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FROM PHOTOGRAPHY TO *.GRAPHIES: UNCONVENTIONAL IMAGING TECHNIQUES

A short course at Tampere University of Technology,
Tampere, Finland, Sept. 3 – Sept. 14, 2001

Lecture 8.

METHODS AND MEANS FOR RECORDING COMPUTER GENERATED HOLOGRAMS

Lecture 8.
METHODS AND MEANS FOR RECORDING SYNTHESIZED HOLOGRAMS

Classification of recording media and methods

The samples of Fourier and Fresnel form, so to say, mathematical holograms; that is, arrays of, generally, complex numbers defining the phase and amplitude of the synthesized wave front. In order to transform the mathematical hologram into a physical one capable of generating the required wave front, these numbers need to be transformed into parameters of optical media modulating the amplitude and phase of light wave used for hologram reconstruction.

The existing optical media may be classified into three categories: amplitude media, phase media, and combined or amplitude-phase media.

In amplitude media, the controlled parameter is the light intensity transmission coefficient. This is the most common and available class, whose typical representatives are the standard halogen-silver emulsions used in photography and optical holography.

In phase media, light intensity transmission is not controllable, but the optical thickness of such media can be controllable, for example, by varying their refractive index, or physical thickness, or both. These include thermoplastic materials, photo resists, bleached photographic materials, media based on bichromized gelatin, photo polymers, etc.

Combined media allow independent control of the light intensity transmission factor and optical thickness. Currently, these are photographic materials with two or more layers sensitive to radiation of different wavelengths. This permits the user to control of the transparency of certain layers and the optical thickness of others by exposing each layer to its wavelength independently/

Special digitally controlled hologram recording devices are required for the control of optical parameters of these media based on computations of the wave field. Today, no such a special purpose device exists, and different displays designed for the output of characters, graphs, and grey-scale pictures are used instead.

The distinctive feature of alphanumeric and graphical displays is that they perform only binary or two level modulation of the medium's optical parameters. Therefore, amplitude and phase media employed in these devices will be referred to as binary; their controlled optical parameters may assume only two values. The use of amplitude and phase media in the binary mode is ineffective in terms of information capacity because the possibility of writing information on them is defined only by their spatial degrees of freedom (their resolution power), whereas in amplitude and phase media, in principle, the degrees of freedom related to a transmission (reflection, refraction) factor may be used as well. The major advantage of the binary mode is simplicity for media exposure and copying, and the possibility of using the most widespread printers and graphic displays.

The disadvantages of the binary mode may be overcome by using grey-scale displays for recording on amplitude and phase media. For recording holograms on combined media, color display devices may be used. In these devices, medium optical parameters are most commonly controlled in these devices by an intensity-modulated beam of light or electrons affecting separate elementary areas (resolution cells) and writing (successively or simultaneously into several cells) appropriately coded samples of the mathematical hologram.

The most important characteristics of recorders are their sampling step; that is, the distance $\Delta\xi$ and $\Delta\eta$ between neighboring, separately exposed, resolution cells, and the total number of cells which may be exposed. The sampling step defines the angular dimensions of the reconstructed image.

The existing methods for recording synthesized holograms on amplitude, phase, binary, and combined media may be classified with respect to different features. Here, the representation methods of complex numbers describing samples of the mathematical hologram is chosen as the classification key. (Fig. 1).

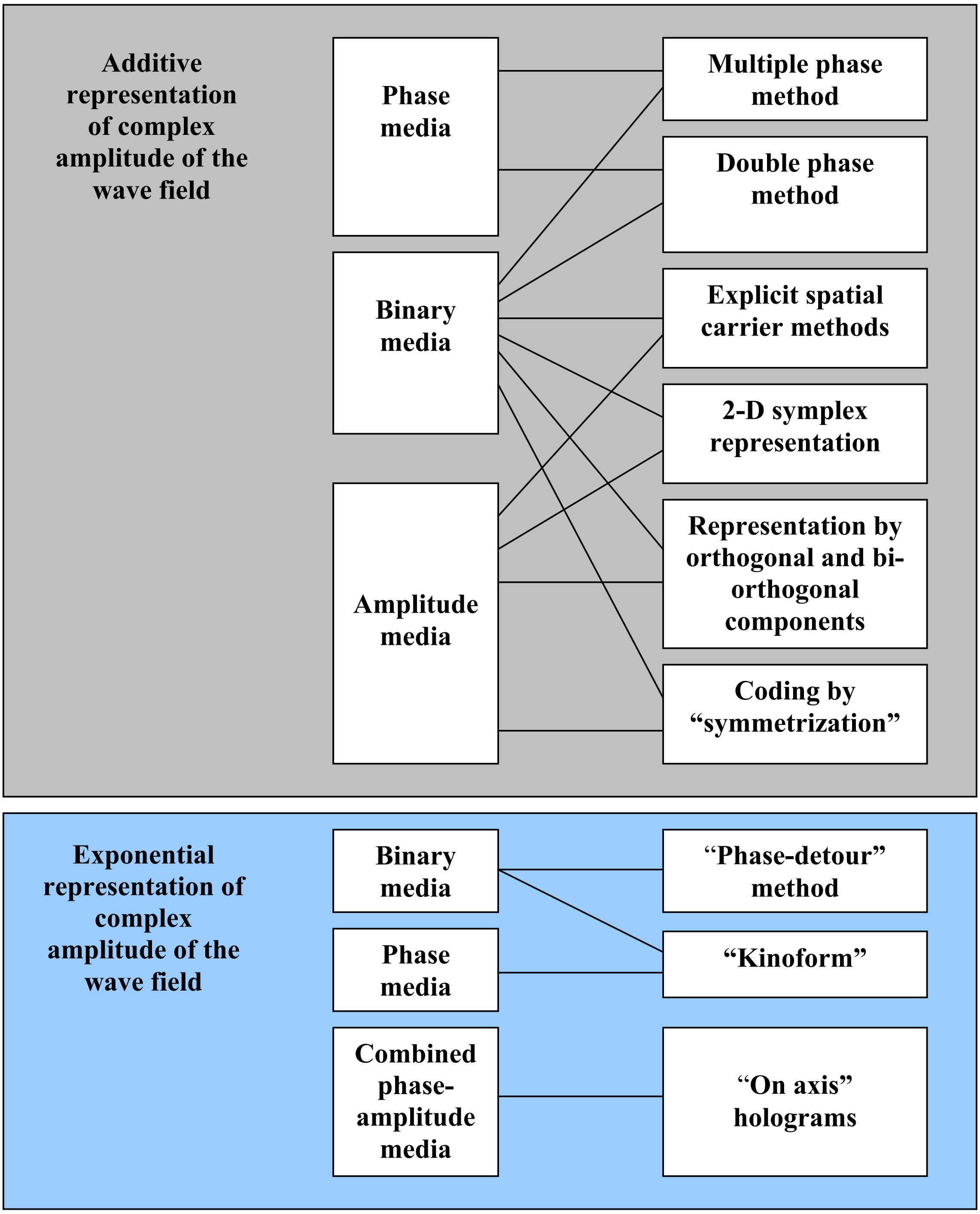


Fig. 1 Classification of methods for recording computer generated holograms

Complex numbers may be represented in two ways: exponentially or additively (Fig.1). From the physical standpoint, the most natural representation of complex numbers seems to be the exponential form ***Aexp (iφ),*** where ***A*** and ***φ*** are, respectively, the amplitude and phase of the numbers. The combined media are most suitable for this representation.

