Lect. 6. Optical Reconstruction of CGHs

6.1. Introduction

In the reconstruction stage, computer generated holograms synthesized with the use of discrete representations of wave propagation transformations and recorded using one of the hologram encoding methods (see Lect. 5) are subjected, to analog optical transformations. The discrepancy between them and their discrete representation affects in a certain way the hologram reconstruction result. Transformations of a digital signal into a physical hologram according to the method of hologram recording have its influence as well. In order to illustrate techniques for analysis of the hologram reconstruction that account for these factors, we consider the reconstruction, in an analog optical set-ups performing optical Fourier transform (Fig. 6.1), of computer generated Fourier holograms for three methods of hologram encoding: for symmetrization method, for orthogonal encoding method and for double phase recording on a phase medium.

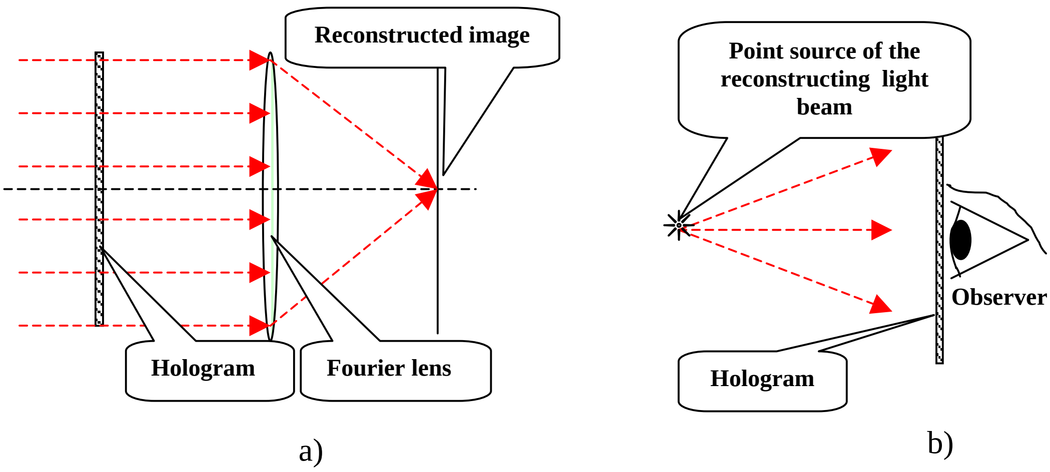


Fig. 6.1. Schemes of optical (a) and visual (b) reconstruction of computer generated holograms

6.2. Definitions and denotations

The following characteristics of the hologram recording device affect the reconstruction result: type of the sampling grid, sampling intervals, recording aperture and physical size of recorded holograms (see Fig. 6.2). In what follows, we will use the following denotations for them:

(ξ,η) - physical coordinates on the hologram recording medium,

$(\Delta\xi,\Delta\eta)$ - sampling intervals of the rectangular sampling grid along coordinates (ξ,η),

(ξ_0,η_0) -shift parameters that depend on the geometry of positioning the hologram in the reconstruction set up,

$h_{rec}(\xi,\eta)$ -hologram recording device aperture function and

$w(\xi,\eta)$ -a window function that defines physical size of the recorded hologram: $w(\xi,\eta) \geq 0$, when (ξ,η) belongs to the hologram area and $w(\xi,\eta) = 0$, otherwise.

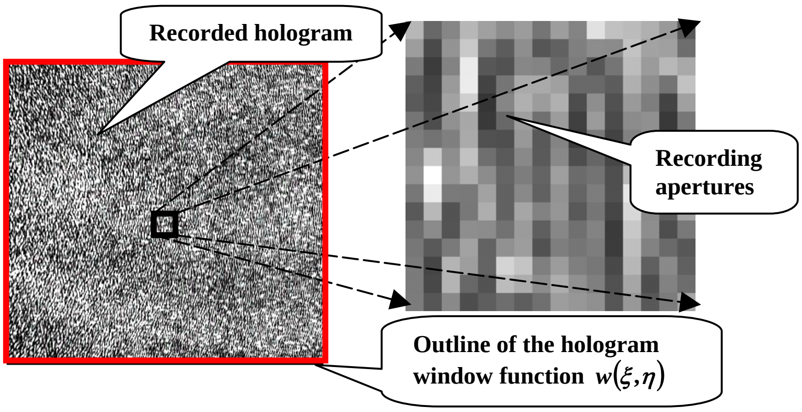


Figure 6.2. Definitions related to recorded physical computer generated holograms

6.3. Reconstruction of Fourier holograms synthesized using the symmetrization method

For the symmetrization method, the following equation describes conversion of the numerical matrix of the mathematical hologram $\{\Gamma_{r,s}\}$ into the physical hologram $\tilde{\Gamma}(\xi,\eta)$ recorded on an amplitude-only medium:

$$\tilde{\Gamma}(\xi,\eta) = w(\xi,\eta)\sum_r \sum_s (\Gamma_{r,s} + b) h_{rec}(\xi - \xi_0 - r\Delta\xi, \eta - \eta_0 - s\Delta\eta), \qquad (6.3.1)$$

where b is a constant bias required for eliminating negative values in recording samples $\{\Gamma_{r,s}\}$ of the mathematical hologram and indices (r,s) run over all available hologram samples. Eq. 6.3.1 may be rewritten in a form of the convolution:

$$\tilde{\Gamma}(\xi,\eta) = w(\xi,\eta)\cdot\left\{h_{rec}(\xi,\eta) \otimes \sum_{r=-\infty}^{\infty}\sum_{s=\infty}^{\infty}(\Gamma_{r,s} + b)\delta(\xi - \xi_0 - r\Delta\xi)\delta(\eta - \eta_0 - s\Delta\eta)\right\}, \qquad (6.3.2)$$

where $\otimes$ stands for the convolution and summation limits are changed to $[-\infty,\infty]$ bearing in mind that hologram window function $w(\xi,\eta)$ selects available hologram samples from virtual infinite number of samples.

