

L. Yaroslavsky. Course 0510.7211 “Digital Image Processing: Applications”

Lect. 7. Discrete representation of imaging transformations

The conformity principle between continuous and discrete signal transformations

Digital filters: $\mathbf{b} = \{b_k\} = \mathbf{H} \cdot \mathbf{a} = \left\{ \sum_n h_{k,n} a_n \right\}$. Shift-inv. DF: $b_k = \sum_n a_n h_{k-n} = \sum_n h_n a_{k-n}$;

Continuous and discrete impulse response and frequency response of digital filters:

$$h_{cnt}(x, \xi) = \sum_n \sum_m h_n \varphi_d [x - (k-n)\Delta x] \varphi_r (\xi - n\Delta x); h_{dscr} = \{h_n\} \quad H_{dscr}(f) = \left[\sum_{n=0}^{N-1} h_n \exp(i2\pi f n \Delta x) \right]$$

$$H_{cnt}(f, p) = \left[\sum_{n=0}^{N-1} h_n \exp(i2\pi p n \Delta x) \right] \cdot \frac{\sin[\pi(f-p)N_b \Delta x]}{\sin[\pi(f-p)\Delta x]} \cdot \Phi_r(-p) \cdot \Phi_d(f)$$

Boundary effects and methods for signal extension.

Discrete Fourier Transform as a discrete representation of the Fourier integral:

$$\alpha(f) = \int_{-\infty}^{\infty} a(x) \exp(i2\pi f x) dx \Leftrightarrow \alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp(i2\pi k r / N); a_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \alpha_r \exp(-i2\pi k r / N)$$

Basic properties:

Cyclicity: $\alpha_r = \alpha_{(r) \bmod N}$; $a_k = a_{(k) \bmod N}$ Symmetries: $\{\pm a_{N-k}\} \Leftrightarrow \{\pm \alpha_{N-s}\}$; $\{a_k = \pm a_{N-r}^*\} \Leftrightarrow \{\alpha_r = \pm \alpha_{N-r}\}$

Shift and convolution theorems: $a_{(k+p) \bmod N} \Leftrightarrow \alpha_r \exp(-i2\pi p r / N)$; $\left\{ \sum_n a_n b_{(k-n) \bmod N} \right\} \Leftrightarrow \sqrt{N} \alpha_r \beta_r$

Sampling theorem: $\frac{1}{L} \alpha_{(r) \bmod N} \Leftrightarrow a_{k_1} \delta(k_2); k = Lk_1 + k_2$;

$$L \left\{ \left[1 - \text{rect} \frac{r - N/2}{LN - N} \right] \alpha_{(r) \bmod N} \right\} \Leftrightarrow \sum_{n=0}^{N-1} a_n \frac{\sin \left[\pi \frac{N-1}{N} (k-nL) \right]}{N \sin[\pi(k-nL)/LN]}; N \text{ - even number}; k = 0 \dots LN - 1$$

$$L \left\{ \left[1 - \text{rect} \frac{r - (N+1)/2}{LN - N - 1} \right] \alpha_{(r) \bmod N} \right\} \Leftrightarrow \sum_{n=0}^{N-1} a_n \frac{\sin[\pi(k-nL)]}{N \sin[\pi(k-nL)/LN]}; N \text{ - odd number}; k = 0 \dots LN - 1$$

Multidimensional DFTs: separability. 2-D DFT $\alpha_{r,s} = \frac{1}{\sqrt{NM}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} a_k \exp(i2\pi(kr + ls)/NM)$.

Shifted DFTs: $\alpha_r^{u,v} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k^{u,v} \exp(i2\pi k(r+v)/N) \exp(i2\pi u k / N)$

Discrete cosine transform: $DCT = SDF_{1/2,0}(a_k = a_{2N-1-k})$: $\alpha_r = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{(k+1/2)r}{N}\right)$

$$IDCT = ISDF_{1/2,0}(\alpha_r = -\alpha_{2N-r}): a_k = \frac{1}{\sqrt{2N}} \left\{ \alpha_0 + 2 \sum_{r=1}^{N-1} \alpha_r \cos\left(\pi \frac{(k+1/2)r}{N}\right) \right\}$$

