L. Yaroslavsky. Fundamentals of Digital Image Processing. Course 0555.3230

Lec. 8 Othogonal transforms in image processing

8.1 Discrete cosine and sine transforms

There is a number of special cases of SDFTs worth of a special discussion. All of them are associated with representation of signal and/or spectra that exhibit certain symmetry. In digital image processing the number of image samples is most commonly even. If signals are symmetrical with respect to a certain point, their corresponding discrete signal with even number of samples will retain signal symmetry if the samples are shifted with respect to the signal symmetry center by half of the discretization interval. Therefore semi-integer shift parameters of SDFT represent especial interest in this respect.

The most important special case is that of ***Discrete Cosine Transform*** (***DCT***). Let, for a signal $\{a_k\}$, $k = 0,1,..., N-1$, form an auxiliary signal

$$\tilde{a}_k = \begin{cases} a_k, & k = 0,1,..., N-1 \\ a_{2N-k-1}, & k = N,...,2N-1 \end{cases} \tag{8.1}$$

and compute its SDFT(1/2,0):

$$\mathbf{SDFT}_{1/2,0}\{\tilde{a}_k\} = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{k=N}^{2N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] \right\} =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{k=N}^{2N-1} a_{2N-k-1} \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] \right\} =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(2N-k-1/2)r}{2N}\right] \right\} =$$
$$\frac{1}{\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{k=0}^{N-1} a_k \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] \right\} =$$
$$\frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left[\pi \frac{(k+1/2)r}{N}\right]. \tag{8.2}$$

In this way we arrive at the Discrete Cosine Transform defined as:

$$\alpha_r^{DCT} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \cos\left[\pi \frac{(k+1/2)r}{N}\right]. \tag{8.3}$$

DCT signal spectrum is, as one can see from Eq.(4.3.58), always an odd (anti-symmetric) sequence if regarded outside its base interval $[0, N-1]$:

$$\alpha_r^{DCT} = -\alpha_{2N-r}^{DCT}; \alpha_N = 0 \tag{8.4}$$

while signal is, as it follows from Eq. 4.3.56, even (symmetric):

$$a_k = a_{2N-k-1}. \tag{8.5}$$

From the above derivation of the DCT it also follows that, for the DCT, signals are regarded as periodical with period $2N$:

$$a_k = a_{(k) \bmod 2N}. \tag{8.6}$$

The inverse DCT can easily be found as the inverse SDFT(1/2,0) of anti-symmetric spectrum of Eq.(8.4):

$$\tilde{a}_k = \frac{1}{\sqrt{2N}} \sum_{r=0}^{2N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] =$$

$$\frac{1}{\sqrt{2N}} \left\{ \alpha_0^{DCT} + \sum_{r=1}^{N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{r=N+1}^{2N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] = \right.$$

$$\frac{1}{\sqrt{2N}} \left\{ \alpha_0^{DCT} + \sum_{r=1}^{N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{r=1}^{N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)(2N-r)}{2N}\right] = \right.$$

$$\frac{1}{\sqrt{2N}} \left\{ \alpha_0^{DCT} + \sum_{r=1}^{N-1} \alpha_r^{DCT} \exp\left[-i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{r=1}^{N-1} \alpha_r^{DCT} \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] = \right.$$

$$\frac{1}{\sqrt{2N}} \left(\alpha_0^{DCT} + 2\sum_{r=1}^{N-1} \alpha_r^{DCT} \cos\left[\pi \frac{(k+1/2)r}{N}\right] \right) \qquad (8.7)$$

Coefficient α_0^{DCT} is, similarly to the case of DFT, proportional to the signal dc-component:

$$\alpha_0^{DCT} = \sqrt{2N} \left(\frac{1}{N} \sum_{k=0}^{N-1} a_k \right). \qquad (8.8)$$

Coefficient α_{N-1}^{DCT}

$$\alpha_{N-1}^{DCT} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k (-1)^k \sin\left(\pi \frac{k+1/2}{N} \right) \qquad (8.9)$$

represents signal's highest frequency.