Let now (x,y) be coordinates in reconstructed image plane situated at a distance Z from the hologram plane and λ be wavelength of the reconstructing light beam. As it was mentioned, we assume that recorded computer generated holograms $\tilde{\Gamma}(\xi,\eta)$ are subjected at reconstruction to optical integral Fourier Transform. According to the convolution theorem of the integral Fourier transform, its resulting reconstructed wave front $A_{rcstr}(x,y)$ the can be represented as:

$$A_{rcstr}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\Gamma}(\xi,\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta=$$

$$W(x,y)\otimes\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}(\xi,\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta\times\right.$$

$$\left.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(\Gamma_{r,s}+b)\delta(\xi-\xi_0-r\Delta\xi)\delta(\eta-\eta_0-s\Delta\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta\right\}=$$

$$W(x,y)\otimes\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}(\xi,\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta\times\right.$$

$$\left.\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(\Gamma_{r,s}+b)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)\delta(\xi-\xi_0-r\Delta\xi)\delta(\eta-\eta_0-s\Delta\eta)d\xi d\eta\right\}=$$

$$W(x,y)\otimes\left\{H_{rec}(x,y)\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(\Gamma_{r,s}+b)\exp\left(-i2\pi\frac{r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right)\right. \qquad (6.3.3)$$

where

$$W(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}w(\xi,\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta \qquad (6.3.4)$$

and

$$H_{rec}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}(\xi,\eta)\exp\left[-i2\pi\frac{x(\xi-\xi_0)+y(\eta-\eta_0)}{\lambda Z}\right]d\xi d\eta \qquad (6.3.5)$$

are Fourier transforms of the hologram window function and of the aperture function of the hologram recording device, the latter being found with an account of the shifts (ξ_0,η_0) in positions of recorded hologram samples with respect to optical axis of the optical set up used for reconstruction of the hologram.

Introduce now an array $\overline{A}_{k,l}^{(o)}$, $k=0,...,2N_1-1$, $l=0,1,...,N_2-1$, of samples of the object wave front. In the symmetrization method, the array is symmetrical as defined by Eq. 5.1.19 (Lect.5):

$$\overline{A}_{k,l}^{(o)}=\begin{cases}\overline{A}_{k,l}, 0\le k\le N_1-1;\ 0\le l\le N_2-1\\ \overline{A}_{2N-k,l},\ N_1\le k\le 2N_1-1;\ 0\le l\le N_2-1\end{cases} \qquad (6.3.6)$$

For Fourier holograms, mathematical hologram $\Gamma_{r,s}$ is computed as Shifted DFT of this array:

$$\Gamma_{r,s}\propto\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left\{i2\pi\left[\frac{(k+u)(r+p)}{2N_1}+\frac{(l+v)(s+q)}{N_2}\right]\right\}, \qquad (6.3.7)$$

where (u,v) and (p,q) are shift parameters. Substitute Eq. 6.3.7 into Eq. 6.3.3 and obtain:

$$A_{rcstr}(x,y) \propto W(x,y) \otimes \{H_{rec}(x,y) \times$$

$$\left\{ \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left\{ \sum_{k=0}^{2N_1-1} \sum_{l=0}^{N_2-1} \overline{A}_{k,l}^{(o)} \exp\left\{ i2\pi \left[\frac{(k+u)(r+p)}{2N_1} + \frac{(l+v)(s+q)}{N_2} \right] \right\} + b \right\} \times \right.$$

$$\left. \exp\left(-i2\pi \frac{r\Delta\xi x + s\Delta\eta y}{\lambda Z} \right) \right\} = W(x,y) \otimes \{H_{rec}(x,y) \times$$

$$\left\{ \sum_{k=0}^{2N_1-1} \sum_{l=0}^{N_2-1} \overline{A}_{k,l}^{(o)} \exp\left[i2\pi \left(\frac{k+u}{2N_1} p + \frac{l+v}{N_2} q \right) \right] \sum_{r=-\infty}^{\infty} \exp\left[i2\pi \left(\frac{k+u}{2N_1} - \frac{\Delta\xi x}{\lambda Z} \right) r \right] \times \right.$$

$$\left. \sum_{s=-\infty}^{\infty} \exp\left[i2\pi \left(\frac{l+v}{2N_1} - \frac{\Delta\eta y}{\lambda Z} \right) s \right] + b \sum_{r=-\infty}^{\infty} \exp\left(-i2\pi \frac{\Delta\xi x}{\lambda Z} r \right) \sum_{s=-\infty}^{\infty} \exp\left(-i2\pi \frac{\Delta\eta y}{\lambda Z} s \right) \right\}. \quad (6.3.8)$$

At this stage one can see that selection of shift parameters $p = q = 0$ will be appropriate to remove phase shift factors at samples $\{\overline{A}_{k,l}^{(o)}\}$ of the object wave front. Then, using the Poisson's summation formula,:

$$\sum_{r=-\infty}^{\infty} \exp(-i2\pi fr) = \sum_{do=-\infty}^{\infty} \delta(f - do) \qquad (6.3.9)$$

obtain:

$$A_{rcstr}(x,y) \propto W(x,y) \otimes$$

$$\left\{ H_{rec}(x,y) \left\{ \sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} \sum_{k=0}^{2N_1-1} \sum_{l=0}^{N_2-1} \overline{A}_{k,l}^{(o)} \delta\left(\frac{k+u}{2N_1} - \frac{\Delta\xi x}{\lambda Z} - do_x \right) \delta\left(\frac{l+v}{N_2} - \frac{\Delta\eta y}{\lambda Z} - do_y \right) + \right. \right.$$

$$\left. \left. b \sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} \delta\left(\frac{\Delta\xi x}{\lambda Z} - do_x \right) \delta\left(\frac{\Delta\eta y}{\lambda Z} - do_y \right) \right\} \right\} = .$$

$$\sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} \tilde{\overline{A}}^{(o)}(x, do_x; y, do_y) +$$

$$b \sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} W\left(x - \frac{\lambda Z}{\Delta\xi} do_x, y - \frac{\lambda Z}{\Delta\xi} do_y \right) \tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\frac{\lambda Z}{\Delta\xi} do_x, \frac{\lambda Z}{\Delta\xi} do_y \right), \qquad (6.3.10)$$