Symmetrization method

The *symmetrization method* requires that, prior to hologram synthesis, the object be symmetrized, so that its Fourier hologram contains only real valued samples and, thus, may be recorded on purely amplitude media. As real numbers may be both positive and negative, holograms should be recorded in the amplitude medium with constant positive bias, making all the recorded values positive.

The philosophy of this method is based on the well known property of the integral Fourier transform which may be formally written as follows: if object amplitude exhibits symmetry

$$\overline{A}_0(x,y) = \overline{A}_0^*(x,y) = \overline{A}_0(-x,y), \tag{1a}$$

then its Fourier hologram is real valued and also symmetrical

$$\Gamma_F(\xi,\eta) = \Gamma_F^*(\xi,\eta) = \Gamma_F(-\xi,\eta) \tag{1b}$$

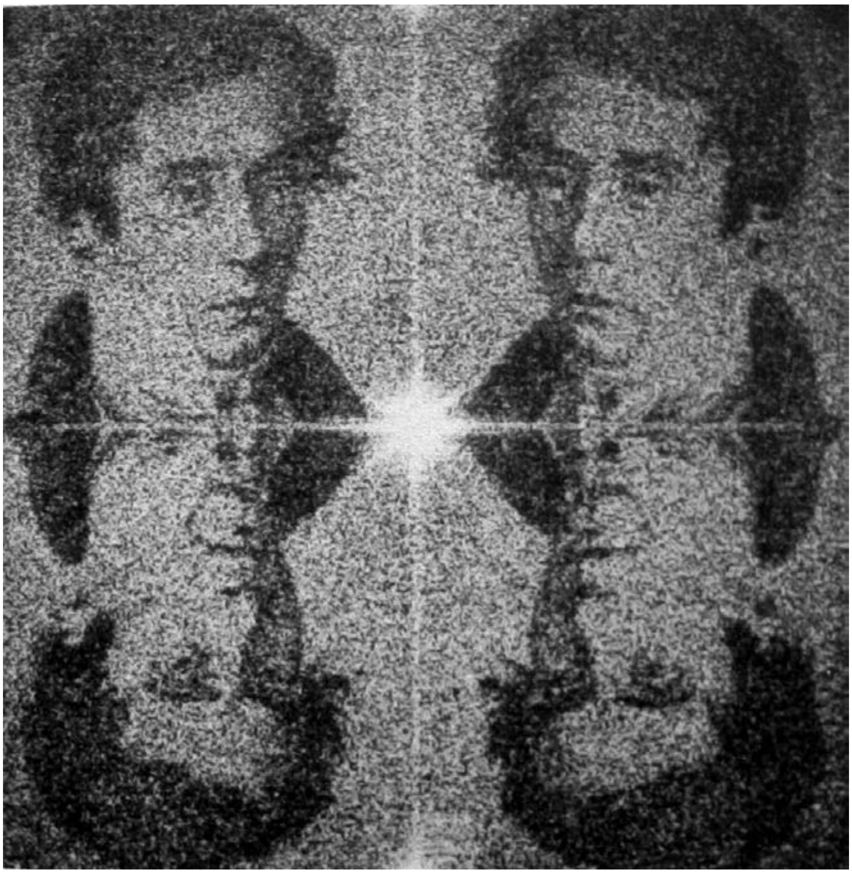
where * stands for the complex conjugate. For SDFT as discrete representation of the integral Fourier transform, a similar property requires symmetrization through a rule depending on the shift parameters ***u, v, p***, and ***q***. For example, with integer 2***u*** and 2***v*** and **3*p* =3*q*=0**, this rule becomes

$$A_{sym}(k,l) = \begin{cases} \overline{A}_0(k,l), & 0 \le k \le N-1; 0 \le l \le M-1 \\ \overline{A}_0(2N-1-k,l), & N \le k \le 2N-1; 0 \le l \le M-1 \end{cases} \tag{2}$$

which implies symmetrization by object duplication. In doing so, the number of samples of the object and, correspondingly, its Fourier hologram, is twice that of the original object. It is this two-fold redundancy that enables one to avoid recording the phase component. Symmetrization by quadruplication is possible as well. This consists in symmetrizing the object according to the rule of Eq.(2) with respect to both indices ***k*** and ***l***. Hologram redundancy becomes four-fold. The hologram of symmetrized objects are also symmetrical and reconstruct duplicated or quadruplicated objects depending on the particular symmetrization (Fig.2). Symmetrization exemplifies optimal coding of the information regarding an object necessary for matching a hologram with the properties of the recording medium.

Notably, duplication and quadruplication do not imply a corresponding increase in computer time for execution of SDFT at the hologram calculation step because, for computation of SDFT, one may use combined algorithms making use of signal redundancy for accelerating computations.

a)



b)

Fig. 2 Images reconstructed from synthesized holograms under duplicated symmetrization (a) and under quadruplicated symmetrization (b).

Binary holograms and "phase detour" coding

Historically, one of the first methods for recording synthesized holograms was that proposed for binary holograms by A.Lohmann and his collaborators. In this method, an elementary cell of a binary medium is allocated for reproducing the amplitude and phase of each sample of a mathematical hologram, the modulus of the complex number being represented by the size of the opening (aperture) in the cell and the phase - by the position of the opening within the cell. All the cells corresponding to mathematical hologram samples are arranged over a regular (usually,rectangular) raster. A shift of aperture by $\Delta\xi$ in a given cell with respect to its raster node corresponds to a phase detour for this cell equal to $(2\pi\Delta\xi\cos\theta_{\xi}/\lambda$ for hologram reconstruction at an angle θ_{ξ} to the system's optical axis perpendicular to the hologram plane (Fig. 3). The use of the aperture spatial shift of the representing complex number phase was named the "*detour phase*" method.