The DCT can be regarded as a discrete analog of integral cosine transform. It was introduced by Ahmed and Rao as a transform for image coding. It has a very good energy compaction property and is very frequently considered to be an approximation to Karhunen-Loeve transform. From the above derivation, it becomes clear that the energy compaction property of the DCT has a very simple and fundamental explanation. It should be attributed to the fact that the DCT is a shifted DFT of a signal that is extended outside the base interval $[0, N-1]$ in a way (4.3.56) that removes the signal discontinuity at the signal border, the main cause of poor convergence property of DFT and other bases defined on a finite interval. The absence of discontinuity at signal borders makes the DCT to be attractive not only as a good transform for signal discretization but also as a substitute for DFT for the implementation of signal digital convolution in transform domain with the use of Fast Transforms.

The DCT has its complement sine transform, the ***DcST***:

$$\alpha_0^{DcST} = \frac{2}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1/2)r}{N}\right]. \qquad (8.10)$$

The DcST is an imaginary part of the SDFT(1/2,0) of a signal

$$\tilde{a}_k = \begin{cases} a_k, & k = 0,1,..., N-1 \\ -a_{2N-k-1}, & k = N, N+1,...,2N-1 \end{cases} \qquad (8.11)$$

extended to interval $[0, 2N-1]$ in an odd symmetric way:

$$\frac{1}{i\sqrt{2N}} \sum_{k=0}^{2N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] =$$

$$\frac{1}{i\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] + \sum_{k=N}^{2N-1} \tilde{a}_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] \right\} =$$

$$\frac{1}{i\sqrt{2N}} \left\{ \sum_{k=0}^{N-1} a_k \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] - \sum_{k=N}^{2N-1} a_{2N-k-1} \exp\left[i2\pi \frac{(k+1/2)r}{2N}\right] \right\} =$$

$$\frac{1}{i\sqrt{2N}}\left\{\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+1/2)r}{2N}\right]-\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(2N-k-1/2)r}{2N}\right]\right\}=$$
$$\frac{1}{i\sqrt{2N}}\left\{\sum_{k=0}^{N-1} a_k \exp\left[i2\pi\frac{(k+1/2)r}{2N}\right]-\sum_{k=0}^{N-1} a_k \exp\left[-i2\pi\frac{(k+1/2)r}{2N}\right]\right\}=$$
$$\frac{2}{\sqrt{2N}}\sum_{k=0}^{N-1} a_k \sin\left[\pi\frac{(k+1/2)r}{N}\right]. \qquad (8.12)$$

DcST spectrum exhibits even symmetry when regarded in interval $[0,2N-1]$:

$$\alpha_r^{DcST}=\alpha_{2N-r}^{DcST}. \qquad (8.13)$$

and assumes periodical replication, with a period $2N$, of the signal complemented with its odd symmetrical copy:

$$\tilde{a}_k=\tilde{a}_{(k)\mathrm{mod}\,2N}. \qquad (8.14)$$

In distinction from the DFT and the DCT, the DcST does not contain signal's dc-component:

$$\alpha_0^{DcST}=0 \qquad (8.15)$$

and the inverse DcST :

$$a_k=\frac{1}{\sqrt{2N}}\left(\alpha_N^{DcST}+2\sum_{r=1}^{N-1}\alpha_r^{DcST}\sin\left[\pi\frac{(k+1/2)r}{N}\right]\right) \qquad (8.16)$$

involves spectral coefficients with indices $\{1,2,...,N\}$ rather then $\{0,1,...,N-1\}$ for the DCT.

In a similar way one can introduce different modifications of the SDFT for different shifts and different types of signal symmetry. For instance, SDFT(1/2,1/2) of a signal extended in an anti-symmetric way (Eq. 8.11) is a modification of the DCT

$$\alpha_r^{DCTIV}=\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} a_k \cos\left[\pi\frac{(k+1/2)(r+1/2)}{N}\right] \qquad (8.17)$$

that obtained a name DCT-IV. One can easily verify that the DCT-IV spectrum also exhibits an odd symmetry:

$$\left\{\alpha_r^{DCTIV}=-\alpha_{2N-r-1}^{DCTIV}\right\}. \qquad (8.18)$$

Therefore the inverse DCT-IV is identical to the direct one

$$a_k^{DCTIV}=\sqrt{\frac{2}{N}}\sum_{r=0}^{N-1} \alpha_r \cos\left[\pi\frac{(k+1/2)(r+1/2)}{N}\right]. \qquad (8.19)$$