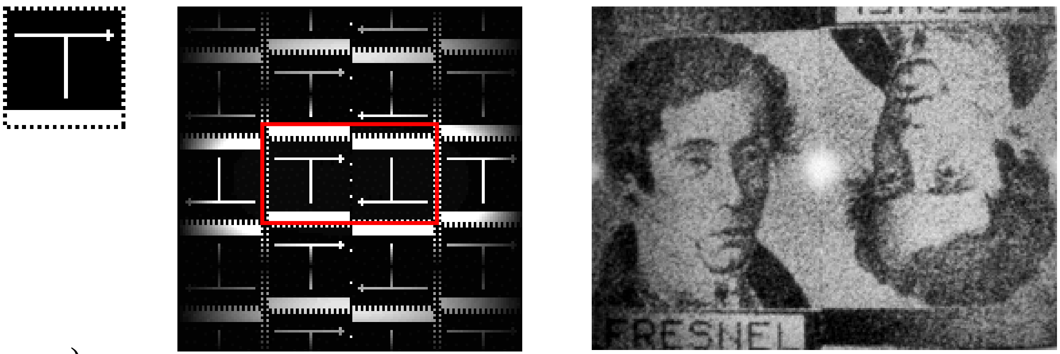
where it is denoted:

$$\tilde{\bar{A}}^{(o)}\left(x,do_x;y,do_y\right)=\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\overline{A}_{k,l}^{(o)}\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right), \qquad (6.3.11)$$

More detailed derivation of this formula is provided in Appendix.
It follows from this formula that

- the hologram placed into the optical Fourier system reconstructs the original object wave front $\tilde{\bar{A}}^{(o)}\left(x,do_x;y,do_y\right)$ in several diffraction orders defined by indices do_x and do_y;
- the pattern of the reconstructed images samples in the diffraction orders is masked by a function $\tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\frac{\lambda Z}{\Delta\xi}do_x,\frac{\lambda Z}{\Delta\xi}do_y\right)$, frequency response of the hologram recording device.
- the masked object wave front is reconstructed by interpolation of its samples with the interpolation function $W\left(x-\frac{\lambda Z}{\Delta\xi}do_x,y-\frac{\lambda Z}{\Delta\xi}do_y\right)$ equal to Fourier transform of the hologram window function $w(\xi,\eta)$;
- constant bias in hologram recording results in appearance in the reconstructed image of bright spots in the center of each diffraction order (the last term in Eq. 6.3.9)

Arrangement of diffraction orders in the reconstructed image plane under shift parameters $u=-N_1,v=-N_2/2$ and an example of image optically reconstructed from a computer-generated hologram synthesized using object symmetrization by duplication are shown in Fig. 6.3.



a) b) c)

Figure 6.3. Reconstruction of a hologram synthesized using image symmetrization by duplication: a) - a test object, b) - diffraction orders of reconstruction; the zero order is outlined by the red rectangle; c) – an example of optical reconstruction of a computer generated hologram synthesized using symmetrization by duplication.

Note that Eq. 6.3.9 can be extended to the case of hologram recording on combined amplitude and phase medium. For this one can simply discard in the formula the term containing constant bias b:

$$A_{rcstr}(x,y) = \sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\overline{A}_{k,l}^{(o)}\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right)=\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}\tilde{A}^{(o)}\left(x,do_x;y,do_y\right)$$

(6.3.11)

6.4. Reconstruction of holograms recorder using the orthogonal encoding method

In case of orthogonal encoding, the encoded hologram $\tilde{\Gamma}_{m,n}$ is formed from samples $\Gamma_{r,s}$ of the mathematical hologram as

$$\tilde{\Gamma}_{m,n} = \frac{1}{2}(-1)^r(i)^{m_0}\left[(-1)^{m_0}\Gamma_{r,s}+\Gamma_{r,s}^*\right]+b \qquad (6.4.1)$$

where $m = 2r + m_0$, $m_0 = 0,1$, $n = r$ (Lect.5, Eq. 5.1.4). Then for the physical hologram $\tilde{\Gamma}(\xi,\eta)$ in continuous coordinates (ξ,η) of the recording medium one obtains, in the same denotations that were adopted above:

$$\tilde{\Gamma}(\xi,\eta) = \frac{1}{2}w(\xi,\eta)\left\{\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m_0=0}^{1}\left\{(-1)^r(i)^{m_0}\left[(-1)^{m_0}\Gamma_{r,s}+\Gamma_{r,s}^*\right]+b\right\}\times\right.$$
$$\left.h_{rec}(\xi-\xi_0-r\Delta\xi,\eta-\eta_0-s\Delta\eta)\right\}=$$
$$w(\xi,\eta)\left\{\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m_0=0}^{1}\left\{\mathrm{Re}\left[(-1)^r(-i)^{m_0}\Gamma_{r,s}\right]+b\right\}h_{rec}(\xi+\xi_0-r\Delta\xi,\eta+\eta_0-s\Delta\eta)\right\}=$$
$$w(\xi,\eta)\left\{h_{rec}(\xi,\eta)\otimes\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m_0=0}^{1}\left\{\mathrm{Re}\left[(-1)^r(-i)^{m_0}\Gamma_{r,s}\right]+b\right\}\times\right.$$
$$\left.\delta(\xi-\xi_0-r\Delta\xi)\delta(,\eta-\eta_0-s\Delta\eta)\right\}, \qquad (6.4.2)$$

where $\mathrm{Re}(.)$ is real part of the variable.

