Lect. 6. Discrete sampling theorem.

Whittaker-Nyquist-Kotelnikov-Shannon sampling theorem. Sampled data in computers.
Signals $\{a_k\}_{k=0,...,N-1}$ presented via transform

$$\boldsymbol{\Phi}_N = \{\varphi_r(k)\}_{r=0,1,...,N-1} : \mathbf{A}_N = \{a_k\} = \boldsymbol{\Phi}_N \boldsymbol{\Gamma}_N = \left\{\sum_{r=0}^{N-1} \gamma_r \varphi_r(k)\right\}_{k=0,1,...N-1}$$

Let only K of N signal samples are available. They define N equations for signal transform coefficients

$$\tilde{\mathbf{A}}_K = \left\{a_{\tilde{k}} = \sum_{r=0}^{N-1} \gamma_r \varphi_r(\tilde{k})\right\}_{\tilde{k}\in\tilde{K}}$$

Introduce a signal band-limited approximation $\hat{\mathbf{A}}_N^{BL} = \left\{\hat{a}_k = \sum_{\tilde{r}\in R} \gamma_{\tilde{r}} \varphi_{\tilde{r}}(k)\right\}$, where $\tilde{\gamma}_r = \begin{cases}\gamma_r, & r \in \tilde{R} \\ 0, & r \notin \tilde{R}\end{cases}$

$$\tilde{\mathbf{A}}_K = \mathbf{KofN}_{\Phi} \cdot \tilde{\Gamma}_K = \left\{\tilde{a}_{\tilde{k}} = \sum_{\tilde{r}\in R} \gamma_{\tilde{r}} \varphi_{\tilde{r}}(\tilde{k})\right\} \Rightarrow \tilde{\Gamma}_K = \{\tilde{\gamma}_r\} = \mathbf{KofN}_{\Phi}^{-1} \cdot \tilde{\mathbf{A}}_K \text{, where } \mathbf{KofN}_{\Phi} = \{\varphi_{\tilde{r}}(\tilde{k})\}$$

<u>Discrete sampling theorem:</u>

<u>Statement 1.</u> For any discrete signal of N samples defined by its K<N sparse and not necessarily regularly arranged samples, its band-limited, in terms of certain transform , approximation can be obtained with mean square error $MSE = \left\|A_N - \hat{A}_N\right\|^2 = \sum_{k=0}^{N-1} |a_k - \hat{a}_k|^2 = \sum_{r\notin R} |\gamma_r|^2$ ***provided positions of the samples secure the existence of the matrix inverse to the sub-transform matrix*** $\mathbf{KofN}_{\Phi}$ ***that corresponds to the band-limitation. The approximation error can be minimized by using a transform with the best, for this class of signals, energy compaction property.***

<u>Statement 2.</u> Any signal of samples that is known to have only K non-zero transform coefficients for certain transform T (T- "band-limited" signal) can be fully recovered from exactly K of its samples provided positions of the samples secure the existence of the matrix inverse to the sub-transform matrix $\mathbf{KofN}_{\Phi}$ ***that corresponds to the band-limitation.***

<u>Analysis of Transforms:</u>

DFT: Low pass band-limitation: $\mathbf{K}of\mathbf{N}_{DFT}^{LP}$ - Vandermonde matrix

$$\mathbf{K}of\mathbf{N}_{DFT}^{LP} = \left\{\exp\left(i2\pi \frac{\tilde{k}\tilde{r}_{LP}}{N}\right)\right\};\ \tilde{r}_{LP} \in \tilde{R}_{LP} = \{[0,1,...,(K-1)/2, N-(K-1)/2,..., N-1]\}$$

<u>Theorem 1</u>.

