathbf{MellTrsf}(a(z))$

Fresnel Transform

$$\alpha_{Fr}^{D}(f)=\int_{-\infty}^{\infty} a(x)\exp\left(-i\pi(x-f)^2/D^2\right)dx;\ a(x)=\int_{-\infty}^{\infty}\alpha_{Fr}^{D}(f)\exp\left(i\pi(x-f)^2/D^2\right)df;\ D^2=\lambda z$$

$$\delta(x,\xi)=\exp\left(-i\pi(x^2-\xi^2)\right)\lim_{F\to\infty} 2F\operatorname{sinc}(2\pi F(x-\xi))$$

Radon Transform.

$$\alpha(\xi_1,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x_1,x_2)\delta(x_1\cos\theta+x_2\sin\theta-\xi_1)dx_1dx_2$$

$$=\int_{-\infty}^{\infty} a(\xi_1\cos\theta-\xi_2\sin\theta,\xi_1\sin\theta+\xi_2\cos\theta)d\xi_2$$

Projection theorem and computer tomography:

$$\int_{-\infty}^{\infty}\alpha(\xi_1,\theta)exp(i2\pi f\xi_1)d\xi_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x_1,x_2)exp(i2\pi(fx_1\cos\theta+fx_2\sin\theta))dx_1dx_2$$

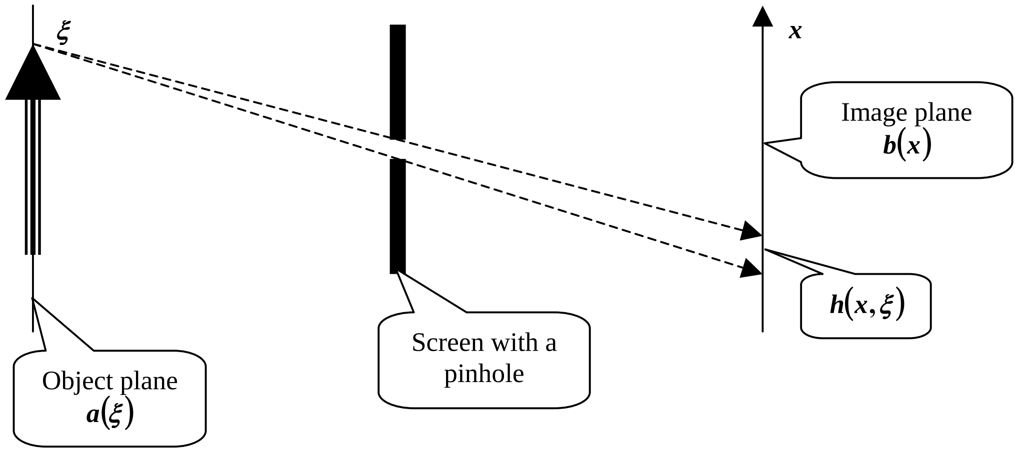
Stochastic transformations. Mathematical models of random noise interferences
Stochastic linear systems as models of imaging systems.

Signal dependent transformations

Problems for self-testing:

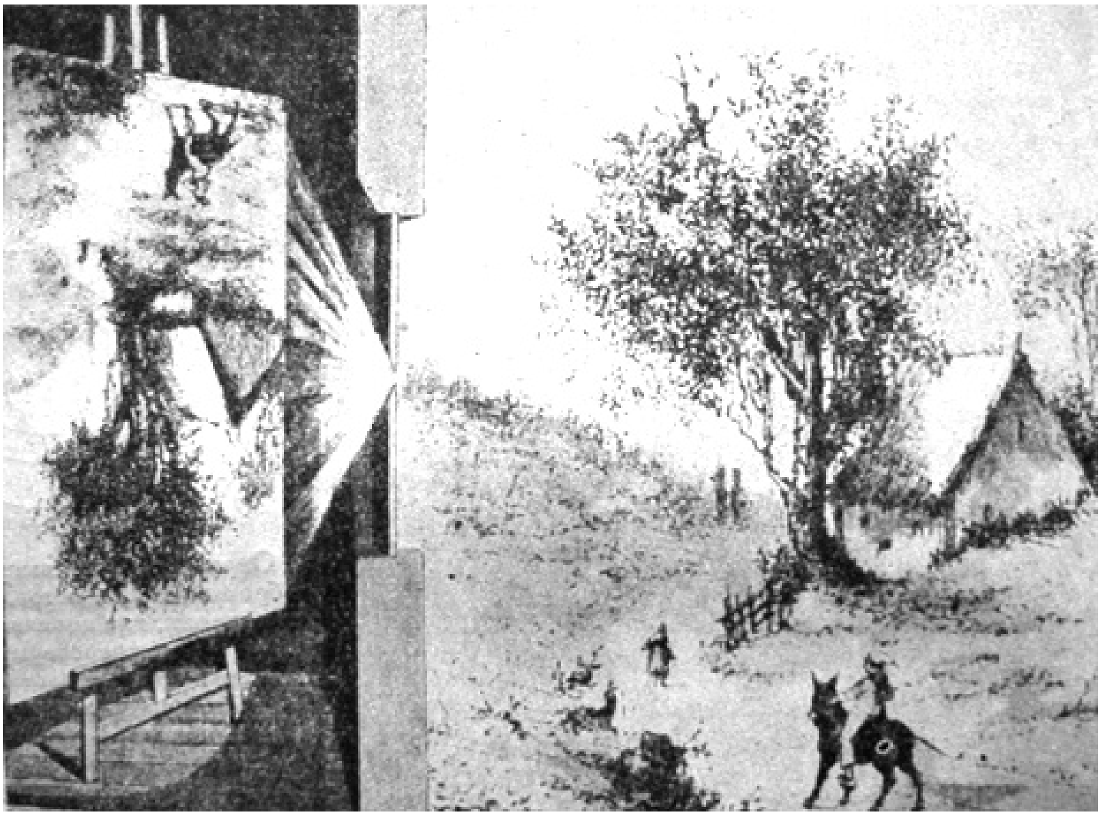
1. Explain basic types of signal transformations in terms of signal mappings in the signal space.
2. What are linear system impulse and frequency responses in shift invariant and shift variant cases.
3. List and comment basic properties of Integral Fourier Transform.
4. How Hartley, Hankel, Fresnel Transforms and Melline Transfor are linked with Fourier Transform
5. Formulate and prove projection theorem for Radon Transform.
6. Describe transforms that are used for modeling signal random interferences and distortions.

Direct imaging

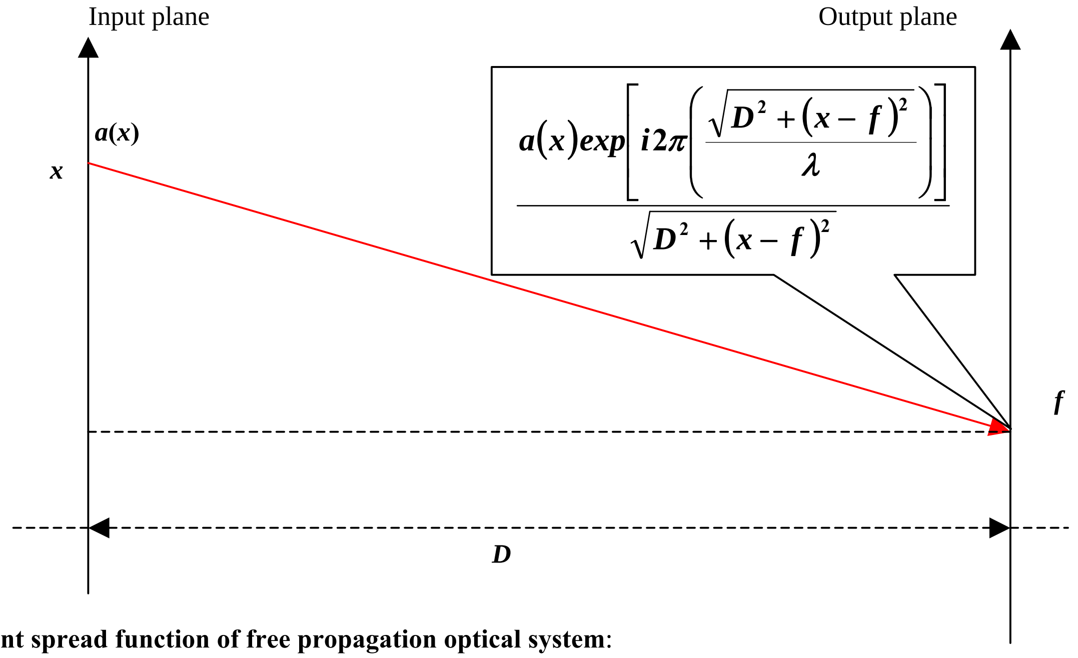


$$b(x) = \int_{-\infty}^{\infty} a(\xi) h(x,\xi) d\xi$$

An example: camera-obscura (pinhole camera) (*Ibn Al Haytam, X century*):



Optical systems with free propagation:
Kirchhoff, Fresnel and Fourier integrals



Point spread function of free propagation optical system:

$$PSF(x,f)=\frac{exp\left[i2\pi\left(\dfrac{\sqrt{D^2+(x-f)^2}}{\lambda}\right)\right]}{\sqrt{D^2+(x-f)^2}}$$