Following the reasoning and settings used in deriving Eq.(6.3.9), one can obtain that, in the Fourier transform reconstruction scheme, such a hologram reconstructs the following wave front:

$$A_{rcstr}(x,y) \propto W(x,y) \otimes \{H_{rec}(x,y) \times$$
$$\left\{ \cos\left[\pi\left(\frac{1}{4}+\frac{\Delta\xi x}{\lambda Z}\right)\right] \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} \overline{A}_{k,l}^{(o)} \delta\left(\frac{k+u}{N_1}-\frac{\Delta\xi x}{\lambda Z}-do_x+\frac{1}{2}, \frac{l+v}{N_2}-\frac{\Delta\eta y}{\lambda Z}-do_y\right)+\right.$$
$$\sin\left[\pi\left(\frac{1}{4}-\frac{\Delta\xi x}{\lambda Z}\right)\right] \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} \overline{A}_{N_1-k,N_2-l}^{(o)\,*} \delta\left(\frac{k+u}{N_1}-\frac{\Delta\xi x}{\lambda Z}-do_x+\frac{1}{2}, \frac{l+v}{N_2}-\frac{\Delta\eta y}{\lambda Z}-do_y\right)+$$
$$\left.\left. b\cos\left(\pi\frac{\Delta\xi x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta\left(\frac{\Delta\xi x}{\lambda Z}-do_x+\frac{1}{2}, \frac{\Delta\eta y}{\lambda Z}-do_y\right)\right\}\right\} =$$
$$\cos\left[\pi\left(\frac{1}{4}+\frac{\Delta\xi x}{\lambda Z}\right)\right]\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{\overline{A}}_{dir}^{(o)}(x,do_x;y,do_y)+$$
$$\sin\left[\pi\left(\frac{1}{4}-\frac{\Delta\xi x}{\lambda Z}\right)\right]\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{\overline{A}}_{conj}^{(o)}(x,do_x;y,do_y)+$$
$$\left.\left. b\cos\left(\pi\frac{\Delta\xi x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} W\left(\frac{\Delta\xi x}{\lambda Z}-do_x+\frac{1}{2}, \frac{\Delta\eta y}{\lambda Z}-do_y\right)\right\}\right\}, \quad (6.4.3)$$

where

$$\tilde{\overline{A}}_{dir}^{(o)}(x,do_x;y,do_y) = \sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi}, \left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\overline{A}_{k,l}^{(o)} \times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi}; x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right), \quad (6.4.4)$$

$$\tilde{\overline{A}}_{conj}^{(o)}(x,do_x;y,do_y) = \sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi}, \left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\overline{A}_{N_1-k,N_2-l}^{(o)} \times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi}; x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right), \quad (6.4.5)$$

and, as above, (u,v) are shift parameters of SDFT used in computing the mathematical hologram (as in Eq. 6.3.7 with (p,q) set, as above, to zero) and N_1 and N_2 are dimensions of the array $\{\overline{A}_{k,l}^{(o)}\}$.

One can see from Eq. (6.4.3) that, similarly to the above case, the reconstructed image contains a number of diffraction orders, masked by the function $\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi}, \left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]$, spatial frequency response of the hologram recording device, and a central spot in several diffraction orders due to the constant bias in the hologram. But here, in contrast to the above case, each diffraction order contains two superimposed images of the object, - a direct image and its conjugate rotated by 180^o with respect the former. Each of them is additionally masked by functions $\left\{\cos\left[\pi\left(\frac{1}{4}+\frac{\Delta\xi x}{\lambda D}\right)\right]\right.$ and $\left\{\sin\left[\pi\left(\frac{1}{4}-\frac{\Delta\xi x}{\lambda D}\right)\right]\right.$, respectively. Therefore, in the central region of the direct image the conjugate image is suppressed, but at its periphery the aliasing conjugate image has an intensity comparable with that of the direct image. This is caused by the fact, that here additional intermediate samples hologram needed for representing the spatial carrier are obtained by zero order interpolation of samples of the mathematical hologram.

The pattern of diffraction orders of the direct and conjugate images and their corresponding masking functions are shown in Fig. 6.4 for $u=-N_1/2$ and $v=-N_2/2$. One can easily see the direct image and its conjugate, an aliasing image whose contrast increases along the horizontal axis from the center to the periphery. Additionally, Fig. 6.4, represents an example of an image optically reconstructed from a computer generated hologram with orthogonal encoding, in which one can clearly see the aliasing artifacts.

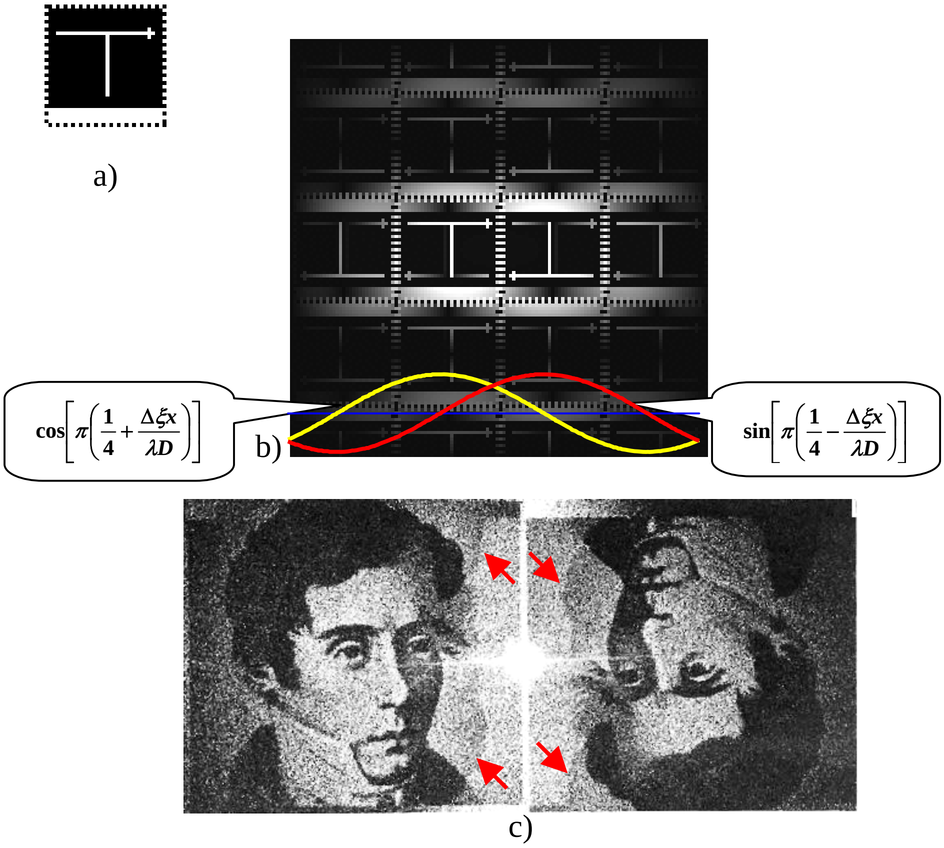


Figure 6.4. Test image (a), its reconstruction (b) for the orthogonal encoding method (at the bottom, spatial weighting functions of the direct and conjugate images are depicted) and an example (c) of optical reconstruction of a computer generated hologram synthesized using the orthogonal coding method; red arrows indicate aliasing images superposed on the reconstructed image.