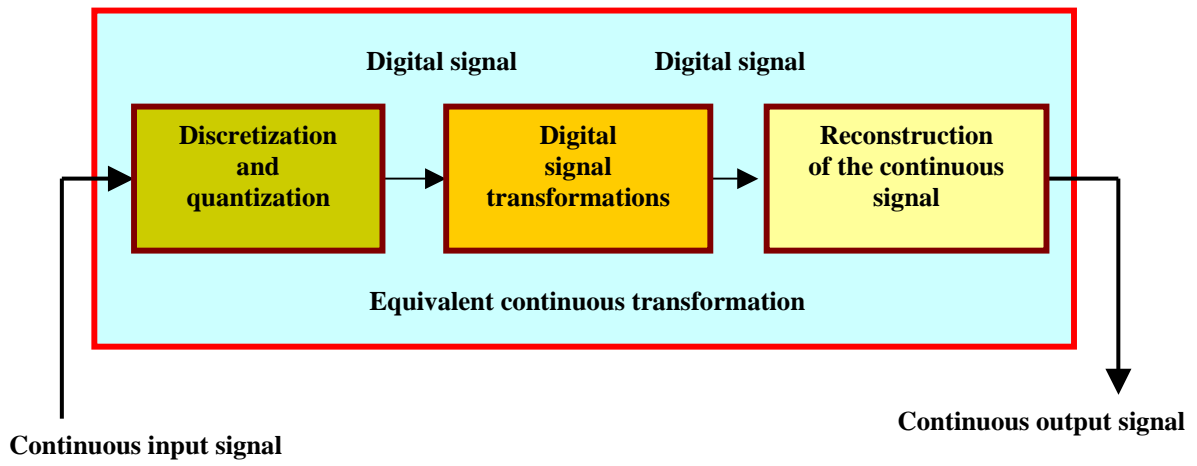
Discrete Sine Transform: $DST = SDF_{1,1}(a_k = -a_{2N-k}; a_n = 0)$: $\alpha_r = \frac{1}{\sqrt{2N+1}} \sum_{k=0}^{N-1} a_k \sin\left(2\pi \frac{(k+1)(r+1)}{2N+1}\right)$

Other orthogonal transforms: Walsh transform: $\alpha_r = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(2\pi i \sum_{m=0}^{\log N - 1} k_m r_m^{Gr} \log_{N-1-m} / 2\right)$

Haar Transform: $\alpha_r = 2^{(R-\log N)} \sum_{k=0}^{N-1} a_k \exp(2\pi i k_{\log N - R - 1} / 2) \delta([k]_R \oplus [r]_R)$; R is index of the nonzero most significant bit in the binary representation of r ; $[k]_R, [r]_R$ are a binary numbers composed of R most and least significant bits of the binary representation of k and r , respectively.

Problems for self-testing

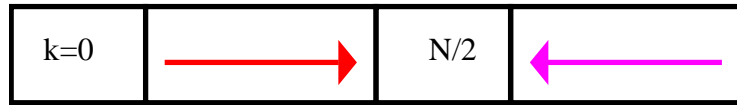
1. Explain the conformity principle between analogue and digital signal transformations.
2. Derive the general formula of digital filtering in signal domain. What are continuous and discrete impulse and frequency responses of digital filters.
3. From the Fourier integral, derive basic formula of DFT and demonstrate similarities and dissimilarities between DFT and integral Fourier Transform.
4. Explain discrete sampling theorem and the optimality of signal discrete sinc-interpolation.
5. Derive and explain Shifted Discrete Fourier Transform and its relation to DFT.
6. Describe other orthogonal transforms and their relationships with DFT and its modifications.



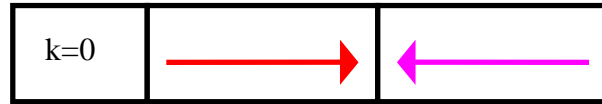
The conformity principle between analogue and digital signal transformations