Kirchhoff equation:

$$\alpha(f)=\int_X a(x)\frac{exp\left[i2\pi\left(\dfrac{\sqrt{D^2+(x-f)^2}}{\lambda}\right)\right]}{\sqrt{D^2+(x-f)^2}}dx$$

For $D >> max|x-f|$ (“near zone” propagation), *Fresnel approximation*:

$$\alpha(f)=\int_X a(x)\frac{exp\left[i2\pi\left(\dfrac{\sqrt{D^2+(x-f)^2}}{\lambda}\right)\right]}{\sqrt{D^2+(x-f)^2}}dx \cong C\int_X a(x)exp\left[i\pi\frac{(x-f)^2}{\lambda D}\right]dx$$

with C as an irrelevant constant.

If $exp\left(i\pi x^2/D^2\right)\cong 1$ and $exp\left(i\pi f^2/D^2\right)\cong 1$, (“far zone” propagation) *Fraunhofer approximation*:

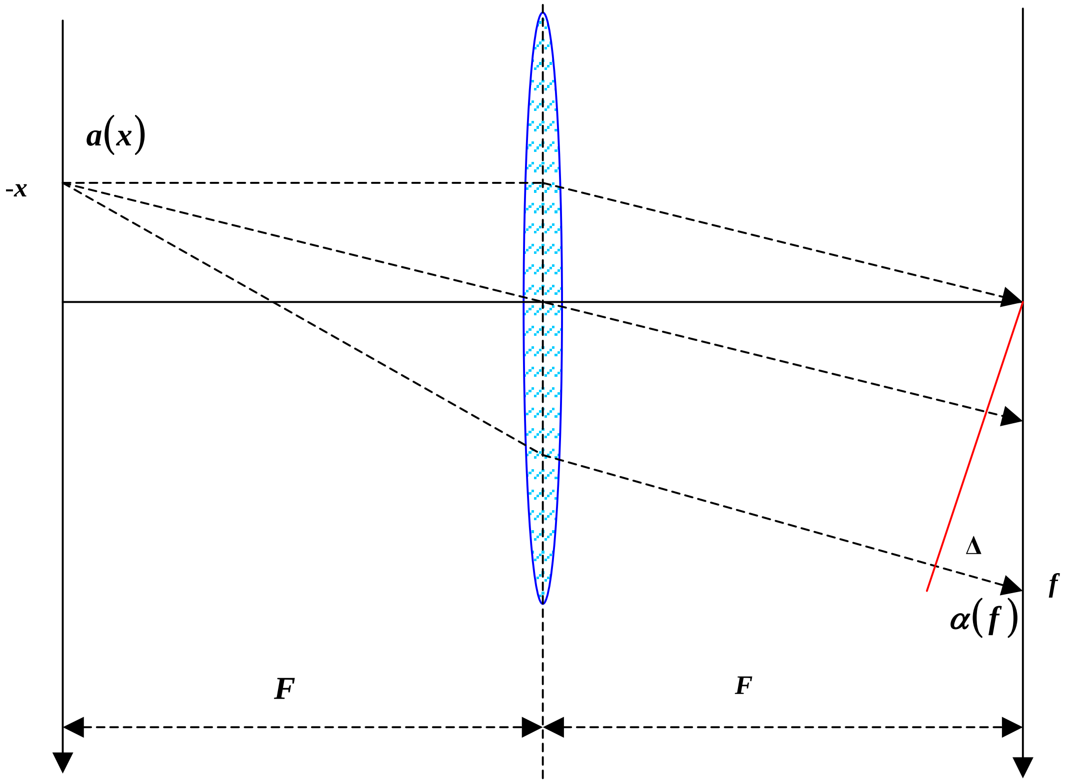
$$\alpha(f)\cong C\int_X a(x)exp\left[-i2\pi\frac{xf}{\lambda D}\right]dx$$

Lens a Fourier Transformer

Two reciprocal main properties of a lens (a definition of a lens) are:

- lens focuses parallel beam of light into a point;
- light from a point source in the lens focal plane is converted by the lens into a parallel beam propagating along the line passing through the point source and center of the lens.

One can use this property to determine point spread function of the optical system with a lens and input image plane and output image plane in front and rear focal planes of the lens.