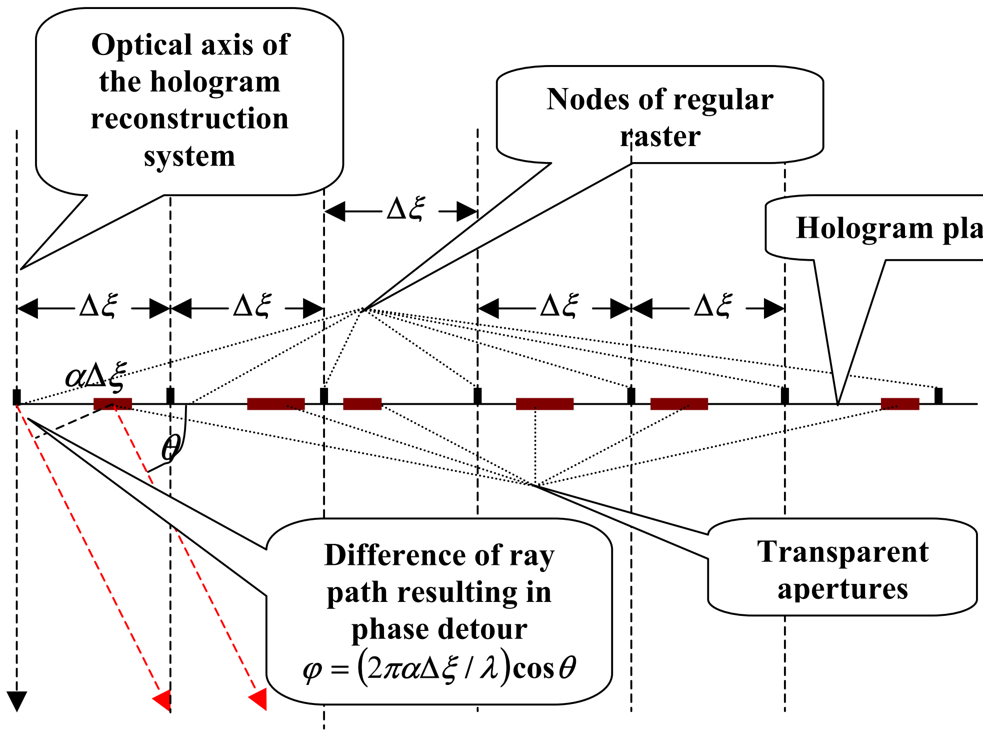
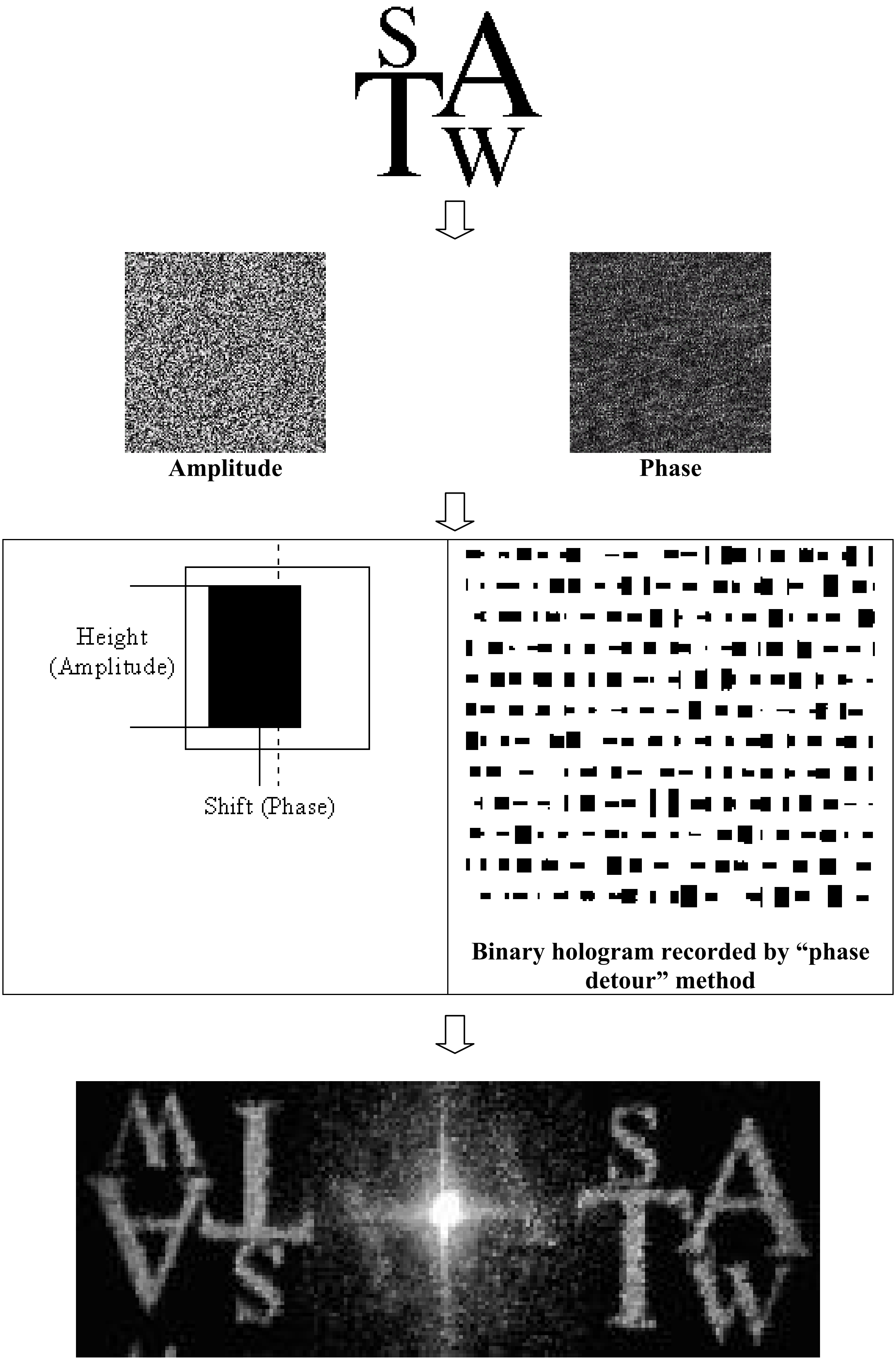


Fig. 3. Phase coding by spatial displacement of the transparent aperture

As was already noted, only spatial degrees of freedom are used for recording in phase media; therefore, the number of binary medium cells should exceed the number of hologram samples by a factor equal to the product of the amplitude and phase quantization levels. This product may run into several tens or even hundreds. The low effectiveness of using the degrees of freedom of the hologram carrier is the major drawback of the binary hologram method. But its merits, such as simpler recording technology, photo chemical processing and copying of synthesized holograms, and the possibility of using widespread computer plotters, accounts for its popularity. Several modifications of the method are known today, oriented to different types of plotters and different ways of taking into account boundary effects that occur when the aperture crosses the elementary cell boundaries.



Hologram synthesis and recording on binary media by “phase detour” method

Amplitude media: orthogonal and bi-orthogonal coding

With an additive representation, the complex number (regarded as a vector in the complex plane) is the sum of several components. For recording in amplitude media these components should be assigned standard directions in a complex plane (phase angle) and controllable length (amplitude). The simplest case is the representation of a vector $\boldsymbol{\Gamma}$ by its orthogonal components, say, real $\boldsymbol{\Gamma_{re}}$ and imaginary $\boldsymbol{\Gamma_{im}}$ parts (Fig. 4):

$$\boldsymbol{\Gamma = \Gamma_{re} e_{re} + \Gamma_{im} e_{im}} \tag{3}$$

where $\boldsymbol{e_{re}}$ and $\boldsymbol{e_{im}}$ are orthogonal unit vectors. When recording a hologram, the phase angle between the orthogonal components may be coded by the detour phase method, with $\boldsymbol{\Gamma_{re}}$ and $\boldsymbol{\Gamma_{im}}$ being recorded into hologram resolution cells neighbouring in a raster rows as shown in Fig.5, a. In doing so, the reconstructed image will be observed at an angle $\boldsymbol{\theta_\xi}$ to the axis $\boldsymbol{\xi}$ defined as follows:

$$\boldsymbol{\Delta\xi \cos\theta_\xi = \lambda/4}\,, \tag{4}$$

where $\boldsymbol{\lambda}$ is the light wavelength used for hologram reconstruction. For recording negative values of $\boldsymbol{\Gamma_{re}}$ and $\boldsymbol{\Gamma_{im}}$ a constant positive bias may be added to recorded values.