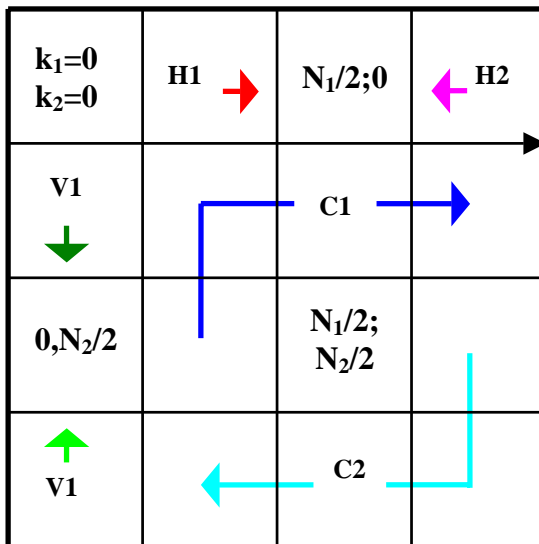
N - even number



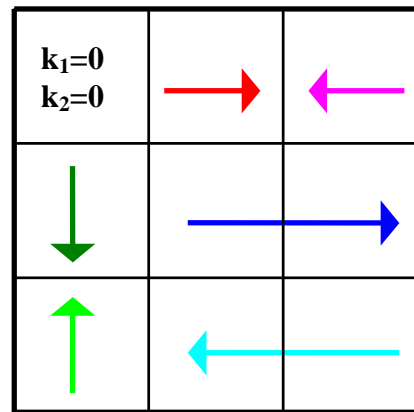
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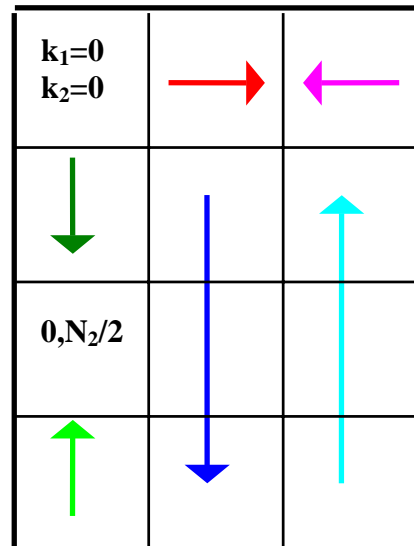
N_1, N_2 - even numbers