Because of 1/2-shift in frequency domain, the DCT-IV of a signal assumes periodical replication, with alternating sign, of the auxiliary signal of Eq. 8.11.
Similarly to the DCT-IV, one can define the DST-IV:

$$\alpha_r^{DSTIV}=\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} a_k \sin\left[\pi\frac{(k+1/2)(r+1/2)}{N}\right] \qquad (8.20)$$

that is obviously an imaginary part of the SDFT(1/2,1/2) of a symmetrized auxiliary signal of Eq. 8.11.
In the same way one can show that the SDFT(1,1) of an auxiliary signal $\{\tilde{a}_k\}$ formed from a signal $\{a_k, k=0,1,...,N-1\}$ by extending it in an odd way to $2N+2$ samples:

$$\tilde{a}_k=\begin{cases} a_k, & 0\le k\le N-1 \\ 0, & k=N \\ -a_{2N-k}, & N\le k\le 2N \\ 0, & k=2N+1 \end{cases} \qquad (8.21)$$

reduces to the transform

$$\alpha_r^{DST} = \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} a_k \sin\left[\pi \frac{(k+1)(r+1)}{N}\right] \qquad (8.22)$$

known as ***Discrete Sine Transform*** (***DST***). DST spectrum is also and in the same way anti-symmetric:

$$\left\{\alpha_r^{DST} = -\alpha_{2N-r}^{DST}\right\};\ \alpha_N^{DST} = \alpha_{2N+1}^{DST} = 0 \qquad (8.23)$$

and, therefore, inverse DST is also identical to the direct one:

$$a_k = \frac{1}{\sqrt{2N}} \sum_{r=0}^{N-1} \alpha_r^{DST} \sin\left[\pi \frac{(k+1)(r+1)}{N}\right]. \qquad (8.24)$$

DST of a signal assumes periodical replication, with a period $2N+2$ of the auxiliary signal of Eq. 8.21.

2-D and multi-dimensional DCTs and DSTs are usually defined as separable to 1-D transforms on each of the coordinates. For instance, 2-D DCT of a signal $\{a_{k,l}\}$, $k = 0,1,..., N_1 - 1$, $l = 0,1,..., N_2 - 1$ is defined as

$$\alpha_{r,s}^{DCT} = \frac{2}{\sqrt{N_1 N_2}} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} a_{k,l} \cos\left[\pi \frac{(k+1/2)r}{N_1}\right] \cos\left[\pi \frac{(l+1/2)s}{N_2}\right] \qquad (8.22)$$

that corresponds to 4-fold image symmetry illustrated in Fig.8.1.