Since two resolution elements are used here for recording one hologram sample, such a hologram has double redundancy, as in symmetrization with duplication. With such coding and recording of a hologram, one must take into account that the optical path difference $\boldsymbol{\lambda}/2$ and $3\boldsymbol{\lambda}/4$ will correspond to the next pair of hologram resolution cells (see Fig.5, a); that is, values of $\boldsymbol{\Gamma_{re}}$ and $\boldsymbol{\Gamma_{im}}$ for each odd sample of the mathematical hologram should be recorded with opposite signs.

This coding technique may formally be described as follows. Let $\boldsymbol{(r,s)}$ be indices characterizing the number of the mathematical hologram sample $\boldsymbol{\Gamma(r,s)}$, $\boldsymbol{r} = \boldsymbol{0,1,\ldots, M\text{-}1}$; let $\boldsymbol{m,n}$ be indices of the number of the medium resolution cell; and let $\boldsymbol{\tilde{\Gamma}(m,n)}$ be the hologram coded for recording. The the coded hologram is written as

$$\boldsymbol{\tilde{\Gamma}(m,n) = (-1)^r i^{m_0} \left[(-1)^{m_0} \Gamma(r,s) + \Gamma^*(r,s)\right] / 2 + c}\,, \tag{5}$$

where $\boldsymbol{m=2r+m_0}$; $\boldsymbol{m_0}$=0 or 1; $\boldsymbol{n=s}$, and $\boldsymbol{c}$ is a positive bias constant. Indeed,

$$\boldsymbol{\tilde{\Gamma}(m,n) = \begin{cases} (-1)^r \, Re[\Gamma(r,s)], & m_0 = 0; \\ (-1)^r \, Im[\Gamma(r,s)], & m_0 = 1. \end{cases}}$$

In order to avoid constant biasing of $\boldsymbol{\Gamma_{re}}$ and $\boldsymbol{\Gamma_{im}}$ at their recording, and resulting loss of energy effectiveness, Lee proposed allocating four neighbouring-in-raster medium resolution cells for recording one hologram sample (Fig.5 b). With a reconstruction angle as defined by Eq. (4), a phase detours **0, π/2, π**, and **3π/2** correspond to them. Therefore, these cells should be written in the following order: .$\boldsymbol{(\Gamma_{re}+|\Gamma_{re}|)/2}$; $\boldsymbol{(\Gamma_{im}+|\Gamma_{im}|)/2}$; $\boldsymbol{(\Gamma_{re}-|\Gamma_{re}|)/2}$; $\boldsymbol{(\Gamma_{im}-|\Gamma_{im}|)/2}$. One may see that in this case all

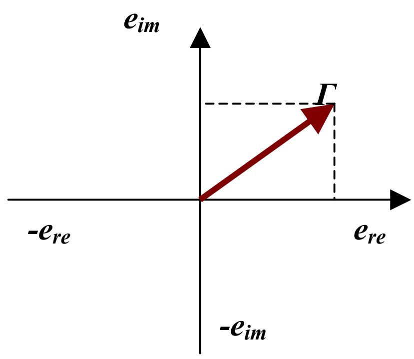


Fig. 4 Orthogonal and bi-orthogonal representation of complex numbers

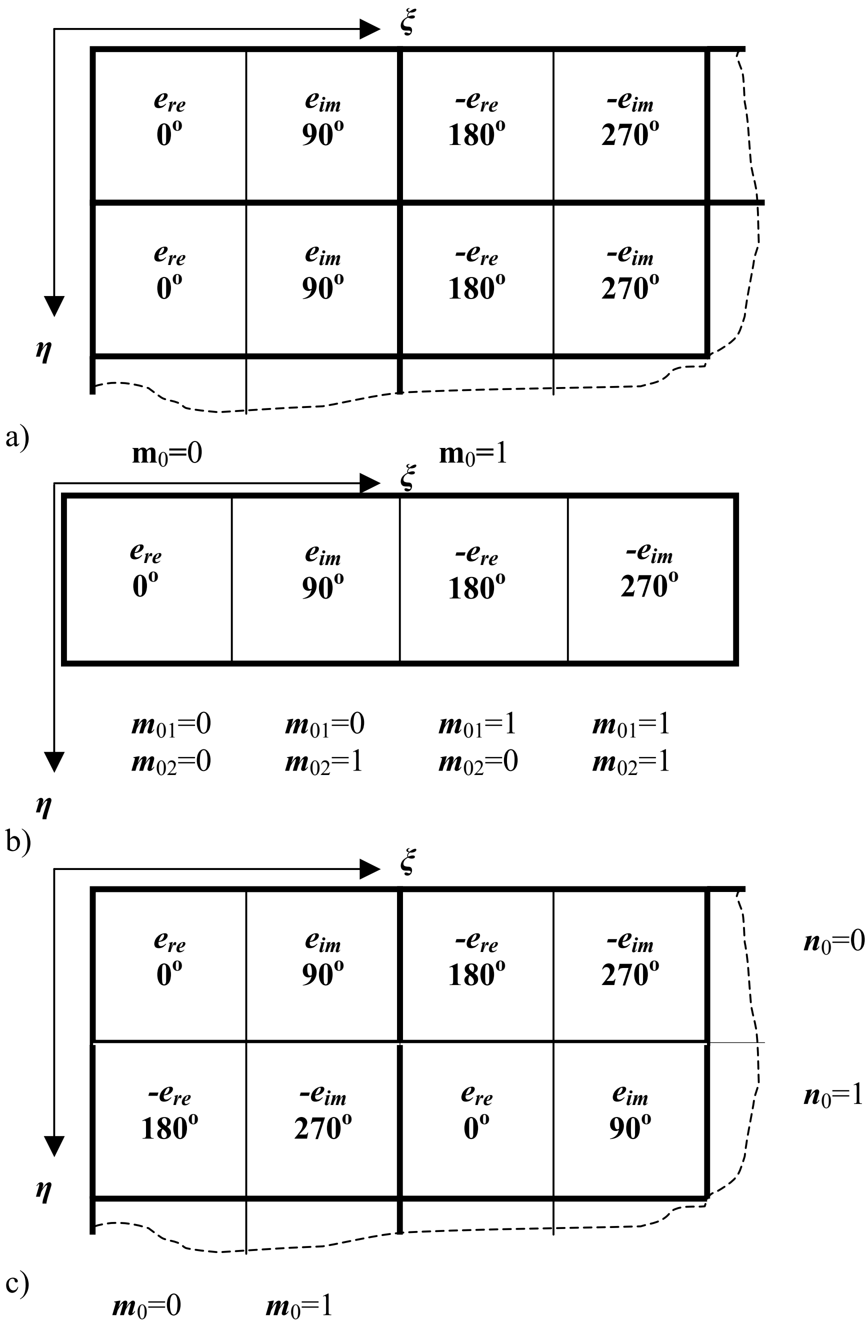


Fig. 5 Hologram coding by decomposition of complex numbers in orthogonal (a,b) and bi-orthogonal (c) bases Resolution cells corresponding to one hologram sample are outlined by a bold line

the recorded values are nonnegative and the method represents a vector in the complex plane in a bi-orthogonal basis $\{\boldsymbol{e_{re}}, \boldsymbol{e_{im}}, -\boldsymbol{e_{re}}, -\boldsymbol{e_{im}}\}$ (Fig. 4):

$$\Gamma=. e_{re}\,(\Gamma_{re}+|\Gamma_{re}|)/2+ e_{im}\,(\Gamma_{im}+|\Gamma_{im}|)/2+(-e_{re})\,(\Gamma_{re}-|\Gamma_{re}|)/2+(-e_{im})(\Gamma_{im}-|\Gamma_{im}|)/2 \qquad (6)$$