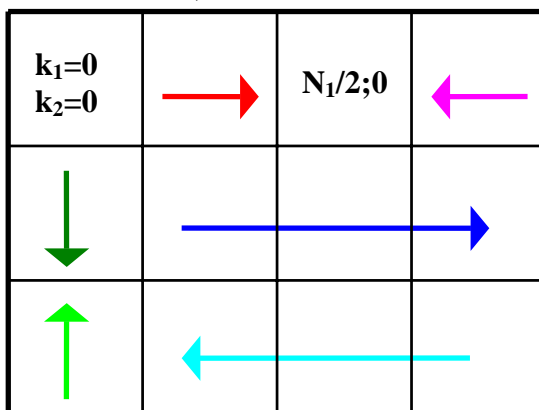
N_1, N_2 - odd numbers



N_1 - odd, N_2 - even



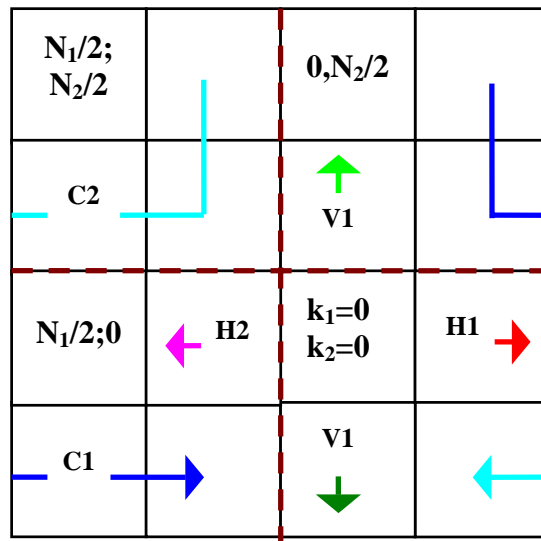
N_1 - even, N_2 - odd number



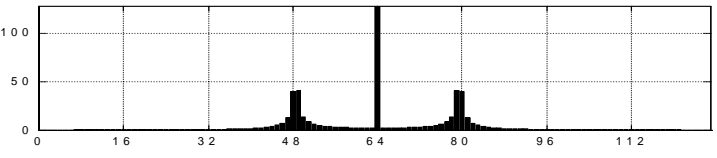
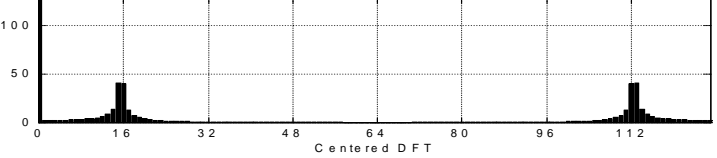
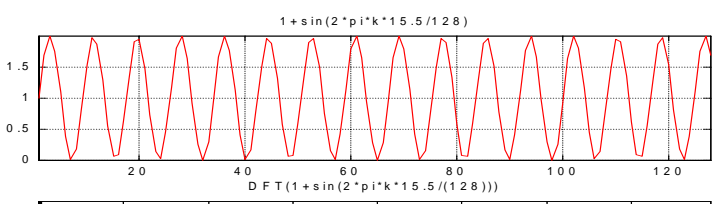
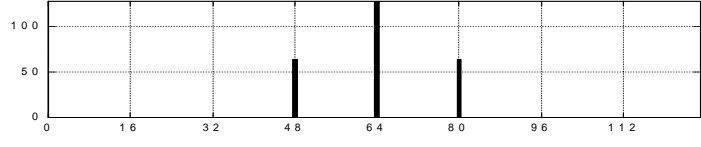
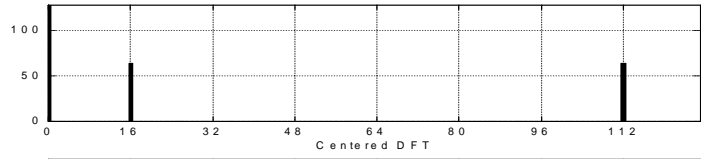
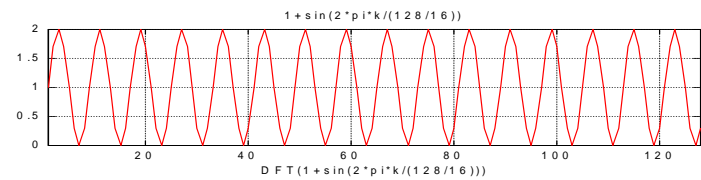
are complex conjugate to

Types of the DFT spectra symmetry for one and two-dimensional real-valued signals

Centering DFT spectrum representation



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DFTs of sinusoidal signals with integer and non-integer number of periods

**Discrete sampling theorem:
Discrete sinc-interpolation as a gold standard for convolution based discrete
signal interpolation**