Low-pass DFT band-limited signals of N samples with only K nonzero low frequency DFT coefficients can be precisely recovered from exactly K of their samples taken in arbitrary positions

DFT: High pass band-limitation

$$\mathbf{K}of\mathbf{N}_{DFT}^{HP} = \left\{\exp\left(i2\pi \frac{\tilde{k}\tilde{r}_{HP}}{N}\right)\right\};\ \tilde{R}_{HP} = \{[(N-K+1)/2, (N-K+3)/2,.., (N+K-1)/2]\}$$

<u>Theorem 2</u>.***High-pass DFT band-limited signals of N samples with only K nonzero high frequency DFT coefficients can be precisely recovered from exactly K of their arbitrarily taken samples.*** **DCT:**

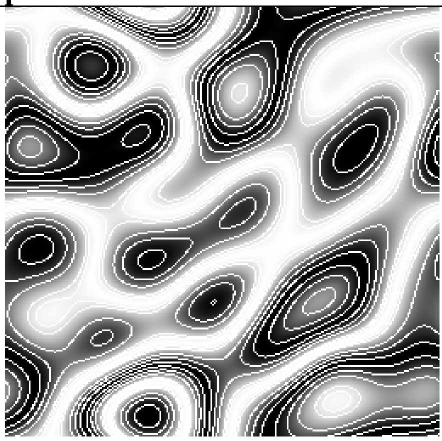
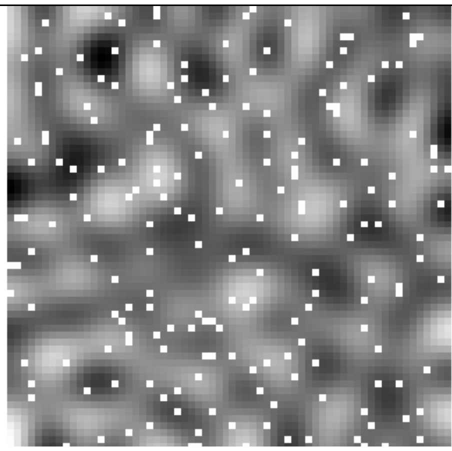
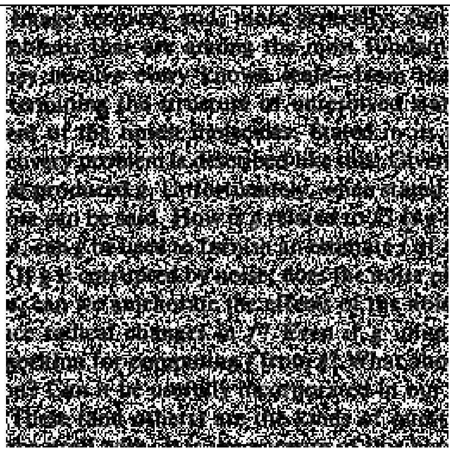
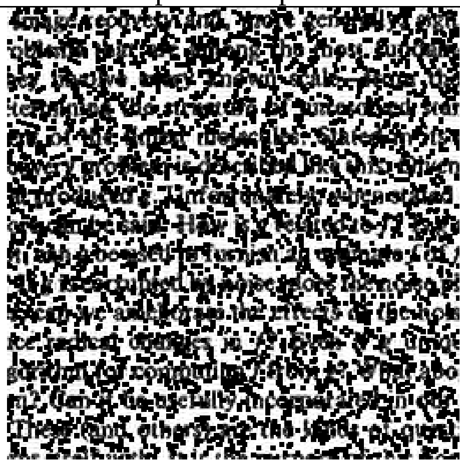
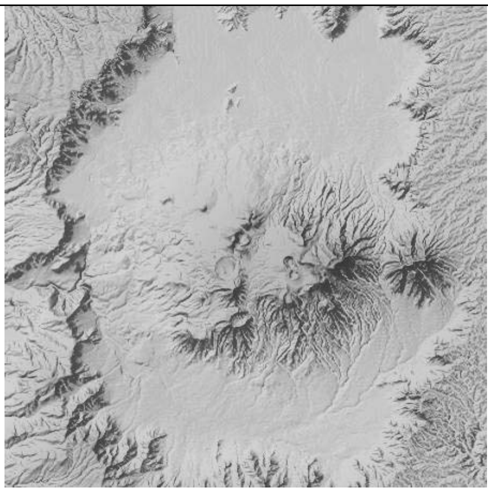
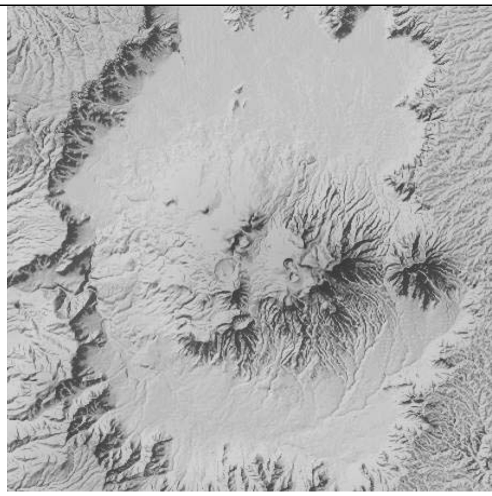
N-point Discrete Cosine Transform of a signal is equivalent to 2*N*-point Shifted Discrete Fourier Transform (SDFT) with shift parameters (1/2,0) of 2*N*- sample signal obtained from the initial one by its mirror reflection. KofN-trimmed matrix of SDFT(1/2,0)