In the notation of (5), the Lee's method may be described as follows:

$$\tilde{\Gamma}(m,n)=\left\{i^{m_{02}}\left[(-1)^{m_{02}}\,\Gamma(r,s)+\Gamma^{*}(r,s)\right]+(-1)^{m_{01}}\,i^{m_{02}}\left[(-1)^{m_{02}}\,\Gamma(r,s)+\Gamma^{*}(r,s)\right]\right\}/4 \qquad (7)$$

In coding by the Lee method, it is required, that the size of the medium resolution cell in one direction be four times smaller than that in the perpendicular one. In this way the proportions of the reconstructed picture can be preserved. One can allocate two resolution cells in two neighbouring raster rows (Fig.5,c) for each sample of the mathematical hologram coded by this method; that is, to write the hologram samples according to the following relation

$$\tilde{\Gamma}(m,n)=\left\{i^{m_{0}}\left[(-1)^{m_{0}}\,\Gamma(r,s)+\Gamma^{*}(r,s)\right]+(-1)^{n_{0}}\,i^{m_{0}}\left[(-1)^{m_{0}}\,\Gamma(r,s)+\Gamma^{*}(r,s)\right]\right\}/4 \qquad (8)$$

where $m=2r+2m_0$; $n=2s+n_0$; and $m_0, n_0=0,1$.. In doing so, the image is reconstructed along a direction making angles θ_ξ and θ_η in the hologram plane and defined as follows:

$$\Delta\xi\cos\theta_\xi=\lambda/2;\quad \Delta\eta\cos\theta_\eta=\lambda/2. \qquad (9)$$

Amplitude media: simplex coding

Vector representation in the biorthogonal basis of Eq.(6) is redundant, which manifests itself in the fact that two of the four components are always zero. This redundancy may be reduced if complex numbers to be decomposed with respect to a two-dimensional simplex (e_0, e_{120}, e_{240}) (Fig.6):

$$\Gamma=\Gamma_0\, e_0+\Gamma_1\, e_{120}+\Gamma_{-1}\, e_{240} \qquad (10)$$

Similar to the bi-orthogonal basis, this basis is not linearly independent because

$$e_0+e_{120}+e_{240}=0 \qquad (11)$$

and its redundancy may be exploited in order to insure that the components Γ_0, Γ_1, \ Γ_{-1} be nonnegative. There exist two versions of hologram coding by two-dimensional simplex, proposed by Burckhardt and Chavel and Hugonin.

Burckhardt's idea is to represent an arbitrary vector in a complex plane as the sum of two components directed along those of three vectors e_0, e_{120}, e_{240}), which confine the part of plane where this vector is situated. It follows that of the three

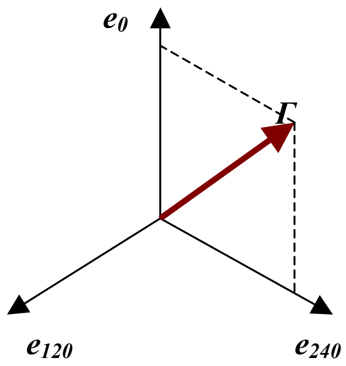


Fig. 6 Symplex representation of complex numbers

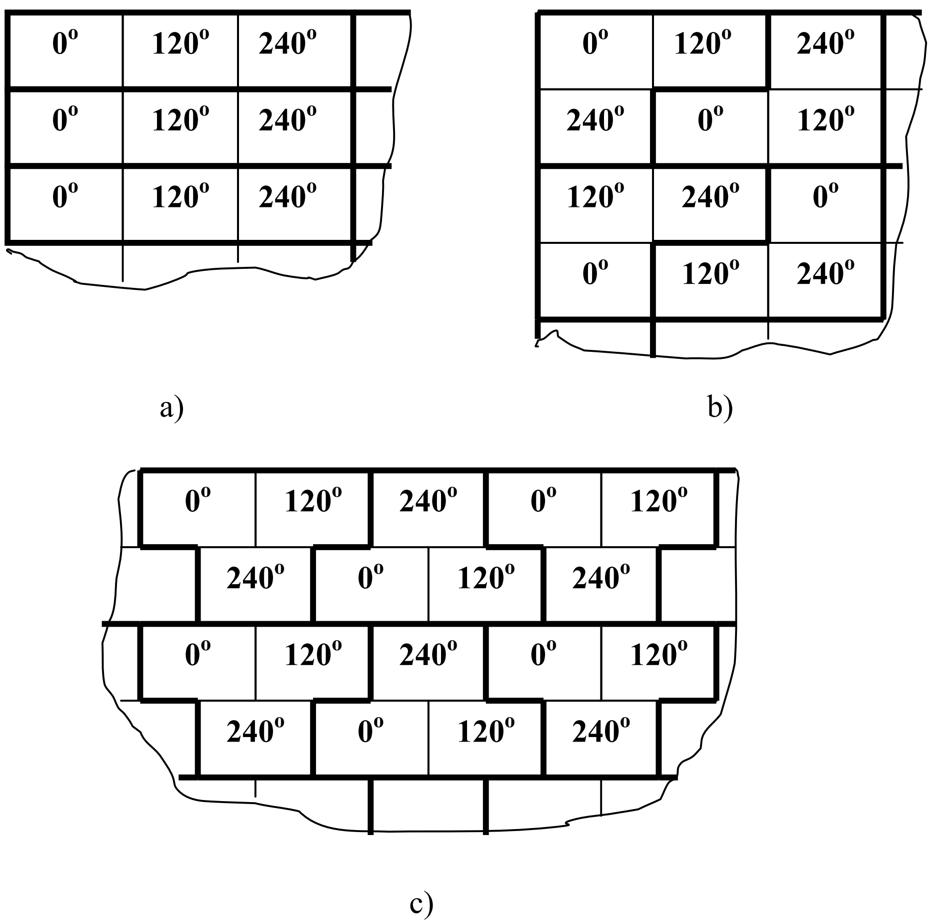


Fig. 7. Hologram coding by decomposition of complex numbers with respect to a simplex: Burkhardt's method (a); a version of the method with allocation of cells representing vectors $\boldsymbol{e_0}$, $\boldsymbol{e_{120}}$ and $\boldsymbol{e_{240}}$ along the rectangle raster (b); a version with allocation along the hexagonal raster. Triples of medium resolution cells used for encoding one hologram sample are outlined with a bold line.