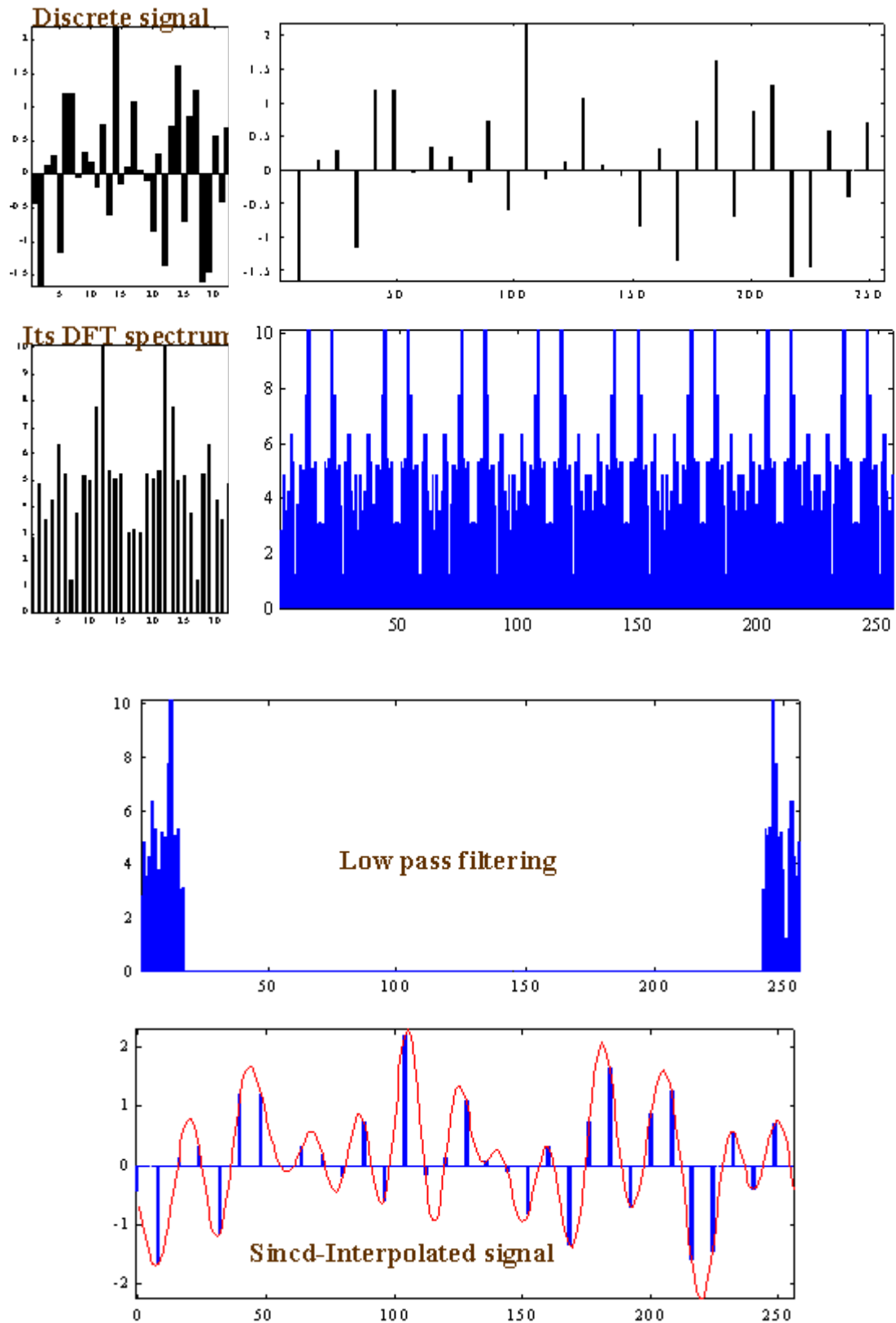