6.5. Reconstruction of holograms recorded on phase media with double-phase method

The phase recording of the hologram on phase-only media according Eq. 5.2.3 (Lect. 5) is described, in the same denotations, as

$$\tilde{\Gamma}(\xi,\eta) = w(\xi,\eta)\left\{\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m_o=0}^{1}\exp\left\{ i\left[\varphi_{r,s} + (-1)^{m_0}\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right]\right\}\times\right.$$

$$h_{rec}\left[\xi+\xi_0-(2r+m_0)\Delta\xi, \eta+\eta_0-s\Delta\eta\right] =$$

$$w(\xi,\eta)\left\{h_{rec}(\xi,\eta)\otimes\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m_o=0}^{1}\exp\left\{i\left[\varphi_{r,s} + (-1)^{m_0}\arccos\left(\frac{|\Gamma_{r,s}|}{2A_0}\right)\right]\right\}\times\right.$$

$$\left.\delta\left[\xi-\xi_0-(2r+m_0)\Delta\xi\right]\delta(\eta-\eta_0-s\Delta\eta)\right\}. \qquad (6.5.1)$$

In a Fourier hologram reconstruction setup, this hologram is Fourier transformed and reconstructs the wave front described as

$$A_{rcstr}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\Gamma}(\xi,\eta)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta =$$

$$W(x,y)\otimes\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}(\xi,\eta)\exp\left[-i2\pi\frac{x(\xi-\xi_0)+y(\eta-\eta_0)}{\lambda Z}\right]d\xi d\eta\times\right.$$

$$\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{m_0=0}^{1}\exp\left\{i\left[\varphi_{r,s} + (-1)^{m_0}\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right]\right\}\times$$

$$\left.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-i2\pi\frac{x\xi+y\eta y}{\lambda Z}\right)\delta\left[\xi-\xi_0-(2r+m_0)\Delta\xi\right]\delta(\eta-\eta_0-s\Delta\eta)d\xi d\eta\right\} =$$

$$W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{m_0=0}^{1}\exp\left\{i\left[\varphi_{r,s} + (-1)^{m_0}\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right]\right\}\times\right.$$

$$\left.\exp\left[i2\pi\frac{(2r+m_0)\Delta\xi x + s\Delta\eta y}{\lambda Z}\right]\right\} =$$

$$W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\exp\left\{i\left[\varphi_{r,s} + (-1)^{m_0}\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right]\right\}\times\right.$$

$$\left.\exp\left(i2\pi\frac{2r\Delta\xi x + s\Delta\eta y}{\lambda Z}\right) + \exp\left[i2\pi\frac{(2r+1)\Delta\xi x + s\Delta\eta y}{\lambda Z}\right]\right\} =$$

$$W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left\{\exp\left[i\left(\varphi_{r,s}+\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right)\right]\exp\left[i2\pi\frac{2r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right]+\right.\right.$$

$$\left.\left.+\exp\left[i\left(\varphi_{r,s}-\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right)\right]\exp\left[i2\pi\frac{(2r+1)\Delta\xi x+s\Delta\eta y}{\lambda Z}\right]\right\}\right\}=$$

$$W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\exp(i\varphi_{r,s})\exp\left(i2\pi\frac{2r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right)\times\right.$$

$$\left[\exp\left(i\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)+\exp\left(-i\arccos\frac{|\Gamma_{r,s}|}{2A_0}\right)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)\right]=$$

$$2W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\exp(i\varphi_{r,s})\exp\left(i2\pi\frac{2r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right)\times\right.$$

$$\cos\left(\arccos\frac{|\Gamma_{r,s}|}{2A_0}+\pi\frac{\Delta\xi x}{\lambda Z}\right)=$$

$$2W(x,y)\otimes\left\{\tilde{H}_{rec}(x,y)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)\times\right.$$

$$\left[\cos\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{|\Gamma_{r,s}|}{2A_0}\exp(i\varphi_{r,s})\exp\left(i2\pi\frac{2r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right)\right.$$

$$\left.\sin\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sqrt{1-\frac{|\Gamma_{r,s}|^2}{4A_0^2}}\exp(i\varphi_{r,s})\exp\left(i2\pi\frac{2r\Delta\xi x+s\Delta\eta y}{\lambda Z}\right)\right].\quad(6.5.2)$$

By substituting $\Gamma_{r,s}$ with its expression through $SDFT(u,v;p,q)$:

$$|\Gamma_{r,s}|\exp(i\varphi_{r,s})=\sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left\{i2\pi\left[\frac{(k+u)(r+p)}{N_1}+\frac{(l+v)(s+q)}{N_2}\right]\right\}\quad(6.5.3)$$

and introducing an auxiliary function $\tilde{A}_{k,l}^{(0)}$ defined through equation

$$\sqrt{4A_0^2-|\Gamma_{r,s}|^2}\exp(i\varphi_{r,s})=\sum_{k=0}^{N_1-1}\sum_{l=0}^{N_2-1}\tilde{A}_{k,l}^{(o)}\exp\left\{i2\pi\left[\frac{(k+u)(r+p)}{N_1}+\frac{(l+v)(s+q)}{N_2}\right]\right\}\quad(6.5.4)$$

obtain after some transformations and setting $p=q=0$:

$$A_{rcstr}(x,y)\propto W(x,y)\otimes\left\{H_{rec}(x,y)\exp\left(i\pi\frac{\Delta\xi x}{\lambda Z}\right)\times\right.$$

$$\left[\cos\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{N_1}\sum_{l=0}^{N_2}\overline{A}_{k,l}^{(o)}\,\delta\left(\frac{k+u}{N_1}-\frac{2\Delta\xi\, x}{\lambda Z}-do_x,\frac{l+v}{N_2}-\frac{\Delta\eta\, y}{\lambda Z}-do_y\right)+\right.$$

$$\left.\left.\sin\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{N_1}\sum_{l=0}^{N_2}\tilde{A}_{k,l}^{(o)}\,\delta\left(\frac{k+u}{N_1}-\frac{2\Delta\xi\, x}{\lambda Z}-do_x,\frac{l+v}{N_2}-\frac{\Delta\eta\, y}{\lambda Z}-do_y\right)\right]\right\}=$$

$$\cos\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{\overline{A}}^{(o)}\left(x,do_x;y,do_y\right)+\sin\left(\pi\frac{\Delta\xi\, x}{\lambda Z}\right)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{\tilde{A}}^{(o)}\left(x,do_x;y,do_y\right), \quad (6.5.5)$$

where

$$\tilde{\overline{A}}^{(o)}\left(x,do_x;y,do_y\right)=\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\overline{A}_{k,l}^{(o)}\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right), \quad (6.4.4)$$

$$\tilde{\tilde{A}}^{(o)}\left(x,do_x;y,do_y\right)=\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\tilde{A}_{k,l}^{(o)}\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right). \quad (6.4.5)$$

The result of reconstruction of a hologram recorded on a phase medium by the two phase method is, thus, similar to the reconstruction of an hologram with orthogonal encoding (see Eq. 6.4.3). Image is also reconstructed in a number of diffraction orders masked by the function $H_{rec}(x,y)$. There is also superposition of the aliasing image described by the function $\tilde{A}_{k,l}^{(o)}$ over the original image $\overline{A}_{k,l}^{(o)}$ and the original and aliasing images are additionally masked by the functions $\cos\left(\pi\frac{\Delta\xi\, x}{\lambda D}\right)$ and $\sin\left(\pi\frac{\Delta\xi\, x}{\lambda D}\right)$, respectively. In the center of the proper image aliasing image is fully attenuated but over the peripheral area it may be of the same intensity as the proper image. In contrast to the orthogonal coding method, the aliasing image here is not conjugate to the original one, but is similar to it, in a sense, because, according to Eq. 6.5.4, it has the same phase spectrum and a distorted amplitude spectrum. An example of an aliasing image is shown in Fig. 6.5 for input images with and without adding pseudo-random phase components.

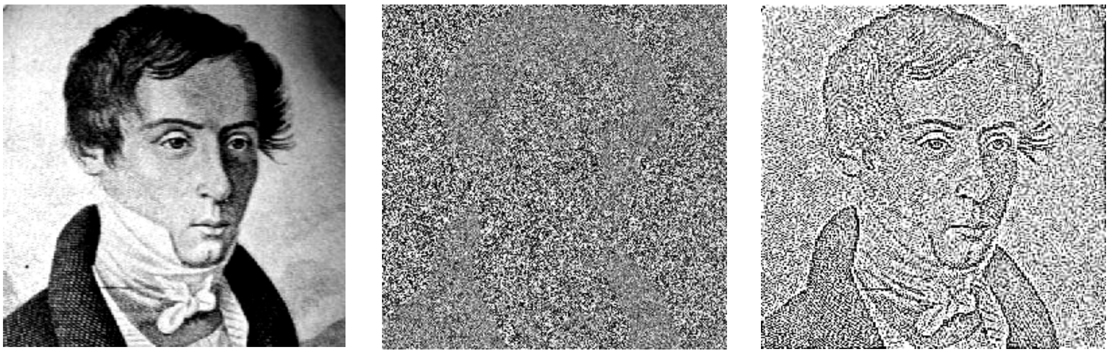


Fig. 6.5 Input image (left) and aliasing images in double phase encoding method for the input image with and without pseudo-random phase component (center and right, correspondingly)

Unlike the holograms coded via the symmetrization or orthogonal methods, double-phase encoding does not produce a central spot in the diffraction orders of the reconstructed image because the hologram is recorded on a phase medium without an amplitude bias.

Appendix.

Derivation of Eq. 6.3.10.

$$\tilde{\Gamma}(\xi,\eta) = w(\xi,\eta)\sum_{r}\sum_{s}\left(\Gamma_{r,s}+b\right)h_{rec}\left(\xi-\xi_0-r\Delta\xi,\eta-\eta_0-s\Delta\eta\right)$$

$$A_{rcstr}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[w(\xi,\eta)\sum_{r}\sum_{s}\left(\Gamma_{r,s}+b\right)h_{rec}\left(\xi-\xi_0-r\Delta\xi,\eta-\eta_0-s\Delta\eta\right)\right]\times$$

$$\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(\bar{\xi},\bar{\eta}\right)\exp\left(i2\pi\frac{\xi\bar{\xi}+\eta\bar{\eta}}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}\times$$

$$\sum_{r}\sum_{s}\left(\Gamma_{r,s}+b\right)h_{rec}\left(\xi-\xi_0-r\Delta\xi,\eta-\eta_0-s\Delta\eta\right)\exp\left(-i2\pi\frac{x\xi+y\eta}{\lambda Z}\right)d\xi d\eta =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(\bar{\xi},\bar{\eta}\right)d\bar{\xi}d\bar{\eta}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left(\Gamma_{r,s}+b\right)\times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}\left(\xi-\xi_0-r\Delta\xi,\eta-\eta_0-s\Delta\eta\right)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)\xi+\left(y-\bar{\eta}\right)\eta}{\lambda Z}\right]d\xi d\eta =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(\bar{\xi},\bar{\eta}\right)d\bar{\xi}d\bar{\eta}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left(\Gamma_{r,s}+b\right)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)\left(\xi_0+r\Delta\xi\right)+\left(y-\bar{\eta}\right)\left(\eta_0+s\Delta\eta\right)}{\lambda Z}\right]\times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{rec}(\xi,\eta)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)\xi+\left(y-\bar{\eta}\right)\eta}{\lambda Z}\right]d\xi d\eta =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(\bar{\xi},\bar{\eta}\right)d\bar{\xi}d\bar{\eta}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left(\Gamma_{r,s}+b\right)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)\left(\xi_0+r\Delta\xi\right)+\left(y-\bar{\eta}\right)\left(\eta_0+s\Delta\eta\right)}{\lambda Z}\right]\times$$