$$\mathbf{K}of\mathbf{N}_{SDFT} = \left\{\exp\left(i2\pi \frac{(\tilde{k}+1/2)\tilde{r}}{2N}\right)\right\} = \mathbf{K}of\mathbf{N}_{DFT_{2N}} \left\{\exp\left(i\pi \frac{\tilde{r}}{2N}\right)\delta(k-r)\right\} \Rightarrow \quad \text{for DCT}$$

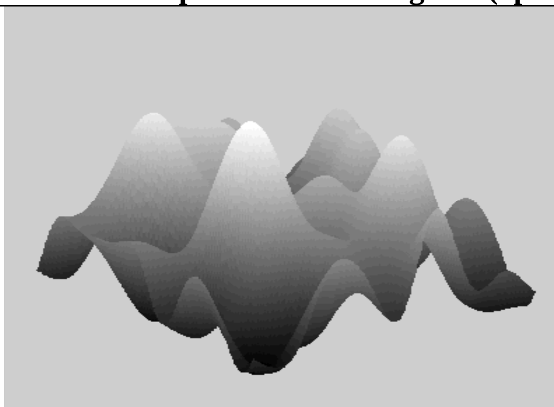
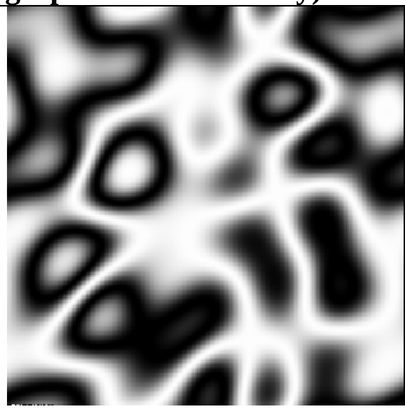
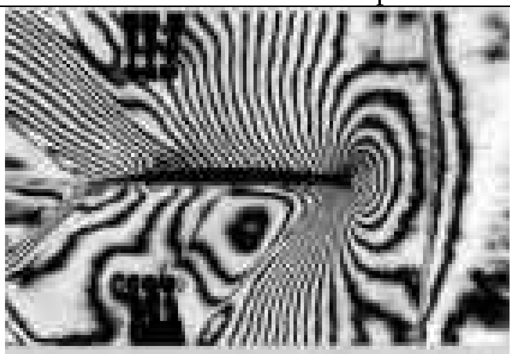
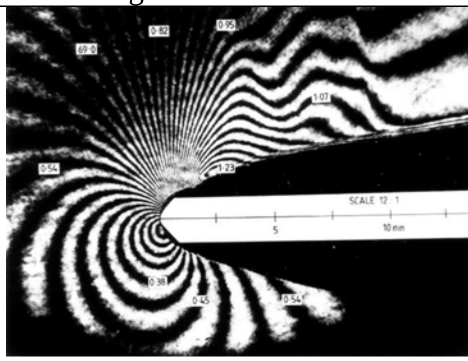
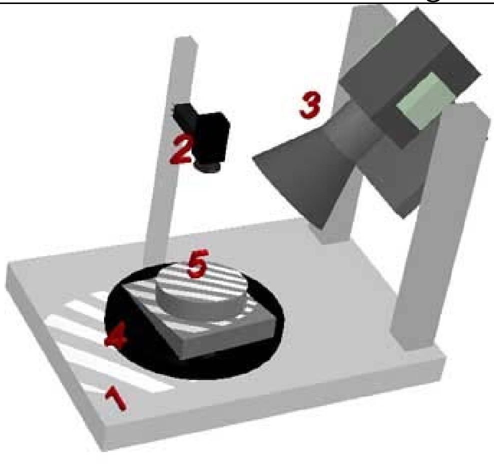
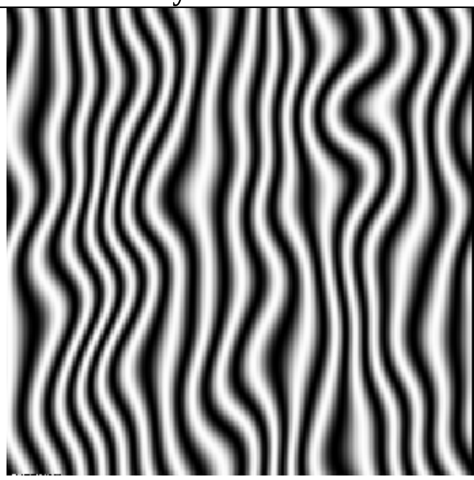
theorems similar to those for DFT hold