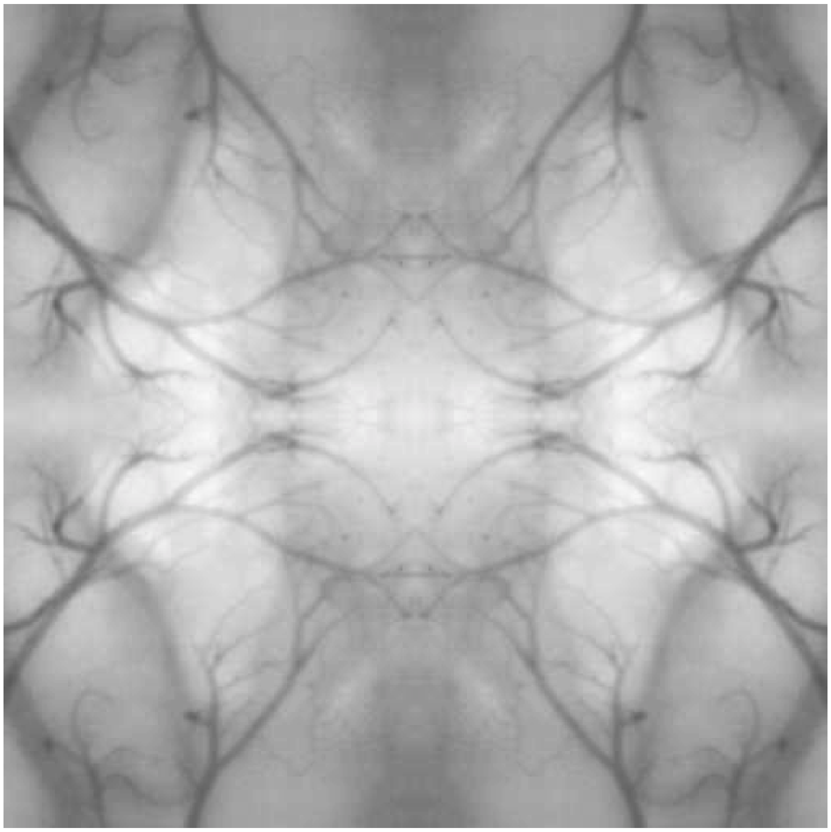


Fig. 8.1 4-fold Image symmetry

8. 2 Walsh functions

For basis of exponential functions $\{\exp(i2\pi kx/X)\}$, generating basis functions by scaling their argument can be treated also as generating by means of multiplying mother function:

$$\exp(i2\pi kx/X) = \prod_{l=1}^{k} \exp(i2\pi x/X) \qquad (8.23)$$

There exists yet another family of orthogonal functions built with the same principle, ***Walsh functions***. Walsh functions are distinguished by the fact that they can assume only two values; either 1, or -1. They are generated by multiplication of clipped sinusoidal functions called the ***Rademacher functions***

$$rad_k(x) = sign\left(\sin\left(2^k \pi x/X\right)\right) \qquad (824)$$

Any two Rademacher functions are mutually orthogonal. However, the system of functions $\{rad_k(x)\}$ is incomplete in the same sense as families of functions $\{\cos(2\pi kx/X)\}$and $\{\sin(2\pi kx/X)\}$ are incomplete. The Walsh functions are actually an extension of Rademacher functions to a complete system. These are defined as

$$wal_k(x) = \prod_{m=0}^{\infty} \left(rad_{m+1}(x)\right)^{k_m^{GC}} , \qquad (8.25)$$

where k_m^{GC} is the ***m***-th digit of the so-called Gray code of number ***k***. Gray code digits are generated from digits $\{k_m\}$ of binary representation of the number

$$k = \sum_{m=0}^{\infty} k_m 2^m; \quad k_m = 0,1 \qquad (8.26)$$

according to the following rule:

$$k_m^{GC} = k_m \oplus k_{m+1} , \qquad (8.27)$$

where $\oplus$ stands for modulo 2 addition.

Formula (8.25) reveals the origin of the Walsh functions. However, for the purpose of calculating their values another representation of the Walsh functions::

$$wal_k(\xi) = \prod_{m=0}^{\infty} \left[(-1)^{\xi_{m+1}}\right]^{k_m^{GC}} = (-1)^{\sum_{m=0}^{\infty} k_m^{GC} \xi_{m+1}} \qquad (8.28)$$

is useful, where

$$\xi = x/X = \sum_{m=0}^{\infty} \xi_m 2^{-m}; \;\; \xi_0 = 0; \qquad \xi_{m>0} = 0,1 \qquad (8.29)$$

The Walsh functions are orthogonal on the interval $[0, X]$. Because of their origin as a product of Rademacher functions, multiplication of two Walsh functions results in another Walsh function with shifted an index or argument. The shift, called ***dyadic shift***, is determined through the bit-by-bit modulo 2 addition of functions' indices or, respectively, arguments:

$$wal_k(\xi) wal_l(\xi) = wal_{k \oplus l}(\xi); \qquad (8.30)$$

$$wal_k(\xi) wal_k(\zeta) = wal_k(\xi \oplus \zeta) \qquad (8.31)$$

One can regard $\sum_{m=0}^{\infty} k_m^{GC} \xi_{m+1}$ in the definition of Walsh as a scalar product of vectors $\{k_m^{GC}\}$ and $\{\xi_{m+1}\}$. In this sense one can treat basis of Walsh functions as a scale basis. Multiplicative nature of Walsh functions and exponential ones justifies yet another name for these families of bases, multiplicative bases.