$$H_{rec}\left(x-\bar{\xi},y-\bar{\eta}\right)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(\bar{\xi},\bar{\eta}\right)H_{rec}\left(x-\bar{\xi},y-\bar{\eta}\right)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)\xi_0+\left(y-\bar{\eta}\right)\eta_0}{\lambda Z}\right]\times$$

$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left(\Gamma_{r,s}+b\right)\exp\left[-i2\pi\frac{\left(x-\bar{\xi}\right)r\Delta\xi+\left(y-\bar{\eta}\right)s\Delta\eta}{\lambda Z}\right]d\bar{\xi}d\bar{\eta} =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)H_{rec}\left(\bar{\xi},\bar{\eta}\right)\exp\left(-i2\pi\frac{\bar{\xi}\xi_0+\bar{\eta}\eta_0}{\lambda Z}\right)\times$$

$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left(\Gamma_{r,s}+b\right)\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi r+\bar{\eta}\Delta\eta s}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}\,. \qquad \text{(A1)}$$

Consider first the term

$$A_{rcstr}^{(\Gamma)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)H_{rec}\left(\bar{\xi},\bar{\eta}\right)\exp\left(-i2\pi\frac{\bar{\xi}\xi_0+\bar{\eta}\eta_0}{\lambda Z}\right)\times$$

$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\Gamma_{r,s}\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi r+\bar{\eta}\Delta\eta s}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}\,; \qquad \text{(A2)}$$

Denoting

$$\tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\bar{\xi},\bar{\eta}\right)=H_{rec}\left(\bar{\xi},\bar{\eta}\right)\exp\left(-i2\pi\frac{\bar{\xi}\xi_0+\bar{\eta}\eta_0}{\lambda Z}\right) \quad \text{(A3)}$$

and substituting into Eq. A2

$$\Gamma_{r,s}\propto\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left\{i2\pi\left[\frac{(k+u)(r+p)}{2N_1}+\frac{(l+v)(s+q)}{N_2}\right]\right\}, \quad \text{(A4)}$$

obtain:

$$A_{rcstr}^{(\Gamma)}(x,y)\propto\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)\tilde{H}_{rec}\left(\bar{\xi},\bar{\eta}\right)\times$$
$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left\{i2\pi\left[\frac{(k+u)(r+p)}{2N_1}+\frac{(l+v)(s+q)}{N_2}\right]\right\}\times$$
$$\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi r+\bar{\eta}\Delta\eta s}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}=$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)\tilde{H}_{rec}\left(\bar{\xi},\bar{\eta}\right)\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left[i2\pi\left(\frac{k+u}{2N_1}p+\frac{l+v}{N_2}q\right)\right]\times$$
$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\exp\left[i2\pi\left(\frac{k+u}{2N_1}r+\frac{l+v}{N_2}s\right)\right]\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi r+\bar{\eta}\Delta\eta s}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}=$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)\tilde{H}_{rec}\left(\bar{\xi},\bar{\eta}\right)\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left[i2\pi\left(\frac{k+u}{2N_1}p+\frac{l+v}{N_2}q\right)\right]\times$$
$$\sum_{r=-\infty}^{\infty}\exp\left[-i2\pi\left(\frac{\bar{\xi}\Delta\xi}{\lambda Z}-\frac{k+u}{N_2}\right)r\right]\sum_{s=-\infty}^{\infty}\exp\left[-i2\pi\left(\frac{\bar{\eta}\Delta\eta}{\lambda Z}-\frac{l+v}{N_2}\right)s\right]d\bar{\xi}d\bar{\eta}. \quad \text{(A5)}$$

Now use Poisson summation formula for the second line of the Eq. 6.3.9 :

$$\sum_{r=-\infty}^{\infty}\exp(-i2\pi fr)=\sum_{do=-\infty}^{\infty}\delta\left(f-do\right), \quad \text{(A6)}$$

obtain

$$A_{rcstr}^{(\Gamma)}(x,y)\propto$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W\left(x-\bar{\xi},y-\bar{\eta}\right)\tilde{H}_{rec}\left(\bar{\xi},\bar{\eta}\right)\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\overline{A}_{k,l}^{(o)}\exp\left[i2\pi\left(\frac{k+u}{2N_1}p+\frac{l+v}{N_2}q\right)\right]\times$$
$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\delta\left(\frac{\bar{\xi}\Delta\xi}{\lambda Z}-\frac{k+u}{N_2}-do_x\right)\delta\left(\frac{\bar{\eta}\Delta\eta}{\lambda Z}-\frac{l+v}{N_2}-do_y\right)d\bar{\xi}d\bar{\eta}\propto$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x-\bar{\xi}, y-\bar{\eta})\tilde{H}_{rec}(\bar{\xi},\bar{\eta})\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\bar{A}_{k,l}^{(o)}\exp\left[i2\pi\left(\frac{k+u}{2N_1}p+\frac{l+v}{N_2}q\right)\right]\times$$
$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\delta\left(\frac{\bar{\xi}\Delta\xi}{\lambda Z}-\frac{k+u}{N_2}-do_x\right)\delta\left(\frac{\bar{\eta}\Delta\eta}{\lambda Z}-\frac{l+v}{N_2}-do_y\right)d\bar{\xi}d\bar{\eta}\propto$$
$$\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\bar{A}_{k,l}^{(o)}\exp\left[i2\pi\left(\frac{k+u}{2N_1}p+\frac{l+v}{N_2}q\right)\right]\times$$
$$\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right). \qquad \text{(A7)}$$