Walsh, Haar and wavelet band-limitation. Peculiarities of 2D band-limitations.

Examples of applications where images are specified by irregular and sparse samples

<table>
<tr><td>
Level lines</td><td>
Sparse samples</td></tr>
<tr><td>
Missing samples: isolated points (P=0.4)</td><td>
Missing samples: 3x3 sqiuares (P=0.4)</td></tr>
<tr><td>
Stereoscopic image (left)</td><td>
Stereoscopic image (right)</td></tr>
<tr><td colspan="2">Shape from stereo</td></tr>
</table>

Other examples:

Shape from interferogram (optical and holographic interferometry)	
	
Surface profile and its "on axis" interferogram	
 Supersonic flow around an isolated airfoil	 Leading edge flow around an aircraft wing.
(adopted from http://www.eng.warwick.ac.uk/OEL/previous/interferometry.htm)	
Structured light illumination Profilometry	
 Schematic diagram of shape measurement by means of structured light illumination (1 – fringe image; 2 – image sensor; 3 – illumination source; 4- support; 5 - object)	 An example of "moiré" pattern

Analysis of transforms

Discrete Fourier Transform

Consider the $\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}^{LP}$ -trimmed $\boldsymbol{DFT_N}$ matrix:

$$\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}^{LP} = \left\{ \mathbf{exp}\left(\boldsymbol{i}2\pi \frac{\tilde{\boldsymbol{k}}\tilde{\boldsymbol{r}}_{LP}}{\boldsymbol{N}} \right) \right\} \quad (1)$$

that corresponds to DFT $\mathbf{K}\boldsymbol{of}\mathbf{N}$ -low-pass band-limited signal. Due to complex conjugate symmetry of DFT or real signals, $\boldsymbol{K}$ has to be an odd number, and the set of frequency domain indices of $\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}$ low-pass band-limited signals in Eq.1 is defined as:

$$\tilde{\boldsymbol{r}}_{LP} \in \tilde{\boldsymbol{R}}_{LP} = \{[\mathbf{0,1,...,}(\boldsymbol{K}-\mathbf{1})/\mathbf{2}, \boldsymbol{N}-(\boldsymbol{K}-\mathbf{1})/\mathbf{2,...,}\boldsymbol{N}-\mathbf{1}]\} \quad (2)$$

For such a case, the following theorems hold:

Theorem 1.

$\mathbf{K}\boldsymbol{of}\mathbf{N}$ -low-pass DFT band-limited signals of $\boldsymbol{N}$ samples with only $\boldsymbol{K}$ nonzero low frequency DFT coefficients can be precisely recovered from exactly $\boldsymbol{K}$ of their samples taken in arbitrary positions.

Proof.

The theorem is proven if matrix $\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}^{LP}$ is invertible. A matrix is invertible if its determinant is nonzero. In order to check whether determinant of the matrix $\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}$ is non-zero, permute the order of columns of the matrix as following:

$$\tilde{\tilde{\boldsymbol{r}}} \in \tilde{\tilde{\boldsymbol{R}}} = \{[\boldsymbol{N}-(\boldsymbol{K}-\mathbf{1})/\mathbf{2,...,}\boldsymbol{N}-\mathbf{1,0,1,...,}(\boldsymbol{K}-\mathbf{1})/\mathbf{2}]\} \quad (3)$$

and obtain matrix

$$\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}^{DFTsh} = \left\{ \exp\left[i2\pi \frac{\tilde{k}\tilde{\tilde{\tilde{r}}}}{N} \right] \right\} = \left\{ \exp\left[i2\pi \frac{N-(K-1)/2}{N} \tilde{k} \right] \delta\left(\tilde{k} - \tilde{\tilde{\tilde{r}}} \right) \right\} \left\{ \exp\left[i2\pi \frac{\tilde{k}\tilde{\tilde{\tilde{r}}}}{N} \right] \right\} \tag{4}$$

where

$$\tilde{\tilde{\tilde{r}}} \in \tilde{\tilde{\tilde{R}}} = \{[0,...,K-1]\} \tag{5}$$