numbers Γ_0, Γ_1, and Γ_{-1} defining Γ through Eq.(41), two always positive and are projections of the vector Γ on corresponding basis vectors, and the third one is zero. The following relations may be obtained for Γ_0, Γ_1 and Γ_{-1} from this condition:

$$\begin{aligned}
\Gamma_0 &= \left[(1+signA)(|B|-B)+(1+signA)(|C|-C)\right]/2\sqrt{3}\ ;\\
\Gamma_1 &= \left[(1+signC)(|A|-A)+(1+signC)(|B|-B)\right]/2\sqrt{3}\ ;\\
\Gamma_{-1} &= \left[(1+signB)(|C|-C)+(1+signB)(|A|-A)\right]/2\sqrt{3}\ ,
\end{aligned} \tag{12}$$

where,

$$\begin{aligned}
A &= \left(\Gamma+\Gamma^*\right)/2\\
B &= \left[\Gamma\, exp(i2\pi/3)+\Gamma^*\, exp(-i2\pi/3)\right]/2\\
C &= \left[\Gamma\, exp(-i2\pi/3)+\Gamma^*\, exp(i2\pi/3)\right]/2\ ,
\end{aligned} \tag{13}$$

where ***signA***, ***signB***, and ***signC*** are signs of A,B, and C.

Chavel and Hugonin noted that by virtue of Eq.(42), the addition of an arbitrary constant V to Γ_0 ,Γ_1 and Γ_{-1} will leave Eq.(41) unchanged. Therefore, they suggested determinating Γ_0 ,Γ_1 and Γ_{-1} in the form

$$\Gamma_k = \Gamma_k^0 + V, \quad k=-1,0,1 \tag{14}$$

by imposing on Γ_k the following constraint: $\sum_{k=-1}^{1}\Gamma_k^0 = 0$, and choosing V so as to make { Γ_k } non-negative. In this case, { Γ_k } are defined as

$$\begin{aligned}
\Gamma_0^0 &= \left(\Gamma+\Gamma^*\right)/2 = 2A/3\ ,\\
\Gamma_1^0 &= \left[\Gamma\, exp(-i2\pi/3)+\Gamma^*\, exp(i2\pi/3)\right]/3 = 2C/3\ ,\\
\Gamma_{-1}^0 &= \left[\Gamma\, exp(-i2\pi/3)+\Gamma^*\, exp(i2\pi/3)\right]/3 = 2B/3\ .
\end{aligned} \tag{15}$$

The bias V, evidently, may differ between different hologram samples. Understandably, the choice of bias value influences the quality of the reconstructed image.

When recording a hologram, one may code the phase angle corresponding to the vectors e_0, e_{120}, e_{240} by means of the detour phase method mentioned above by writing the components Γ_0, Γ_1, and Γ_{-1} into neighboring hologram resolution cells in the raster. If the three components of the simplex-decomposed vector are allocated according to Burckhardt in three row-neighboring raster resolution cells enumerated with respect to m_0 from 0 to 2 (see Fig.7,a), the image is reconstructed at an angle

$$\theta_\xi = arccos(\lambda/3\Delta\xi) \tag{16}$$

to the axis ξ coinciding with the direction of the hologram raster rows.

One may readily demonstrate by using Eq.(12) that the following formula relating values written in cells (m,n) of the medium with samples of the mathematical hologram $\Gamma(r,s)$ corresponds to the Burckhardt coding method:

$$\tilde{\Gamma}(m,n) = \frac{1}{2\sqrt{3}}\left[\left(\frac{Re[\Gamma(r,s)exp(-i2\pi m_0/3)]}{|Re[\Gamma(r,s)exp(-i2\pi m_0/3)]|}\right)\times\right.$$
$$(|Re[\Gamma(r,s)exp(-i2\pi(m_0-1)/3)]| - Re[\Gamma(r,s)exp(-i2\pi(m_0-1)/3)]) +$$
$$\left(1-\frac{Re[\Gamma(r,s)exp(-i2\pi m_0/3)]}{|Re[\Gamma(r,s)exp(-i2\pi m_0/3)]|}\right)\times$$
$$\left.(|Re[\Gamma(r,s)exp(-i2\pi(m_0+1)/3)]| - Re[\Gamma(r,s)exp(-i2\pi(m_0+1)/3)])\right], \qquad (17)$$

wher $Re(z)$ is real part of z, $m=3r+m_0$, m_0=0,1,2; and $n=s$. A hologram coded by the Chavel-Hugonin method may be written in a similar manner.

When recording simplex components along raster rows, three times more hologram resolution cells will be used along raster rows than along columns. Therefore, the scale of the reconstructed image along the coordinate axes will differ by a factor three. In order to equalize the scales, each hologram row may be repeated three times, but this implies excessive use of resolution cells. The redundancy may be decreased by two-dimensional packing of the cells representing the vectors e_0, e_{120}, and e_{240} . Fig.7, b) and c) show two packing (over orthogonal and hexagonal rasters) which provide scale ratios 2:1.5 and $3\sqrt{3}/4$ respectively instead of 1: We omit the appropriate formulas because of their awkwardness.

Evidently, the above methods may be used for hologram recording in binary media as well as in amplitude media. In this case, projections of the complex number on basic vectors are represented by varying the size of the transparent aperture in each of the appropriate resolution cells.

"PCM" – coding

A specifically binary method implementing the idea of additive representation of a complex number is that of "pulse code modulation" (PCM), where each mathematical hologram sample is represented by groups of, say, KxL neighbouring resolution cells (K along raster rows, L along columns), each cell being either completely transparent (reflecting) or opaque. Obviously, the total number of states of such a group is, 2^{KL}, therefore, 2^{KL} different vectors may be coded by each group, and all of them may be computed in advance. It is then required to find for each sample of the mathematical hologram a nearest vector of all possible 2^{KL} vectors and to write the corresponding combination of transparent and opaque cells. The search may be done either by means of selection, which may require up to 2^{KL} steps, or by "weighting", which reduces this estimate to KL.

The PCM method makes much more effective use of the medium degrees of freedom than that of Lohmann or similar binary methods for the exponential representation of complex numbers. Indeed, with the same number KL of medium resolution cells allocated for representation of one mathematical hologram sample, the Lohmann method enables coding of only KL different vectors (e.g. vectors having K and L different values of amplitude and phase, respectively) rather than of 2^{KL} , as with PCM coding.

Recording methods for phase media

For hologram recording on phase media, one can disregard magnitude information and record only the hologram sample phase, which enables the full use of purely phase media. Although such holograms (called "*kinoforms*") contain some distortions when the wave field is reconstructed, they are advantageous in terms of the energy of reconstruction light because the total energy of the reconstruction light is transformed into the energy of the reconstructed wave field without being absorbed in the hologram. Moreover, the distortions may to some degree be reduced by an appropriate choice of the diffusion component of the wave field phase on the object.