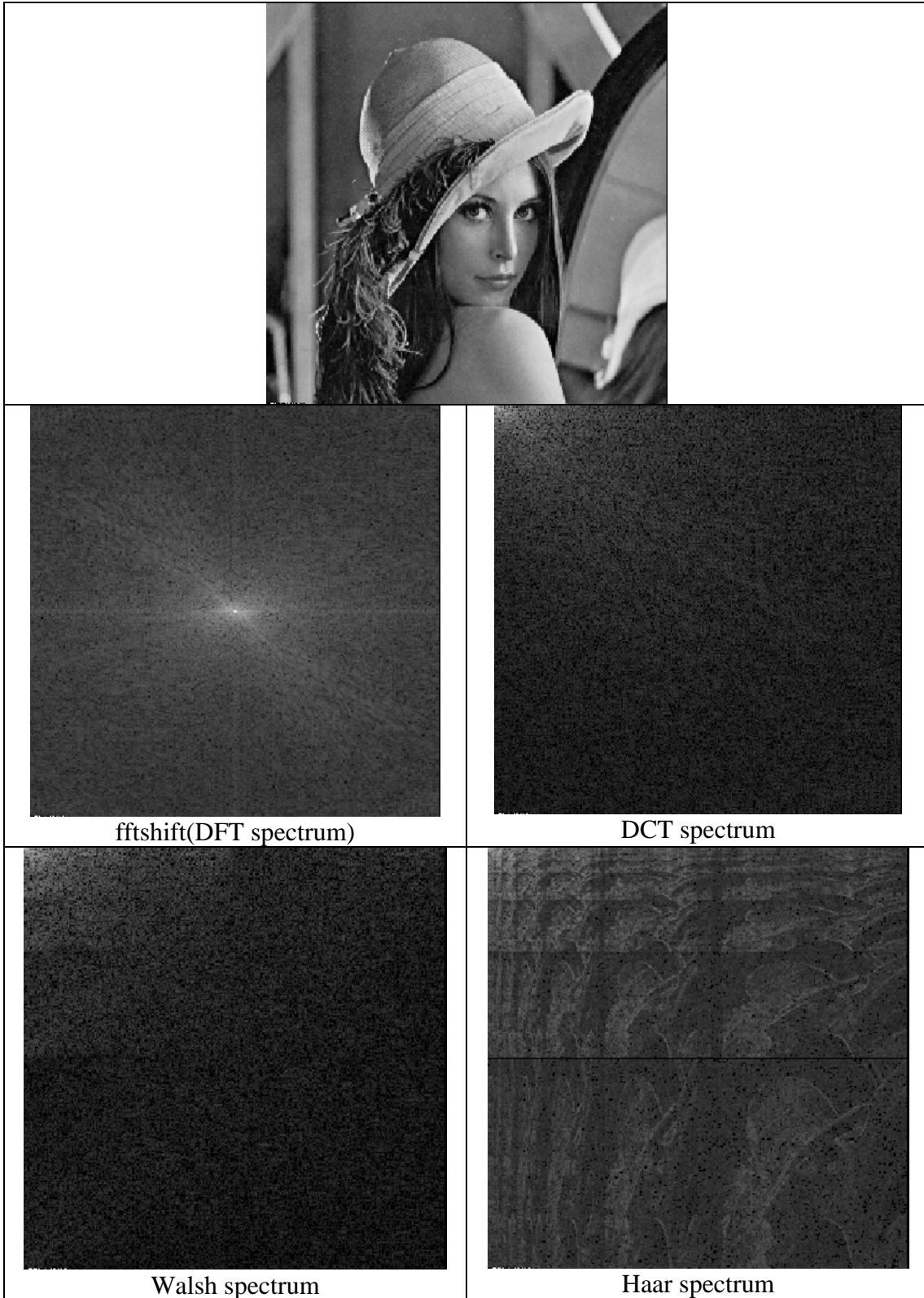