The first matrix in this product of matrices is a diagonal matrix, which is obviously invertible. The second one is a version of Vandermonde matrices, which are also known to have non-zero determinant if, like in our case, its ratios for each row are distinct. As permutation of the matrix columns does not change the absolute value of its determinant, Eq. 4 implies that determinant of $\mathbf{K}\boldsymbol{of}\mathbf{N}$ -trimmed $\boldsymbol{DFT_N}$ matrix of Eq. 1 is also non-zero for arbitrary set $\tilde{K} = \{\tilde{k}\}$ of positions of K available signal samples.

One can easily see that for DFT $\mathbf{K}\boldsymbol{of}\mathbf{N}$ -high-pass band-limited signals, for which

$$\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}^{HP} = \left\{ \exp\left(i2\pi \frac{\tilde{k}\tilde{r}_{HP}}{N} \right) \right\} \tag{6}$$

where

$$\tilde{r}_{HP} \in \tilde{R}_{HP} = \{[(N-K+1)/2, (N-K+3)/2, .., (N+K-1)/2]\} \tag{7}$$

a similar theorem holds

<u>Theorem 2</u>.

$\mathbf{K}\boldsymbol{of}\mathbf{N}$ *-high-pass DFT band-limited signals of* ***N*** *samples with only* ***K*** *nonzero high frequency DFT coefficients can be precisely recovered from exactly* ***K*** *of their arbitrarily taken samples.*

Note that, due to the complex conjugate symmetry of DFT of real signals, ***K*** in this case has to be odd whatever ***N*** is.

Above Theorems 1 and 2 can be extended to a more general case of signal DFT band limitation, when indices $\{\tilde{r}\}$ of nonzero DFT spectral coefficients form arithmetic progressions with common difference other than one such as, for instance,

$$\tilde{r}_{mLP} \in \tilde{R}_{mLP} = \{[0, m, \dots, m(K-1)/2, N - m(K-1)/2, \dots, N - m(K-1)/2 + (K+1)/2]\} \tag{8}$$

Discrete Cosine Transform (DCT)

N-point Discrete Cosine Transform of a signal is equivalent to 2***N***-point Shifted Discrete Fourier Transform (SDFT) with shift parameters (1/2,0) of 2***N***- sample signal obtained from the initial one by its mirror reflection. **K*of*N** -trimmed matrix of SDFT(1/2,0)

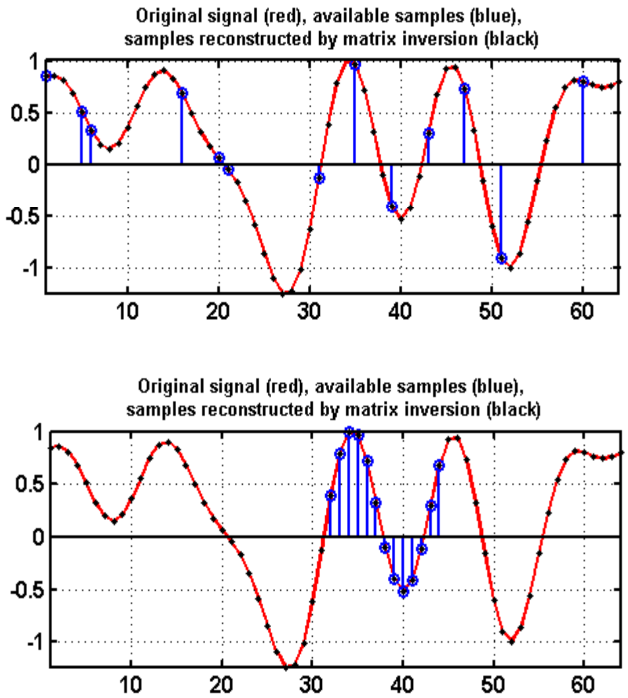
$$\mathbf{K}\boldsymbol{of}\mathbf{N}_{SDFT} = \left\{ \exp\left(i2\pi \frac{(\tilde{k}+1/2)\tilde{r}}{2N} \right) \right\} \tag{9}$$