Other methods for recording in phase media rely upon an additive representation of complex numbers. In this case, component vectors should have standard length and controllable direction (phase angle). The simplest version is a representation of vectors as a sum of such components:

$$\Gamma = A_0 \, exp(i\varphi_1) + A_0 \, exp(i\varphi_2). \tag{18}$$

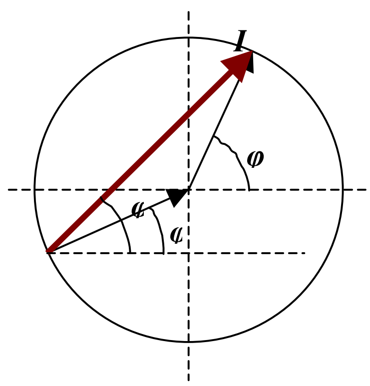


Fig. 8. Additive representation of complex numbers by a sum of equal-length vectors

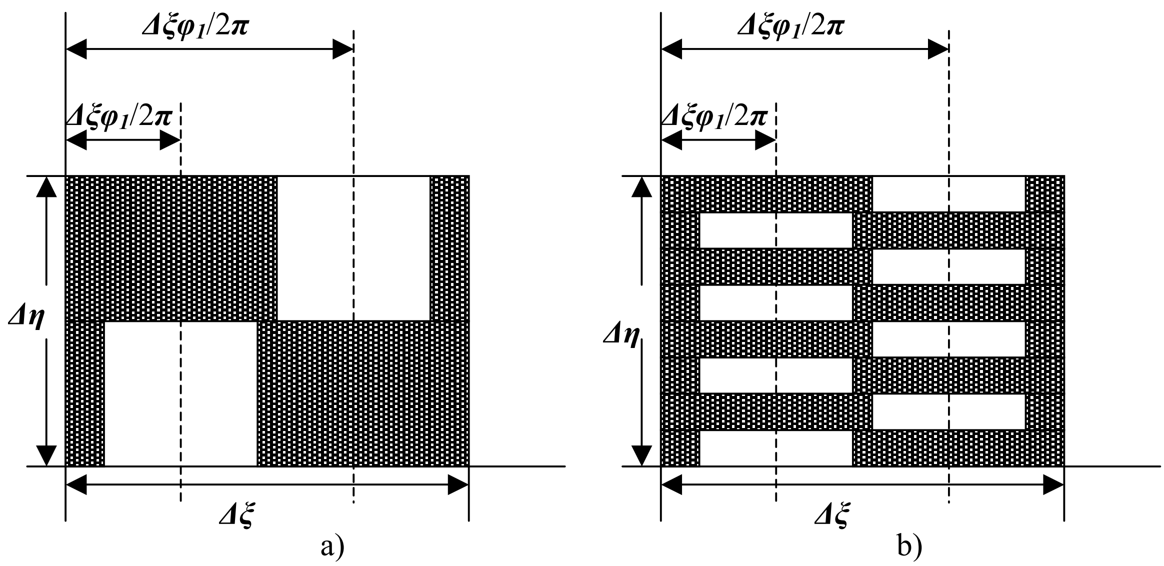


Fig. 9. Two-phase coding for hologram recording on a binary medium: Two separate cells (a) and decomposition of cells into alternative sub-sells.

As may be easily seen from Fig.8 , phase angles φ_1 and φ_2 of the component vectors are defined by

$$\varphi_1 = \varphi - arccos(|\Gamma| / 2A_0) \qquad \varphi_2 = \varphi + arccos(|\Gamma| / 2A_0), \tag{19}$$

where $|\Gamma|$ is the modulus, and φ the phase angle of the recorded hologram sample. This method was named two-phase coding methods. It may be used for hologram recording both in phase and binary media.

When recording in phase media, two neighbouring medium resolution cells may be allocated for representing two vector components (see Fig.5, a). Formally this may be written as

$$\tilde{\Gamma}(m,n) = A_0\, exp\{i[\varphi(r,s) - (-1)^{m_0}\, arccos(|\Gamma(r,s)| / 2A_0)]\}, \tag{20}$$

where $m = 2r + m_0$; $m_0 = 0\ ,\ 1$; $n = s$.

In this case, the image will be reconstructed in a direction normal to the hologram plane, because the optical path difference of rays passing along this direction through the neighbouring resolution cells is zero. This holds, however, only for the central area of the image through which the optical axis of the reconstruction system passes. In the peripheral areas of the image, there is some phase shift between the rays, thus leading to distortions in the peripheral image. This will be discussed in more detail in the section to follow.

Shmaryov proposed writing two holograms of each component vector of Eq.(19) separately and summing the images reconstructed from them in a special optical set up.

Hsue and Sawchuk considered two versions of the double-phase method oriented to binary media for recording with coding of the component vector phases by the detour phase method. In the first version, two elementary cells of the binary medium are allocated to each of the two component vectors, their phases being coded by a shift of the transparent (or completely reflecting) aperture along a direction perpendicular to the line connecting cell centers (Fig.9, a).

This technique exhibits the same distortions as does recording in phase media due to mutual spatial shift of elementary cells. In order to reduce the distortions, Hsue and Sawchuk proposed decomposing each elementary cell into subcells alternating as shown in Fig.9, b. Obviously, the relative shift of elementary cells will now be $\Delta\eta/n$ rather than $\Delta\eta$, and the peripheral image distortions may be cut down significantly. Alternation of subcells is an effective means for decreasing distortion, but like any binary recording, it requires more degrees of freedom in the medium and hologram recording devices.

The two-phase method is readily generalized to the case of multi phase coding by vector decomposition into K equal-length components:

$$\Gamma = \sum_{k=1}^{K} A_0\, exp(i\varphi_k) \tag{21}$$

Γ is a complex number, Eq.(52) represents two equations for K unknown values of phases $\{\varphi_k\}$. They have a unique solution, Eq.(19), only for $K = 2$. At $K > 2$, $\{\varphi_k\}$ may be chosen in a rather arbitrary manner. For example, for odd K it is more convenient to choose $\{\varphi_k\}$ so as to make an arithmetic progression of the phase angles φ_k :

$$\varphi_{k+1} - \varphi_k = \varphi_{k+1} - \varphi_k = \theta. \tag{22}$$

In this case,we obtain the following equations for the increment θ:

$$\frac{\sin K\theta / 2}{\sin\theta / 2} = \frac{|\Gamma|}{A_0} \tag{23}$$

and phase angle φ_k:

$$\varphi_k = \varphi + (k - (K+1)/2)\theta;\ k = 1,2,...,\mathrm{K} \tag{24}$$

For odd K, Eq.(23) boils down to an algebraic equation of power $(K-1)/2$ with respect to $\sin^2\theta/2$. Thus, for $K = 3$, we obtain

$$\theta = 2\arcsin\left(\frac{\sqrt{3}}{2}\sqrt{1-\frac{|\Gamma|}{3A_0}}\right) \tag{25}$$

For even K, it is more expedient to separate all the component vectors into two groups having the same phase angles φ_1 and φ_2 which are defined by analogy with Eq.(19) as

$$\varphi_1 = \varphi - \arccos(|\Gamma| / 2KA_0) \quad \varphi_2 = \varphi + \arccos(|\Gamma| / 2KA_0).$$

If $K > 2$ is chosen, the dynamic range of possible hologram values may be extended because the maximal reproducible amplitude in this case is KA_0. Most interesting of the $K > 2$ cases are those with $K = 3$ and 4 because the two-dimensional spatial degrees of freedom of the medium and hologram recorder may be used more effectively through allocation of the component vectors according to Figs.5, c) , and 7, b) and c).