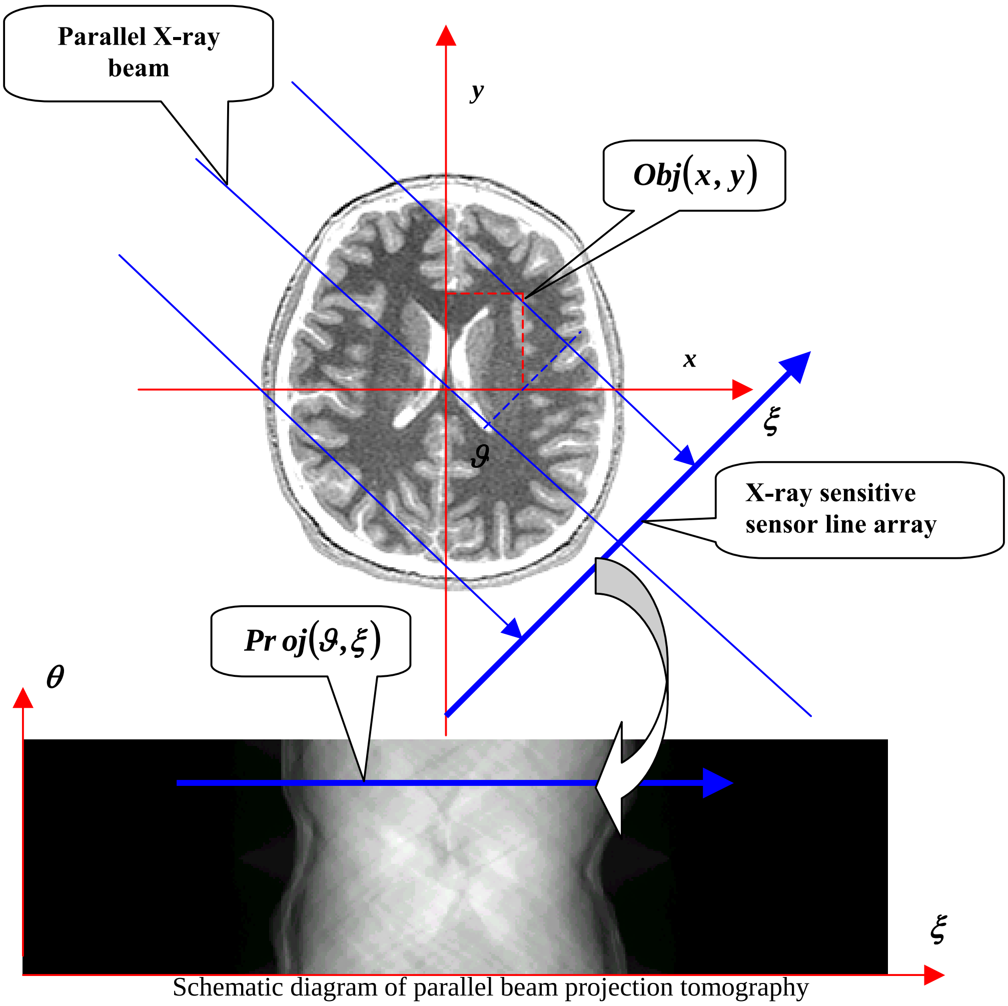
For $x \ll F$ ("paraxial" approximation),

$$\Delta = f\left(\frac{-x}{F}\right)$$

$$PSF(x, f) = exp(i2\pi\Delta / \lambda) = exp\left(-i2\pi\frac{fx}{\lambda F}\right);$$

$$a(f) = \int_X a(x) exp\left(-i2\pi\frac{fx}{\lambda F}\right)dx$$

Radon Transform and Reconstructive Tomography



Schematic diagram of parallel beam projection tomography

In computer tomography, a set of object's projections taken a different observation angles is measured and used for subsequent reconstruction of the object.

The relationship between object $Obj(x,y)$ and its projection

$$Proj(\vartheta,\xi) = \int_X\int_Y Obj(x,y)\,\delta(\xi - x\cos\vartheta - y\sin\vartheta)\,dxdy$$

is an integral one and is called ***Radon Transform***. Since projections are taken over a finite set of the observation angles, the process of projecting can be regarded as a specific type of 2-D signal discretization.

Projection theorem:

$$\Pr Sp(\vartheta, f) = \int_{-\infty}^{\infty} \Pr oj(\vartheta,\xi)\exp(i2\pi f\xi)\,d\xi =$$

$$\int_{-\infty}^{\infty}\left\{\int_X\int_Y Obj(x,y)\,\delta(\xi - x\cos\vartheta - y\sin\vartheta)\,dxdy\right\}\exp(i2\pi f\xi)\,d\xi =$$

$$\int_X\int_Y Obj(x,y)\exp[i2\pi(fx\cos\vartheta + fy\sin\vartheta)]\,dxdy = SpObj(f\cos\vartheta, f\sin\vartheta),$$

where

$$SpObj(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Obj(x,y)\exp[i2\pi(f_x x + f_y y)]\,dxdy$$

Fourier method of tomographic reconstruction