can be represented as a product

$$\mathbf{K}\boldsymbol{of}\mathbf{N}_{SDFT} = \left\{ \exp\left(i2\pi \frac{\tilde{k}\tilde{r}}{2N} \right) \left\{ \exp\left(i\pi \frac{\tilde{r}}{2N} \right) \delta(k-r) \right\} \right\} = \mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT} \left\{ \exp\left(i\pi \frac{\tilde{r}}{2N} \right) \delta(\tilde{k}-\tilde{r}) \right\} \tag{10}$$

of a $2N$ -point DFT matrix and a diagonal matrix $\left\{ \exp\left(i\pi \frac{\tilde{r}}{2N} \right) \delta(\tilde{k}-\tilde{r}) \right\}$. The latter one is invertible and the invertibility of **K*of*N** -trimmed DFT_{2N} matrix $\mathbf{K}\boldsymbol{of}\mathbf{N}_{DFT}$ can be proved, for above described band-limitations, as it was done above for the DFT case. Therefore, for DCT theorems similar to those for DFT hold.

Exact recovery of DFT-band-limited signal by matrix inversion



Restoration of a DFT low pass band-limited signal by matrix inversion for the cases of random (a), upper) and compactly placed signal samples (a), bottom) and by the iterative algorithm (b). Bottom right plot shows standard deviation of signal restoration error as a function of the number of iterations. The experiment was conducted for test signal length 64 samples; bandwidth 13 frequency samples (~1/5 of the signal base band)