Walsh function are akin to complex exponential functions $\{\exp(i2\pi kx/X)\}$ in one more respect. This can be seen if (-1) in Eq. (8.28) is replaced by $\exp(i\pi)$:

$$wal_k(\xi) == \exp\left(i\pi \sum_{m=0}^{\infty} k_m^{GC} \xi_{m+1}\right) \qquad (8.32)$$

and from comparison of graphs of sinusoidal and Walsh functions shown in figure 8.2

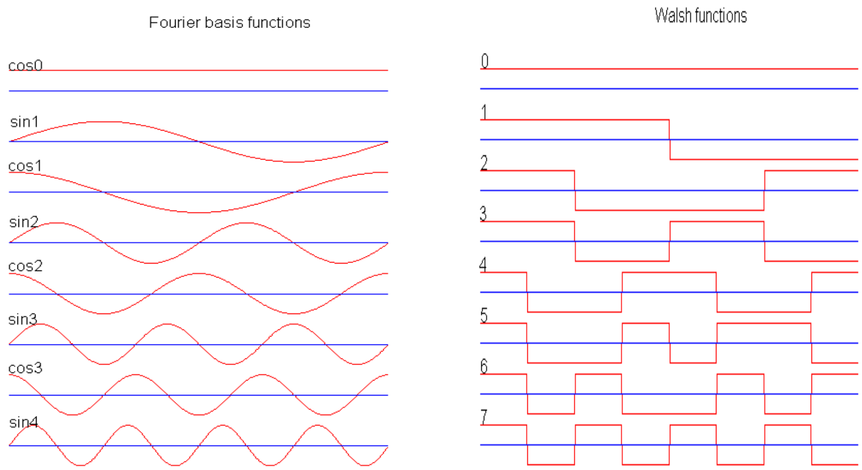


Figure 8.2. Sinusoidal and Walsh functions

As one can see from the figure, an important parameter that unites both families of functions is the number of zero crossing which coincides, for these functions, with their index k . As it was mentioned above, for sinusoidal functions, this parameter has an association with the notion of frequency. For Walsh functions, this parameter obtained name of "***sequency***".

Sequency wise ordering of Walsh functions is not accidental. In principle, ordering basis functions can be arbitrary. The only condition one should obey is matching between signal representation coefficients $\{\alpha_k\}$ and discretization and reconstruction basis functions $\{\varphi_k^{(d)}(x)\}$ and $\{\varphi_k^{(r)}(x)\}$. However almost always natural ways of natural ordering exist. For instance, for shift (convolution) bases the natural ordering is according to successive coordinate shifts. Sinusoidal basis functions are naturally ordered in frequency of signal Fourier transform. This why sequence (number of zero crossing) wise ordering of Walsh functions can also be regarded natural.

However, the advantage of such an ordering is of more fundamental nature. The most natural requirement to ordering basis functions for signal representation is the ordering for which signal approximation error for finite number N of terms in signal expansion over the basis monotonically decreases with the growth of N . For Walsh functions, it happens that, for conventional analog signals, sequency wise ordering satisfies this requirement much better than other methods of ordering. Figure 3-5 illustrates this feature of Walsh function signal spectra. Graphs on the figure show averaged squared Walsh spectral coefficients of image rows as functions of coefficients' indices for sequence wise (Walsh) ordering (left) and for Hadamard ordering (right). In Hadamard ordering, digits $\{k_m\}$ of binary representation of index k (8.26) rather than digits $\{k_m^{GC}\}$ are used for generating Walsh functions:

$$walhad_k(x) = (-1)^{\sum_{m=0}^{\infty} k_m \xi_{m+1}} \qquad (8.33)$$

In conclusion note that signal expansion over Walsh function basis approximates signals with piecewise functions just as it is the case for rectangle impulse basis functions.