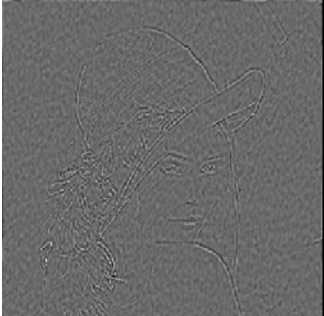



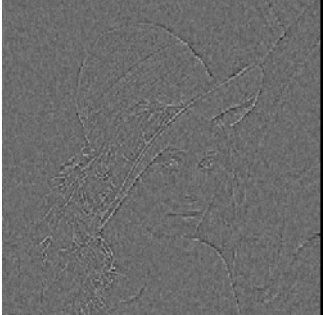




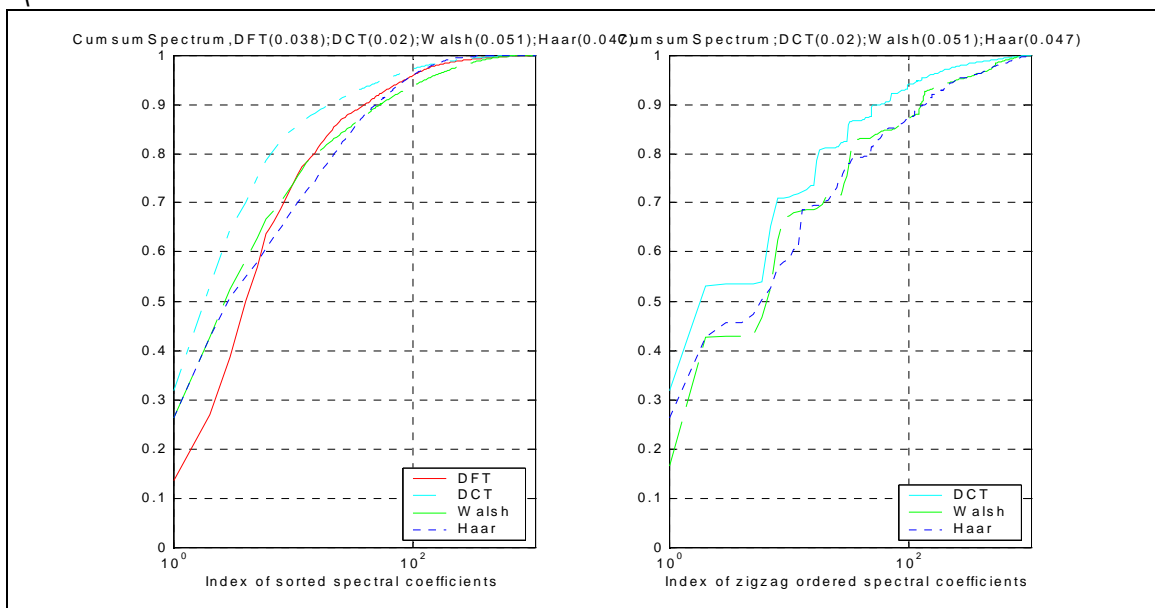
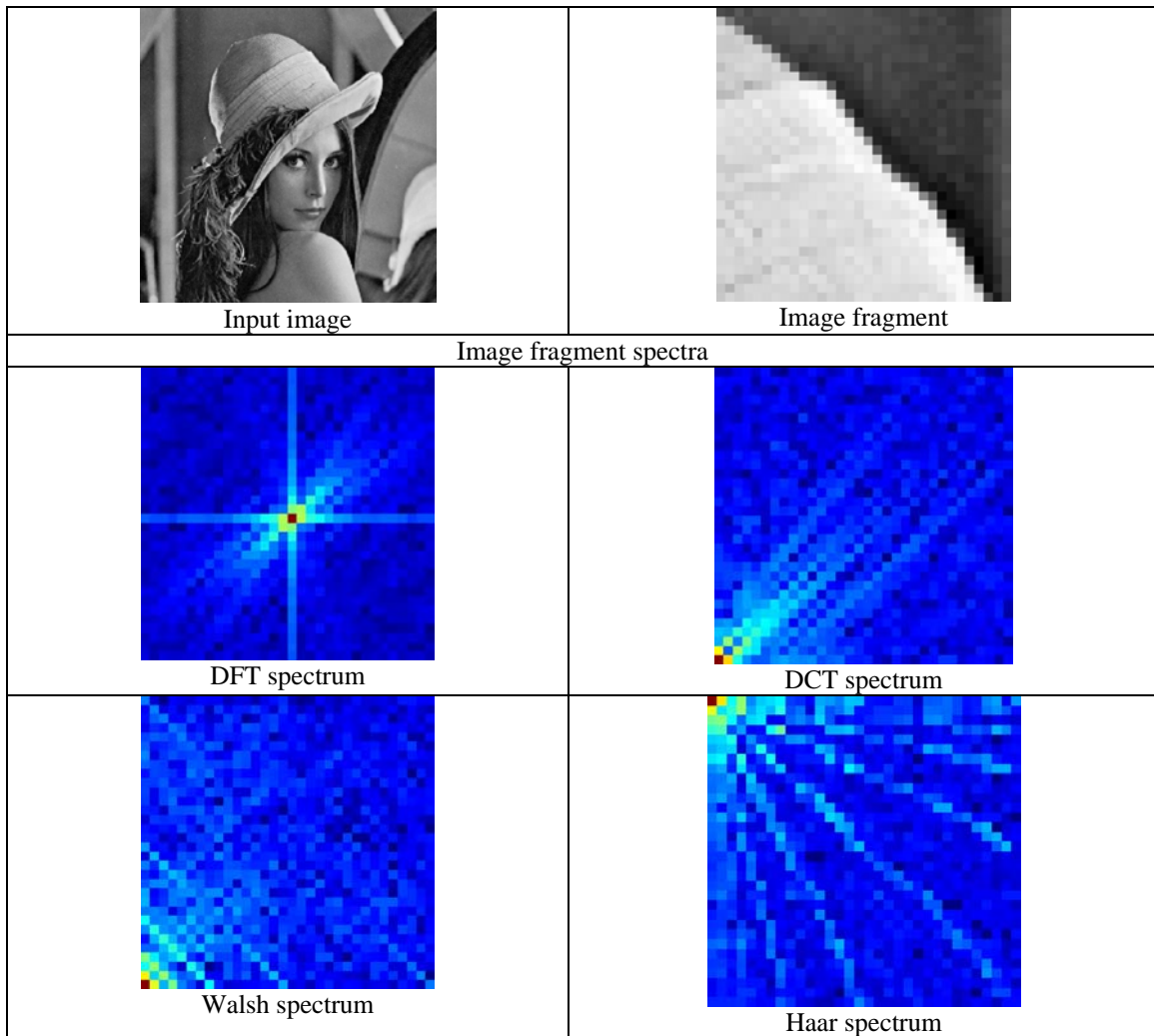
Image spectra in different bases



Comparison of image energy compaction capability of different transforms

Images after spectrum thresholding on energy level 0.01	
 DFT: Residual fraction of image spectrum 0.104	 Reconstruction error; Std_inping/std_err=31db
 DCT: Residual fraction of image spectrum 0.075	 Reconstruction error; Std_inping/std_err=31 db
 Walsh: Residual fraction of image spectrum 0.136	 Reconstruction error; Std_inping/std_err=31 db
 Haar: Residual fraction of image spectrum 0.125	 Reconstruction error; Std_inping/std_err=31 db

Comparison of image local spectra in different bases



Fraction of image fragment energy as a function of magnitude ordered (left) and zig-zag scanned (right) image fragment spectral coefficients