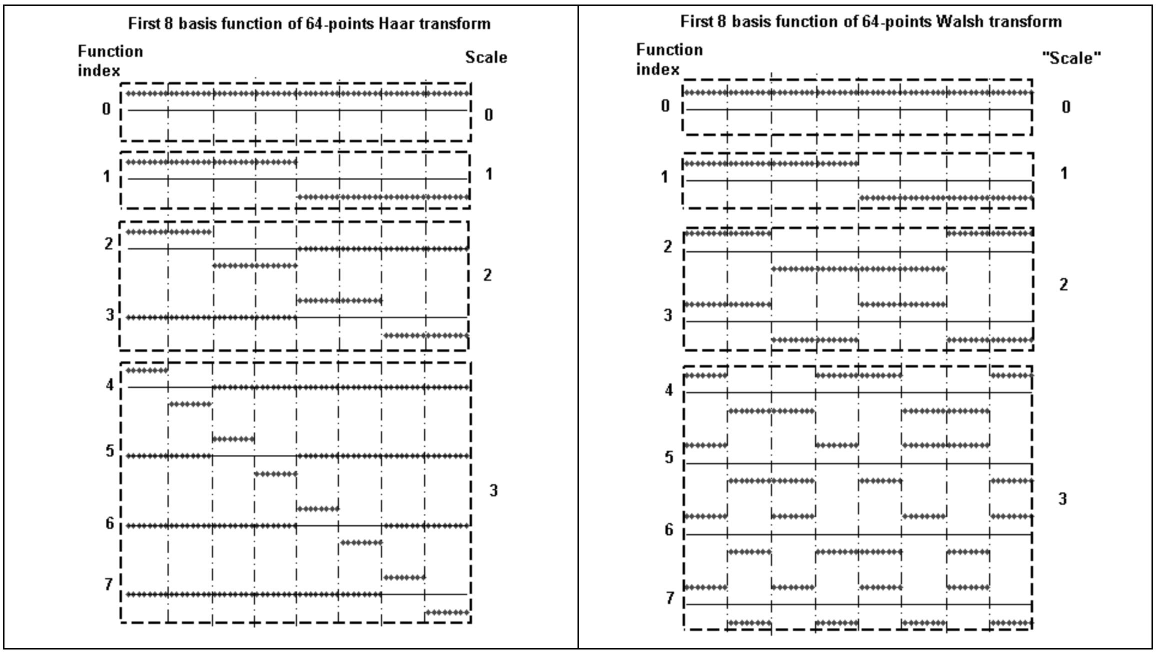
Wavelets and Other Bases

The main peculiarity of wavelet bases is that their basis functions are most naturally ordered in terms of two components: scale and position within the scale. Scale index is analogous to the frequency index for DFT. Position index tells only of the shift of the same basis function within the signal on each scale. Therefore band-limitation for DFT translates to scale limitation for wavelets. Limitation in terms of position is trivial: it simply means that some parts of the signal are not relevant. Commonly, discrete wavelets are designed for signals whose length is an integer power of 2 ($\boldsymbol{N} = \boldsymbol{2^n}$). For such signals, there are $\boldsymbol{s} \le \boldsymbol{n}$ scales and possible "band-limitations".

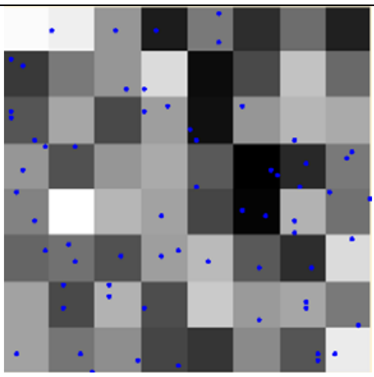
The simplest special case of wavelet bases is Haar basis. Signals with $\boldsymbol{N} = \boldsymbol{2^n}$ samples and only with $\boldsymbol{K}$ lower index non-zero Haar transform (the transform coefficients $\{\boldsymbol{K}, ..., \boldsymbol{N} - \boldsymbol{1}\}$ are zero) are ($\tilde{\boldsymbol{s}} = \left(\lfloor \log_2 (\boldsymbol{K} - \boldsymbol{1}) \rfloor + \boldsymbol{1}\right)$) - "band-limited", where $\lfloor \boldsymbol{x} \rfloor$ is an integer part of $\boldsymbol{x}$. Such signals are piecewise constant within intervals between zero-crossings. The shortest intervals of the signal constancy have $\boldsymbol{2}^{\boldsymbol{n}-\tilde{\boldsymbol{s}}}$ samples. For any two samples that are located on the same interval, all Haar basis function on this and lower scales have the same value. Therefore, having more than one sample per constant interval will not change the rank of the matrix $\mathbf{KofN}$. The condition for perfect reconstruction is, therefore, to have at least one sample on each of those intervals.

For other wavelets as well as for other bases general necessary, sufficient and easily verified condition for the invertibility of $\mathbf{KofN}$-trimmed transform sub-matrix is not known for the present authors. Standard linear algebra procedures for determining matrix rank, can be used for testing invertibility of the matrix.

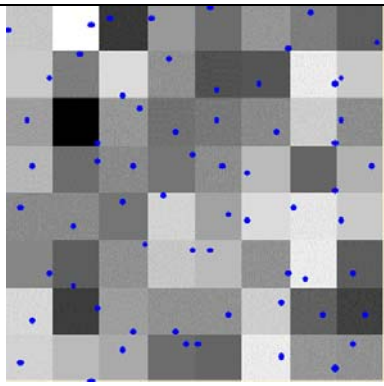
For Walsh basis functions, the index corresponds to the “sequency”, or to the number of zero crossings of the basis function. The sequency carries a certain analogy to the signal frequency. Basis functions ordering according to their sequency, which is characteristic for Walsh transform, preserves, for many real signals, the property of decaying transform coefficients’ energy with their index. Therefore, for Walsh transform the notion of low-pass band-limited signal approximation, similar to the one described in Sect. 0, for DFT, can be used. On the other hand, Walsh basis functions, similarly to Haar basis function, can be characterized by the scale index, which specifies the shortest interval of signal constancy. Signals with $\boldsymbol{N = 2^n}$ samples and band-limitation of $\boldsymbol{K}$ Walsh transform coefficients have shortest intervals of signal constancy of $\boldsymbol{2^{n-\tilde{s}}}$ samples, where $\boldsymbol{\tilde{s} = \left(\lfloor \log_2 (K-1) \rfloor + 1\right)}$. A necessary condition for perfect reconstruction is to have $\boldsymbol{K}$ signal samples taken on different intervals. Unlike the Haar transform case, not all the intervals are needed to be sampled, but only $\boldsymbol{K}$ intervals out off the total number of intervals. For a special case of $\boldsymbol{K}$ equal to a power of 2, there are $\boldsymbol{K}$ intervals, each of which has to be sampled to secure perfect reconstruction, This is the case, when the reconstruction condition for Walsh Transform is identical to that for Haar transform.