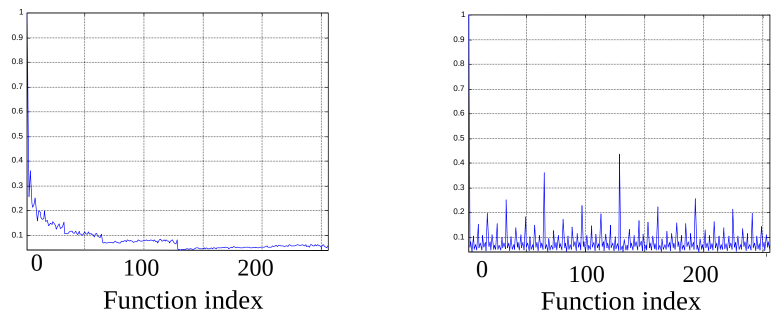


Figure 8.3. Comparison of image Walsh function spectra in Walsh (left) and Hadamard (right) ordering

8.3 Discrete Wavelet Transforms

A distinctive feature of shift (convolution) bases is that, for them, signal representation coefficients depend on signal values in the vicinity of the corresponding sampling point, and therefore they carry local information about signals. In contrast to them, signal discrete representation for scale (multiplicative) bases is “global”: signal representation coefficients depend on the entire signal. Sometimes it is useful to have a combination of these two features in the signal discrete representation. This is achieved with wavelet basis functions built using a combination of shift and scaling of a mother function. A most immediate example of such a combination are ***Haar functions***. Haar functions are generated from from shift basis of rectangle impulse functions $\{rect(x - k\Delta x/\Delta x)\}$ and Rademacher functions $\{sign(\sin(2^k \pi x / X))\}$ in the following way:

$$har_k(x) = 2^{\hat{m}} rad_{\hat{m}+1}(x) rect\left(2^{\hat{m}} \frac{x}{X} - [k] \bmod 2^{\hat{m}}\right), \tag{8.34}$$

where $\hat{m}$ is index of the most significant non-zero digit (bit) in binary representation of k (Eq. (8.26)) and $[k] \bmod 2^{\hat{m}}$ is residual from division of k by $2^{\hat{m}}$.

The Haar functions are orthonormal on the interval $[0, X]$. Graphs of first eight Haar functions are shown in Fig. 8.4. The figure clearly reveals the way in which Haar functions are built. One can see from the figure, that Haar functions form groups in scale and that functions within each group are generated by shift in coordinates with shift interval coordinated with the scale. One can also see that signals reconstructed from their discrete representation over Haar basis functions belong to the same family of piece-wise constant functions as signals reconstructed in the base of rectangle impulse basis functions.

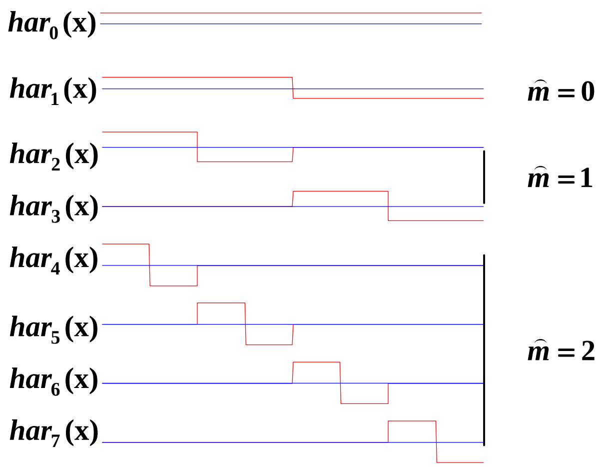


Figure 8.4. First eight Haar functions

Numerical values of Haar functions may be found at each point by expressing them, as in the case of Walsh functions, via a binary code argument representations (3.2.35 and 38):

$$har_k(x) = 2^{\hat{m}} (-1)^{\xi_{\hat{m}+1}} \delta\left([\xi]_{\hat{m}} - (k) \bmod 2^{\hat{m}}\right). \qquad (8.35)$$

At present time, numerous other wavelet bases are known.