At this point one can see that the appropriate selection of shift parameters p and q will be $p=0$ and $q=0$. With this selection obtain:

$$A_{rcstr}^{(\Gamma)}(x,y)\propto$$
$$\sum_{k=0}^{2N_1-1}\sum_{l=0}^{N_2-1}\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}\tilde{H}_{rec}\left[\left(\frac{k+u}{N_1}-do_x\right)\frac{\lambda Z}{\Delta\xi},\left(\frac{l+v}{N_2}-do_y\right)\frac{\lambda Z}{\Delta\xi}\right]\bar{A}_{k,l}^{(o)}\times$$
$$W\left(x-\frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi}-do_x\frac{\lambda Z}{\Delta\xi};x-\frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi}-do_y\frac{\lambda Z}{\Delta\xi}\right). \qquad \text{(A8)}$$

Now consider the second term and obtain similarly:

$$A_{rcstr}^{(b)}(x,y)=b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x-\bar{\xi},y-\bar{\eta})H_{rec}(\bar{\xi},\bar{\eta})\exp\left(-i2\pi\frac{\bar{\xi}\xi_0+\bar{\eta}\eta_0}{\lambda Z}\right)\times$$
$$\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi r+\bar{\eta}\Delta\eta s}{\lambda Z}\right)d\bar{\xi}d\bar{\eta}=$$
$$b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x-\bar{\xi},y-\bar{\eta})H_{rec}(\bar{\xi},\bar{\eta})\exp\left(-i2\pi\frac{\bar{\xi}\xi_0+\bar{\eta}\eta_0}{\lambda Z}\right)\times$$
$$\sum_{r=-\infty}^{\infty}\exp\left(-i2\pi\frac{\bar{\xi}\Delta\xi}{\lambda Z}r\right)\sum_{s=-\infty}^{\infty}\exp\left(-i2\pi\frac{\bar{\eta}\Delta\eta}{\lambda Z}s\right)d\bar{\xi}d\bar{\eta}$$
$$b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x-\bar{\xi},y-\bar{\eta})\tilde{H}_{rec}^{(\xi_0,\eta_0)}(\bar{\xi},\bar{\eta})\sum_{do_x=-\infty}^{\infty}\delta\left(\frac{\bar{\xi}\Delta\xi}{\lambda Z}-do_x\right)\sum_{do_y=-\infty}^{\infty}\delta\left(\frac{\bar{\eta}\Delta\eta}{\lambda Z}-do_y\right)d\bar{\xi}d\bar{\eta}=$$
$$b\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x-\bar{\xi},y-\bar{\eta})\tilde{H}_{rec}^{(\xi_0,\eta_0)}(\bar{\xi},\bar{\eta})\delta\left(\frac{\bar{\xi}\Delta\xi}{\lambda Z}-do_x\right)\delta\left(\frac{\bar{\eta}\Delta\eta}{\lambda Z}-do_y\right)d\bar{\xi}d\bar{\eta}\propto$$
$$b\sum_{do_x=-\infty}^{\infty}\sum_{do_y=-\infty}^{\infty}W\left(x-\frac{\lambda Z}{\Delta\xi}do_x,y-\frac{\lambda Z}{\Delta\xi}do_y\right)\tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\frac{\lambda Z}{\Delta\xi}do_x,\frac{\lambda Z}{\Delta\xi}do_y\right). \qquad \text{(A9)}$$

Thus obtain for the reconstructed object wave front:

$$A_{rcstr}(x,y) = A_{rcstr}^{(\Gamma)}(x,y) + A_{rcstr}^{(b)}(x,y) \propto$$

$$\sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} \sum_{k=0}^{2N_1-1} \sum_{l=0}^{N_2-1} \tilde{H}_{rec}\left[\left(\frac{k+u}{N_1} - do_x\right)\frac{\lambda Z}{\Delta\xi}, \left(\frac{l+v}{N_2} - do_y\right)\frac{\lambda Z}{\Delta\xi}\right] \overline{A}_{k,l}^{(o)} \times$$

$$W\left(x - \frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi} - do_x \frac{\lambda Z}{\Delta\xi}; x - \frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi} - do_y \frac{\lambda Z}{\Delta\xi}\right) +$$

$$b \sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} W\left(x - \frac{\lambda Z}{\Delta\xi} do_x, y - \frac{\lambda Z}{\Delta\xi} do_y\right) \tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\frac{\lambda Z}{\Delta\xi} do_x, \frac{\lambda Z}{\Delta\xi} do_y\right) =$$

$$\sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} \tilde{\overline{A}}^{(o)}(x, do_x; y, do_y) +$$

$$b \sum_{do_x=-\infty}^{\infty} \sum_{do_y=-\infty}^{\infty} W\left(x - \frac{\lambda Z}{\Delta\xi} do_x, y - \frac{\lambda Z}{\Delta\xi} do_y\right) \tilde{H}_{rec}^{(\xi_0,\eta_0)}\left(\frac{\lambda Z}{\Delta\xi} do_x, \frac{\lambda Z}{\Delta\xi} do_y\right), \qquad \text{(A10)}$$

where it is denoted:

$$\tilde{\overline{A}}^{(o)}(x, do_x; y, do_y) = \sum_{k=0}^{2N_1-1} \sum_{l=0}^{N_2-1} \tilde{H}_{rec}\left[\left(\frac{k+u}{N_1} - do_x\right)\frac{\lambda Z}{\Delta\xi}, \left(\frac{l+v}{N_2} - do_y\right)\frac{\lambda Z}{\Delta\xi}\right] \overline{A}_{k,l}^{(o)} \times$$

$$W\left(x - \frac{k+u}{N_1}\frac{\lambda Z}{\Delta\xi} - do_x \frac{\lambda Z}{\Delta\xi}; x - \frac{l+v}{N_2}\frac{\lambda Z}{\Delta\xi} - do_y \frac{\lambda Z}{\Delta\xi}\right). \qquad \text{(A11)}$$