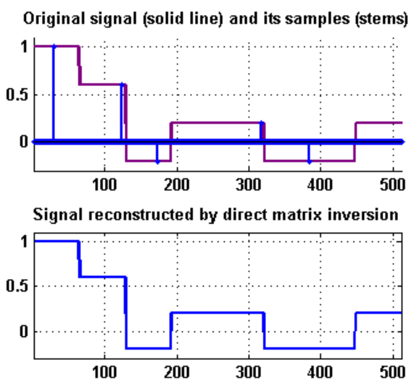
First 8 basis functions of 64 point Haar (a) and Walsh (b) transforms. Intervals of function constancy are outlined by dash-dot lines. Functions that belong to the same scale are outlined by dashed boxes.



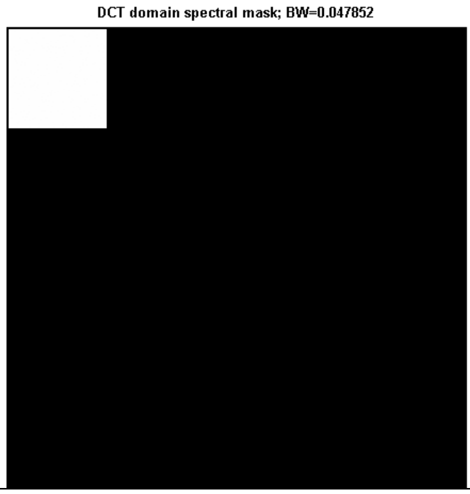
Haar non-recoverable

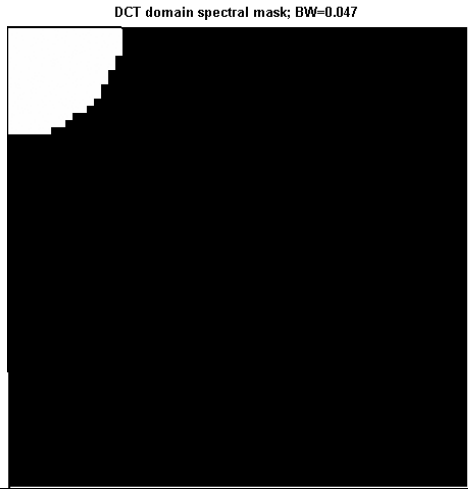


Haar recoverable

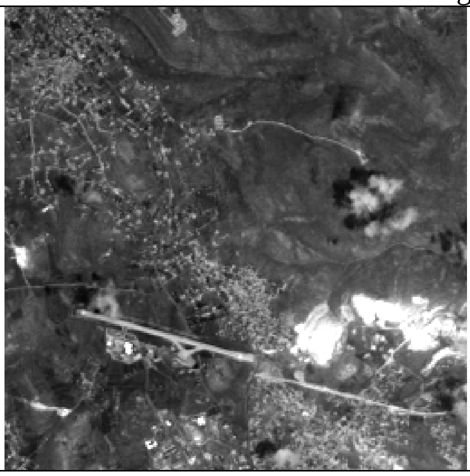
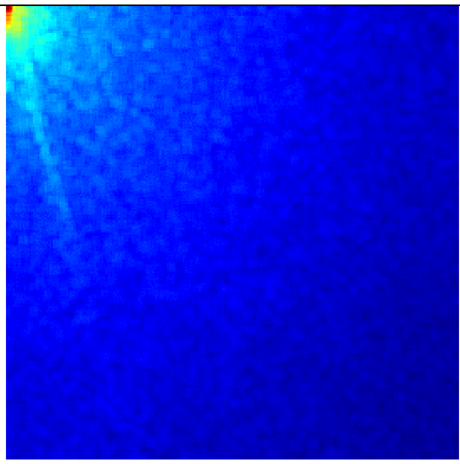


An example for perfect reconstruction in Walsh domain





Rectangular and circular

Real life image and its DCT power spectrum ((5x5 smoothed).^0.15)