8.4 Discrete Orthogonal Transforms: summary

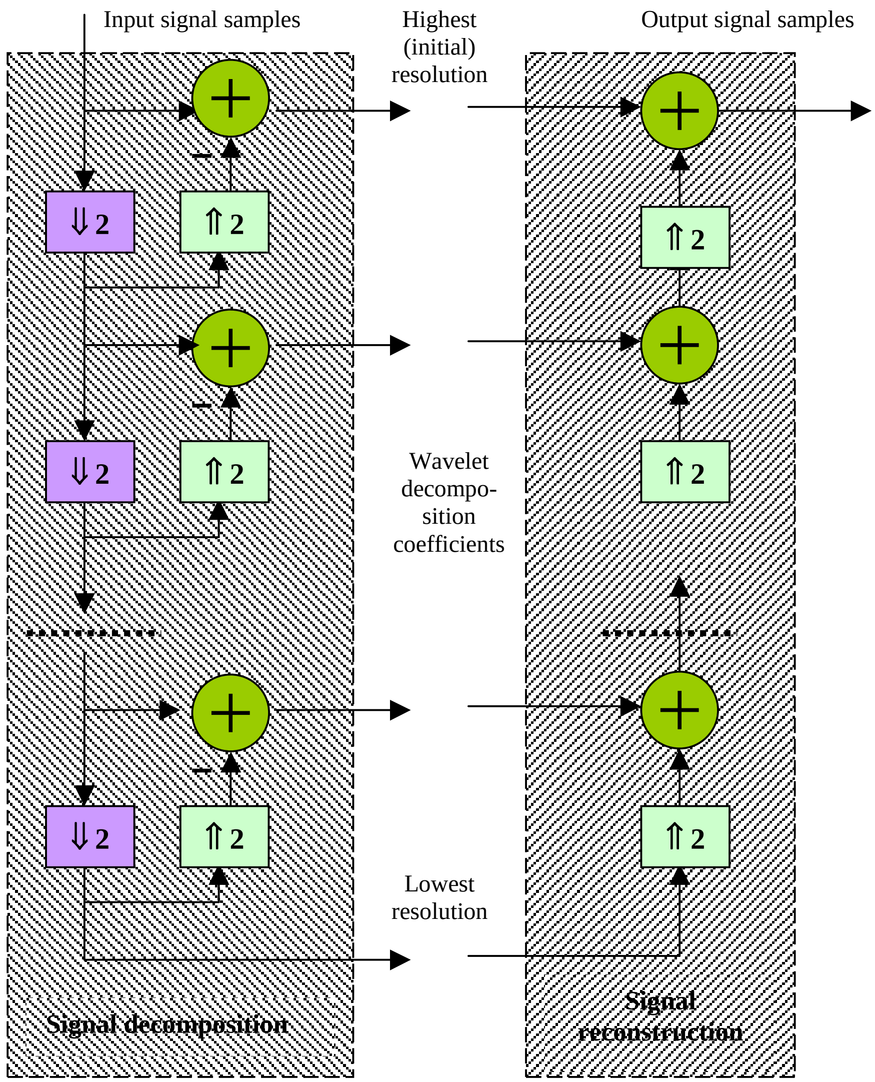
Basis vectors that form matrices of orthogonal transforms are most usually derived from continuous basis functions. Main requirements to discrete orthogonal transforms are basically similar to those to continuous bases. These are good ***energy compaction property*** (low MSE of signal reconstruction from pruned set of transform coefficients, see Sect. 3.2.5) and low computational complexity. They were the driving motives in the developments of fast orthogonal transforms in 1970-1980-th when most of the transforms that are now in use were invented. In Table 8.2 most frequently used orthogonal transforms are listed and their definitions is provided. Computational complexity of transforms listed in the table, except that of Haar Transform, is of the order of magnitude $O(N \log N)$, where N is the transform dimensionality. Computational complexity of Haar Transform is even lower: $O(N)$.

More recently, discrete wavelet transforms gained considerable popularity because of their relatively good energy compaction properties, low computational complexity (similar to that of Haar Transform that is the simplest binary example of wavelets) and their multi resolution property (as it was mentioned, wavelet signal decomposition is hybrid, local and global).