Coding with a spatial carrier

Hologram coding methods that rely on the additive representation of complex numbers have one more important property in common. All of them implicitly introduce some form of spatial carrier and of nonlinear signal transformation of the signal with a spatial carrier similar to the classical method for recording optical holograms.

One can to check that Eqs.(5), (7), (8), (17), and (20) may be rewritten in the following equivalent form explicitly containing hologram samples multiplied by those of the spatial carrier with respect to one or both coordinates

$$\begin{aligned}&\tilde{\Gamma}(m,n) = RE\{\Gamma(r,s)\exp(-i2\pi m/2)\}/4 + c\,;\\ &m = 2r + m_0; \quad m = 0,1\,; n = s\\ &\tilde{\Gamma}(m,n) = rctf\{Re[\Gamma(r,s)\exp(-i2\pi m/2)]\}/2\end{aligned} \tag{5'}$$

$$m = 4r + 2m_{01} + m_{02};\ m_{01}, m_{02} = 0,1;\ n=s \tag{7'}$$

$$\tilde{\Gamma}(m,n) = rctf\{Re[\Gamma(r,s)exp[-i2\pi(m+2n)/2]]\}/2$$
$$m = 2r + m_0;\ \ n = 2s + n_0;\ \ m_0, n_0 = 0,1; \tag{8'}$$

$$\tilde{\Gamma}(m,n) = \frac{1}{3}\sum_{p=0}^{1} hlim\{Re[\Gamma(r,s)exp[-i2\pi(m+p)/3]]\}\times$$
$$rctf\{Re[\Gamma(r,s)exp[-i2\pi(m+p+1/2)/3]]\}$$
$$m = 3r + m_0;\ \ m_0 = 0,1;\ n=s \tag{17'}$$

$$\tilde{\Gamma}(m,n) = A_0\, exp\{i[\varphi(r,s) - cos(2\pi m/2)arccos(|\Gamma(r,s)|/2A_0)]\};$$
$$m = 2r + m_0;\ \ m = 0,1;\ n = s \tag{20'}$$

where ***rctf(z)*** is the "rectifier"

$$rctf(z) = \begin{cases} z, & z \geq 0; \\ 0, & z < 0 \end{cases} \tag{27}$$

and ***hlim(z)*** is the "hard-limiter"

$$h\,lim(z) = \begin{cases} 1, & z \geq 0; \\ 0, & z < 0 \end{cases} \tag{28}$$

As one may see from these expressions, the spatial carrier has a period no greater than one half that of hologram sampling. At least two samples of the spatial carrier correspond to one hologram sample in order to enable reconstruction of amplitude and phase of each hologram sample through the modulated signal of the spatial carrier. This redundancy implies that, in order to modulate the spatial carrier by a hologram, one or more intermediate samples are required between the basic ones. They may be determined by any of variety of hologram sample interpolation methods.

Zero order is the simplest interpolation method. Samples are simply repeated. It is this interpolation that is implied in the recording methods described above. For instance, according to Eq.(5) each hologram sample is repeated two times for two samples of the spatial carrier, in Eq.(7) it is repeated four times for four samples, etc. Of course such an interpolation characteristic found in all the codings based on the detour phase method yields a very rough approximation of intermediate samples. For partial correction of these distortions, mostly for binary coding, several iterative algorithms of hologram calculation have been proposed. All of them are built around an iterative determination of the phase of the hologram sample situated in that place of the elementary cell where, according to the detour phase method, the transparent aperture should be situated.

In order to determine the exact values of the desired intermediate samples, it is necessary at the hologram synthesis to perform ***SDFT(u,v,p,q)*** as many times as many additional samples are required for one basis sample and with appropriate shift parameters ***p*** and ***q***. It should be also noted that the symmetrization method may be regarded as an analog of the method of Eq.(5) with ideal interpolation of intermediate samples. In the symmetrization method, such an interpolation is done automatically and the restored image is free of noise images.

Along with methods that introduce the spatial carriers implicitly, there are methods based on its explicit introduction. The majority of them were proposed in the earliest papers on digital holography as way to simulate optical hologram recording and from the analogy between holograms and interferograms.

Of the methods oriented to amplitude media, let us mention those of Burch who proposed to record a hologram as

$$\tilde{\Gamma}(m,n) = \{1 + |\Gamma(m,n)| cos[\varphi(m,n) + 2\pi m / a]\} / 2 \qquad (29)$$

and of Huang and Prasada, who suggested introducing a constant bias | $\boldsymbol{\Gamma(m,n)}$| in order to enhance the contrast of the useful component of the hologram (second term of the sum in Eq.(29)

$$\tilde{\Gamma}(m,n) = |\Gamma(m,n)| \{1 + cos[\varphi(m,n) + 2\pi m / a]\} / 2\,. \qquad (30)$$

Of binary-media-oriented methods, one may mention that described by

$$\tilde{\Gamma}(m,n) = h\,lim\{cos[arcsin(|\Gamma(m,n)| / A_0)] + cos[\varphi(m,n) + 2\pi m / a]\} \qquad (31)$$

In Eq.(30) and (31), $\boldsymbol{a}$ is the period of the spatial carrier.

Of the phase-media-oriented methods, we may mention that of Kirk and Jones who proposed recording the function

$$\tilde{\Gamma}(m,n) = A_0\, exp\{i[\varphi(m,n) - h(m,n) cos\, 2\pi m / a]\} \qquad (32)$$

where $\boldsymbol{h(m,n)}$ depends somehow on $|\boldsymbol{\Gamma(m,n)}|$ and the number of the diffraction order where the reconstructed image should be obtained. In a sense, this method is equivalent to multi phase coding methods and at $\boldsymbol{a = 2}$ when

$$h(m,n) = arccos(|\Gamma(m,n)| / 2A_0); \qquad (33)$$

it coincides with that of two-phase coding of Eq.(20).

Additional references:

1. L. Yaroslavsky, M. Eden, Fundamentals of Digital Optics, Ch. 10. Synthesis of Holograms, Birkhauser, Boston, 1996.
2. L. Yaroslavsky, N. Merzlyakov, Methods of Digital Holography, Consultens Bureau, N.Y., 1980

Test questions

1. Explain the problem of recording computer generated holograms (CGH) as a specific digital-to-analog conversion problem
2. Characterize the types of recording media for recording CGH
3. Explain the “symmetrisation” method for recording CGH
4. Explain “phase detour” method for coding phase information on amplitude media
5. Describe methods for recording holograms on “binary” media
6. What is “PCM-coding?
7. Explain orthogonal, bi-orthogonal and simplex coding methods
8. Explain amplitude media coding methods in terms of coding with a “spatial carrier”
9. What is “kinoform”?
10. Describe principles of recording amplitude and phase of hologram samples on phase media