There are many ways to introduce and define wavelet transform as well as there is unlimited number of modifications of wavelet bases. Fig. 8.5 represents one of the most simple and straightforward method, the algorithmic one. As it is shown in the figure, signal wavelet decomposition is made in several steps using two basic signal processing procedures: decimation and interpolation. As a result, signal of N samples is decomposed into $\log_2 N$ sub-signals with descending resolution (number of samples) each of which represents different sub-bands of the signal, from high to low frequencies with the lowest level representing signal dc component The way how low pass filtering for decimation and how signal interpolation from its decimated copy are implemented defines type of the wavelets. This method of signal representation is known also by names multiresolution, sub-band representation and pyramidal coding.

Table 8.1. Discrete orthogonal transforms

Transform	Transform matrix $N \times N$
Discrete Fourier Transforms (DFT)	Direct: $\mathbf{DFT} = \left\{ \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(i2\pi \frac{kr}{N} \right) \right\}$
	Inverse: $\mathbf{IDFT} = \left\{ \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \exp\left(-i2\pi \frac{kr}{N} \right) \right\}$
Discrete cosine Transforms (DCT)	Direct: $\mathbf{DCT} = \left\{ \frac{2}{\sqrt{2N}} \cos\left(\pi \frac{(k+1/2)r}{N} \right) \right\}$
	Inverse: $\mathbf{IDCT} = \left\{ \frac{2}{(1+\delta(r))\sqrt{2N}} \cos\left(\pi \frac{(k+1/2)r}{N} \right) \right\}$
DCT-IV	$\mathbf{DCTIV} = \left\{ \frac{\sqrt{2}}{\sqrt{N}} \cos\left(\pi \frac{(k+1/2)(r+1/2)}{N} \right) \right\}$ $\mathbf{IDCTIV} = \mathbf{DCTIV}$
Discrete Sine Transform (DST)	$\mathbf{DST} = \left\{ \frac{1}{\sqrt{2N}} \sin\left(2\pi \frac{(k+1)(r+1)}{2N+1} \right) \right\}$ $\mathbf{IDST} = \mathbf{DST}$
DST-IV	$\mathbf{DSTIV} = \left\{ \frac{1}{\sqrt{2N}} \sin\left(2\pi \frac{(k+1/2)(r+1/2)}{2N} \right) \right\}$ $\mathbf{IDSTIV} = \mathbf{DSTIV}$
Walsh Transform	$\mathbf{WAL} = \left\{ \frac{1}{\sqrt{N}} (-1)^{\sum_{m=0}^{n-1} k_{n-m-1}^{GC} r_m} \right\}$; $n = \log_2 N$; $\{r_m\}$ are digits of binary representation of row index r ; $\{k_{n-m-1}^{GC}\}$ are digits of Gray code of bit reversed digits $\{k_{n-m-1}\}$ of binary representation $\{k_m\}$ of column index; $m = 0,1,...,n-1$; $\{k_s^{GC} = k_s \oplus k_{s+1}\}$; $\oplus$ - modulo 2 binary addition.
Haar Transform	$\mathbf{HAAR} = \left\{ \frac{2^{\hat{m}/2}}{\sqrt{N}} (-1)^{k_{n-\hat{m}-1}} \prod_{m=0}^{\hat{m}-1} \delta(k_{n-m} \oplus r_m) \right\}$; $n = \log_2 N$; $\hat{m}$ - index of the most significant nonzero digit in binary representation of row index r ;



⇓ 2 — Unit that performs signal low pass filtering and 2-fold decimation of signal samples

⇑ 2 — Unit that performs signal 2-fold zooming and interpolation

Figure 8.5. Schematic diagram of discrete wavelet (multi resolution) signal decomposition and reconstruction

Questions for self-testing

1. Derive DCT as a version of SDFT(1/2,0)
2. Review symmetry properies of DCT
3. List and explain modifications of DCT
4. List and explain modifications of DST
5. Explain principle of building basis functions of Walsh Transform
6. Explain the meaning and the role of basis function ordering
7. Explain principle of building basis functions of Haar Transform

Exercise

1. Compare basis functions of DFT, DCT, Walsh and Haar transfor
2. Compare energy compaction properties of DFT, DCT, Walsh and Haar transforms for different images and image pragments

Home work: Compute numerically and compare basis functions of DFT, DCT, Walsh and Hadamard Transforms for Signal length 4 and 8 samples.